22nd International Symposium on
Mathematical Theory of Networks and Systems
July 11-15, 2016. Minneapolis, MN, USA

# Robust Regulation for Port-Hamiltonian Systems of Even Order

Jukka-Pekka Humaloja*, Lassi Paunonen* and Seppo Pohjolainen*

*Abstract*— **We present a controller that achieves robust regulation for a port-Hamiltonian system of even order. The controller is especially designed for impedance energy-preserving systems. By utilizing the stabilization results for port-Hamiltonian systems together with the theory of robust output regulation for exponentially stable systems, we construct a simple controller that solves the Robust Output Regulation Problem for an initially unstable system. The theory is illustrated on an example where we construct a controller for one-dimensional Schrödinger equation with boundary control and observation.**

## I. Introduction

The class of port-Hamiltonian systems includes models of flexible structures, traveling waves, heat exchangers, bioreactors, and, in general, lossless and dissipative hyperbolic systems on one-dimensional spatial domain [7]. Due to this vast coverage of models, the stability and stabilization of port-Hamiltonian systems have been subjects of active research during the past decade [1], [12]. Stability and stabilization properties of systems are essential for robust output regulation problems, which ties robust regulation for port-Hamiltonian systems closely to this field of research.

The Internal Model Principle is the key to understanding how control systems can be robust, i.e., tolerate perturbations in the systems' parameters. The type of robust controller (low-gain controller) proposed by Davison [3] for stable systems has many practical advantages, as the structure of the controller is simple and it can be tuned with input-output measurements from the open loop system. The controller was generalized to infinite-dimensional systems and its tuning process was simplified in [5], [6]. The Internal Model Principle was generalized to regular linear systems in [9], [10]

In this paper, we construct a robust regulating controller for an impedance energy-preserving port-Hamiltonian system of even order. Even though the considered system is initially unstable, by combining output feedback with a typical controller structure we will be able to construct a simple controller that achieves robust output regulation on the system. By robust regulation we mean that the controller exponentially asymptotically tracks the reference signal $y_{ref}$, exponentially asymptotically rejects the boundary disturbance $w$ and allows some perturbations in the plant [6].

As the main contribution of this paper we construct a simple robust regulating controller for an initially unstable system. Using the stability results presented in [1] we derive a sufficient criterion for exponential stability of port-Hamiltonian systems of even order. With the new criterion, we will show that the asymptotically stabilizing output feedback presented in [12] actually achieves exponential stability for impedance energy-preserving port-Hamiltonian systems of even order. When the system is exponentially stabilized, we can utilize the controller structure introduced in [5], [6] for exponentially stable systems, and construct a simple robust regulating controller for a system that was initially unstable.

The structure of this paper is as follows. In Section III we give some required background to port-Hamiltonian systems. Furthermore, we will present a sufficient condition for an even-order port-Hamiltonian system to be exponentially stable, and we will use the result to exponentially stabilize a port-Hamiltonian system. In Section IV we will introduce the control system, and in Section V we present the Robust Output Regulation Problem (RORP) and the Internal Model Principle. In Section VI we will present our main result which will be illustrated in Section VII where we construct a controller for one-dimensional Schrödinger equation. In Section VIII we conclude the paper.

## II. Notation

Here $\mathcal{L}(X, Y)$ denotes the set of bounded linear operators from the normed space $X$ to the normed space $Y$. The domain, range, null space and resolvent of a linear operator $A$ are denoted by $D(A), \mathcal{R}(A)$, $\mathcal{N}(A)$ and $\rho(A)$, respectively. A strongly continuous ($C_0$-) semigroup $T_A(t)$ generated by $A$ is exponentially stable if there are positive constants $M$ and $\alpha$ such that $||T_A(t)|| \leq Me^{-\alpha t}$.

## III. Background on Port-Hamiltonian Systems

A linear port-Hamiltonian system of order $N$ on the spatial interval $\zeta \in [a, b]$ is given by

$$\frac{\partial}{\partial t}x(\zeta,t) = \mathcal{A}x(\zeta,t), \quad x(0) = x_0, \tag{1a}$$

$$u(t) = \mathcal{B}x(\cdot,t), \tag{1b}$$

$$y(t) = \mathcal{C}x(\cdot,t), \tag{1c}$$

where $\mathcal{B}$ and $\mathcal{C}$ are linear operators, and the operator $\mathcal{A}$ is defined by

$$\mathcal{A}x(\zeta,t) := \sum_{k=0}^{N} P_k \frac{\partial^k(\mathcal{H}(\zeta)x(t,\zeta))}{\partial\zeta^k}, \tag{2}$$

where the matrices $P_k \in \mathbb{C}^{n\times n}$ satisfy the condition $P_k^* = (-1)^{k+1}P_k$ for $k \geq 0$, and the matrix $P_N$ is assumed to be invertible [12]. The Hamiltonian density matrix function $\mathcal{H} : [a, b] \to \mathbb{C}^{n\times n}$ is a measurable function such that there

*Tampere University of Technology, Department of Mathematics, P.O.Box 533, 33101 Tampere, Finland, firstname.lastname@tut.fi

exists $0 < m \leq M$ such that for almost every $\zeta \in [a,b]$ we have $\mathcal{H}(\zeta) = \mathcal{H}(\zeta)^*$ and $m|\xi|^2 \leq \xi^*\mathcal{H}(\zeta)\xi \leq M|\xi|^2$ for $\xi \in \mathbb{C}^n$ [1]. The energy state space $X = L^2([a,b],\mathbb{C}^n)$ is equipped with the inner product

$$\langle f, g\rangle_X = \frac{1}{2}\int_a^b g(\zeta)^*\mathcal{H}(\zeta)f(\zeta)d\zeta, \tag{3}$$

and hence, $X$ is a Hilbert space.

Let

$$\begin{aligned}&\Phi : H^N([a,b];\mathbb{C}^n) \to \mathbb{C}^{2nN},\\ &\Phi(x) := (x(b),\ldots,x^{(N-1)}(b), x(a),\ldots,x^{(N-1)}(a))\end{aligned} \tag{4}$$

be the boundary trace operator and introduce the boundary port variables $f_\partial$, $e_\partial$ defined by

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} := \frac{1}{\sqrt{2}}\begin{bmatrix} Q & -Q \\ I & I \end{bmatrix}\Phi(\mathcal{H}x) := R_{ext}\Phi(\mathcal{H}x), \tag{5}$$

where $Q \in \mathbb{C}^{nN\times nN}$ is a block matrix given by

$$Q_{ij} := \begin{cases}(-1)^{j-1}P_{i+j-1}, & i+j \leq N+1 \\ 0, & \text{else}\end{cases}. \tag{6}$$

Note that since $P_N$ is assumed to be invertible, it follows that $Q$ is invertible, and hence, $R_{ext}$ is invertible as well. Using the boundary port variables we define the operators $\mathcal{B}$ and $\mathcal{C}$ as

$$\mathcal{B}x(t) := W_B\begin{bmatrix} f_\partial(t) \\ e_\partial(t)\end{bmatrix}, \tag{7a}$$

$$\mathcal{C}x(t) := W_C\begin{bmatrix} f_\partial(t) \\ e_\partial(t)\end{bmatrix}, \tag{7b}$$

where $W_B, W_C \in \mathbb{C}^{nN\times 2nN}$. [1]

Define the domain of the operator $\mathcal{A}$ as

$$D(\mathcal{A}) = \left\{\mathcal{H}x \in H^N([a,b],\mathbb{C}^n) \,\middle|\, W_B\begin{bmatrix} f_\partial \\ e_\partial\end{bmatrix} = 0\right\}. \tag{8}$$

Since we assumed $P_0$ to be skew-adjoint, it follows from [4, Thm. 4.1] that the operator $A$ generates a contraction semigroup if and only if $W_B\Sigma W_B^* \geq 0$, where

$$\Sigma = \begin{bmatrix} 0 & I \\ I & 0\end{bmatrix}. \tag{9}$$

Furthermore, for $N = 1$ we have from [7, Lem. 9.1.4] that if $W_B\Sigma W_B^* > 0$ then $\mathcal{A}$ generates an exponentially stable semigroup. By utilizing the following proposition [1, Prop. 2.14], we will show that the result of [7, Lem. 9.1.4] holds for $N = 2$ as well.

*Proposition 1:* [1, Prop. 2.14] Let $N = 2$ and $\mathcal{H} \in W^1_\infty([a,b];\mathbb{C}^{n\times n})$, and assume

$$\mathrm{Re}\langle \mathcal{A}x, x\rangle_X \leq -\gamma\left(||(\mathcal{H}x)(a)||^2 + ||(\mathcal{H}x)'(a)||^2 + ||(\mathcal{H}x)(b)||^2\right) \tag{10}$$

for $x \in D(A)$ and for some $\gamma > 0$. Then $\mathcal{A}$ generates an exponentially stable and contractive $C_0$-semigroup.

□

*Lemma 1:* Let $N = 2$. If $W_B\Sigma W_B^* > 0$, then the operator $\mathcal{A}$ with domain (8) generates an exponentially stable $C_0$-semigroup.

*Proof:* Following the proof of [7, Lem. 9.1.4] we write $W_B = S\,[I+V, \quad I-V]$, where $S$ is invertible and $VV^* < I$ (equivalently $V^*V < I$), and define a full rank matrix $W_C = [I+V^*, \quad -I+V^*]$. Thus, the matrix $\begin{bmatrix} W_B \\ W_C\end{bmatrix}$ is invertible.

Let $x \in D(\mathcal{A})$ be arbitrary. By definition of the domain of $\mathcal{A}$ we have that $\begin{bmatrix} f_\partial \\ e_\partial\end{bmatrix} \in \mathcal{N}(W_B)$. Following the proof of [7, Lem. 9.1.4], we may write

$$\begin{bmatrix} f_\partial \\ e_\partial\end{bmatrix} = \begin{bmatrix} I-V \\ -I-V\end{bmatrix} l$$

for some $l \in \mathbb{C}^{2n}$. Since $P_0^* = -P_0$, we have [1, Lem. 2.2] that

$$2\,\mathrm{Re}\langle \mathcal{A}x, x\rangle_X = \mathrm{Re}\langle f_\partial, e_\partial\rangle_{\mathbb{C}^{4n}} = l^*(-I+V^*V)l.$$

Furthermore, we have

$$\begin{aligned} y := W_C\begin{bmatrix} f_\partial \\ e_\partial\end{bmatrix} &= [I+V^*, \quad -I+V^*]\begin{bmatrix} I-V \\ -I-V\end{bmatrix} l \\ &= 2(I-V^*V)l,\end{aligned}$$

from which we obtain

$$\mathrm{Re}\langle \mathcal{A}x, x\rangle_X = \frac{1}{8}y^*\left[-I+V^*V\right]^{-1}y \leq -m_1||y||^2 \tag{11}$$

for some $m_1 > 0$, where we used that $-I+V^*V < 0$.

Using (5), the definition of the domain of $\mathcal{A}$ and the definition of $y$ we obtain

$$\begin{bmatrix} 0 \\ y\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} W_B \\ W_C\end{bmatrix}\begin{bmatrix} Q & -Q \\ I & I\end{bmatrix}\Phi(\mathcal{H}x) := W\Phi(\mathcal{H}x).$$

Since $P_N = P_2$ is assumed to be invertible and $\begin{bmatrix} W_B \\ W_C\end{bmatrix}$ is invertible, the matrix $W$ is invertible and $||Ww||^2 \geq m_2||w||^2$ for every $w \in \mathbb{C}^{4n}$ and some $m_2 > 0$. Taking norms on both sides we obtain

$$\begin{aligned}||y||^2 &= ||W\Phi(\mathcal{H}x)||^2 \\ &\geq m_2||\Phi(\mathcal{H}x)||^2 \\ &\geq m_2\left(||(\mathcal{H}x)(a)||^2 + ||(\mathcal{H}x)'(a)||^2 + ||(\mathcal{H}x)(b)||^2\right),\end{aligned} \tag{12}$$

and finally, by combining (11) and (12) we have reached relation (10), and thus, the operator $\mathcal{A}$ generates an exponentially stable $C_0$-semigroup by Proposition 1. ■

It should be noted that the authors of [1] have also generalized the result of Proposition 1 to port-Hamiltonian systems of even order, where relation (10) becomes

$$\mathrm{Re}\langle \mathcal{A}x, x\rangle_X \leq -\gamma\sum_{\zeta=a,b}\sum_{k=0}^{N-1}\alpha_{\zeta,k}\left|\left|(\mathcal{H}x)^{(k)}(\zeta)\right|\right|^2 \tag{13}$$

for some $\gamma > 0$ and certain $\alpha_{\zeta,k} \geq 0$ [1, Prop. 2.16]. It is easy to see that in the general case $N \in 2\mathbb{N}$ the estimation in equation (12) can be done so that when combined with (11), we obtain relation (13). Furthermore, since the estimation in equation (12) is the only part of the proof of Lemma 1 that depends on the order $N$, the result of Lemma 1 can be generalized to port-Hamiltonian systems of even order as well. We then arrive at the generalization of Lemma 1:

*Lemma 2:* Let $N \in 2\mathbb{N}$. If $W_B \Sigma W_B^* > 0$, then the operator $\mathcal{A}$ with domain (8) generates an exponentially stable $C_0$-semigroup.

□

Using Lemma 2 we can now show that a certain class of port-Hamiltonian systems of even order can be exponentially stabilized by negative output feedback. Consider the class of impedance energy-preserving port-Hamiltonian systems that are systems satisfying the relation [12]

$$\frac{1}{2}\frac{d}{dt}||x(t)||_X^2 = u^*(t)y(t). \tag{14}$$

An impedance energy-preserving system can be identified based on the matrices $W_B$ and $W_C$ by [4, Thm. 4.4]. Essentially the matrices are given a certain structure such that they satisfy

$$W_B \Sigma W_B^* = W_C \Sigma W_C^* = 0, \tag{15a}$$
$$W_B \Sigma W_C^* = W_C \Sigma W_B^* = I, \tag{15b}$$

which can be checked very easily.

Stabilization of impedance energy-preserving systems is considered in [12], where it is shown that negative output feedback asymptotically stabilizes an impedance energy-preserving system. We will now show that, for systems of even order, exponential stability is actually achieved.

*Lemma 3:* Consider the system (1) with $N \in 2\mathbb{N}$ and assume that $u$ and $y$ are such that $W_B$ and $W_C$ satisfy equations (15a)–(15b). Then the system can be exponentially stabilized using negative output feedback.

*Proof:* Using negative output feedback to the system, i.e., $u(t) = r(t) - \kappa y(t)$ where $\kappa > 0$, the closed-loop system is described by [12]

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}x(t), \\ (W_B + \kappa W_C)\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} &= (\mathcal{B} + \kappa\mathcal{C})x(t) = r(t), \\ \mathcal{C}x(t) &= y(t). \end{aligned} \tag{16}$$

Now, consider the operator $\mathcal{A}_s = \mathcal{A}|_{D(\mathcal{A}_s)}$, where

$$D(\mathcal{A}_s) = \left\{ \mathcal{H}x \in H^N([a,b],\mathbb{C}^n) \,\middle|\, W_\kappa \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = 0 \right\}, \tag{17}$$

where $W_\kappa = W_B + \kappa W_C$. It is shown in [12] that $W_\kappa$ satisfies $W_\kappa \Sigma W_\kappa^* = 2\kappa I > 0$, and hence, if $N \in 2\mathbb{N}$, the operator $\mathcal{A}_s$ generates an exponentially stable $C_0$-semigroup due to Lemma 2. ■

## IV. The Plant, Exosystem and Controller

In this section we will present the plant, the exosystem and the controller. The plant is an impedance energy-preserving port-Hamiltonian system of even order given by

$$\dot{x}(t) = \mathcal{A}x(t), \qquad x(0) = x_0, \tag{18a}$$
$$\mathcal{B}x(t) = u(t) + w(t), \tag{18b}$$
$$\mathcal{C}x(t) = y(t), \tag{18c}$$

where $\mathcal{A}$ is given in (2) with $N \in 2\mathbb{N}$, $\mathcal{B}$ and $\mathcal{C}$ are given in (7a)–(7b) with $W_B$ and $W_C$ satisfying (15a)–(15b), and $w(t)$ is a bounded and differentiable disturbance signal.

Since $W_B$ satisfies $W_B \Sigma W_B^* = 0$, the system (18) is a boundary control system [4, Thm. 4.2], and hence there are operators $A : D(A) \to X$ with $D(A) = D(\mathcal{A}) \cap \mathcal{N}(\mathcal{B})$ and $Ax = \mathcal{A}x$ for $x \in D(A)$, and $B \in \mathcal{L}(U,X)$ such that $\mathcal{R}(B) \subset D(\mathcal{A})$ and $\mathcal{B}Bu = u$ [2, Def. 3.3.2]. Using these operators, the transfer function from $u$ to $y$ is given by [6]

$$P(s) = \mathcal{C}(sI - A)^{-1}(\mathcal{A}B - sB) + \mathcal{C}B. \tag{19}$$

The exosystem that generates the boundary disturbance signal $w(t)$ and the reference signal $y_{ref}(t)$ is given by

$$\dot{v}(t) = Sv(t), \qquad v(0) = v_0 \tag{20a}$$
$$w(t) = Ev(t), \tag{20b}$$
$$y_{ref}(t) = -Fv(t) \tag{20c}$$

on a finite-dimensional space $W = \mathbb{C}^q$. Here $S = \mathrm{diag}(i\omega_1, i\omega_2, \ldots, i\omega_q)$ with $\{\omega_i\}_{i=1}^q \subset \mathbb{R}$ and $\omega_i \neq \omega_j$ for $i \neq j$, $E \in \mathcal{L}(W,U)$ and $F \in \mathcal{L}(W,Y)$. Furthermore, we assume that for every $k \in \{1,2,\ldots,q\}$ the transfer function $P(i\omega_k) \in \mathcal{L}(U,Y)$ is surjective, which is crucial to the solvability of the robust output regulation problem.

The dynamic error feedback controller is of the form

$$\dot{z}(t) = \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t) \qquad z(0) = z_0, \tag{21a}$$
$$u(t) = Kz(t) - \kappa y(t), \tag{21b}$$

where $e(t) = y(t) - y_{ref}(t)$ is the error signal, $\kappa > 0$, and the parameters $(\mathcal{G}_1, \mathcal{G}_2, K)$ are to be chosen such that robust output regulation is achieved. Note that in the usual formulation of the controller we have $\kappa = 0$, and hence, the parameter $\kappa$ is not included in the controller parameters. However, the extra term $-\kappa y(t)$ is required to exponentially stabilizing the plant (18). The controller (21) is an abstract linear system on Banach space $Z$. The operator $\mathcal{G}_1 : D(\mathcal{G}_1) \subset Z \to Z$ generates a $C_0$-semigroup on $Z$, $\mathcal{G}_2 \in \mathcal{L}(Y,Z)$ and $K \in \mathcal{L}(Z,U)$ [8].

In order to give the state-space presentation of the closed-loop control system, we define a new variable $\xi = x - B_s r - Gv$, where $r = Kz$, the operator $B_s \in \mathcal{L}(U,X)$ is such that $\mathcal{R}(B_s) \subset D(\mathcal{A})$ and $(\mathcal{B} + \kappa\mathcal{C})B_s r = r$, and the operator $G \in \mathcal{L}(W,X)$ is such that $\mathcal{R}(G) \subset D(\mathcal{A})$ and $(\mathcal{B} + \kappa\mathcal{C})Gv = Ev$. These operators exist as the plant (18) with input $u = Kz - \kappa y$ is a boundary control system [12]. Define now the extended state-space by $X_e := X \times \mathbb{C}^q$, and let $\xi_e(t) := (\xi(t), z(t))$ be the extended state. Following [6], the closed-loop control system can be written as

$$\dot{\xi}_e = A_e \xi_e + Hv + Dy_{ref}, \tag{22}$$

where $D(A_e) = D(A_s) \times \mathbb{C}^q$ and

$$A_e = \begin{bmatrix} A_s - B_sK\mathcal{G}_2\mathcal{C} & \mathcal{A}B_sK - B_sK(\mathcal{G}_1 + \mathcal{G}_2\mathcal{C}B_sK) \\ \mathcal{G}_2\mathcal{C} & \mathcal{G}_1 + \mathcal{G}_2\mathcal{C}B_sK \end{bmatrix},$$
$$H = \begin{bmatrix} \mathcal{A}G - B_sK\mathcal{G}_2\mathcal{C}G - GS \\ \mathcal{G}_2\mathcal{C}G \end{bmatrix},$$
$$D = \begin{bmatrix} B_sK\mathcal{G}_2 \\ -\mathcal{G}_2 \end{bmatrix}, \tag{23}$$

where the operator $A_s$ is given by $A_s : D(A_s) \to X$ with $D(A_s) = D(\mathcal{A}) \cap \mathcal{N}(\mathcal{B}+\kappa\mathcal{C})$ and $A_sx = \mathcal{A}x$ for $x \in D(A_s)$.

## V. The Robust Output Regulation Problem

In this section we formulate the robust output regulation problem and present a few related concepts. We consider perturbations $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{E}, \tilde{F}) \in \mathcal{O}$ of the operators $(\mathcal{A}, \mathcal{B}, \mathcal{C}, E, F)$ where the operators in the class $\mathcal{O}$ of admissible perturbations are such that $(i)$ the perturbed plant $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}})$ is a boundary control system and $(ii)$ $i\omega_k \in \rho(\tilde{\mathcal{A}})$ for $k \in \{1, 2, \ldots, q\}$. It is easy to see that these conditions are satisfied for all bounded and sufficiently small perturbations to $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and for arbitrary bounded perturbations to the operators $E$ and $F$ [10].

The following formulation of the robust output regulation problem is given in [10]:

**The Robust Output Regulation Problem.** Choose the controller $(\mathcal{G}_1, \mathcal{G}_2, K, \kappa)$ in such a way that the following are satisfied:

1) The closed-loop system generated by $A_e$ is exponentially stable.
2) For all initial states $\xi_{e0} \in X_e$ and $v_0 \in W$ the regulation error satisfies $e^{\alpha \cdot} e(\cdot) \in L^2([0, \infty); Y)$ for some $\alpha > 0$.
3) If the operators $(\mathcal{A}, \mathcal{B}, \mathcal{C}, E, F)$ are perturbed to $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{E}, \tilde{F}) \in \mathcal{O}$ in such a way that the closed-loop system remains exponentially stable, then for all initial states $\xi_{e0} \in X_e$ and $v_0 \in W$ the regulation error satisfies $e^{\tilde{\alpha} \cdot} e(\cdot) \in L^2([0, \infty); Y)$ for some $\tilde{\alpha} > 0$.

We say that a controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ incorporates a $p$-copy of the internal model of the exosystem $S$ if for all $k \in \{1, 2, \ldots, q\}$ we have $\dim(\mathcal{N}(i\omega_k - \mathcal{G}_1)) \geq \dim(Y)$ [8]. Since we assumed that the eigenvalues of $S$ are distinct and we have $\dim(Y) < \infty$, the controller incorporates a $p$-copy of the internal model of the exosystem if $\mathcal{G}_1 = \mathrm{diag}(i\omega_1 I_Y, i\omega_2 I_Y, \ldots, i\omega_q I_Y)$. Furthermore, a controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ is said to satisfy the $\mathcal{G}$-conditions if

$$\mathcal{R}(i\omega_k - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}, \tag{24a}$$
$$\mathcal{N}(\mathcal{G}_2) = \{0\}, \tag{24b}$$

for all $k \in \{1, 2, \ldots, q\}$. [8]

## VI. Construction of the Robust Controller

In this section we will prove that a controller of the form (21) with suitably chosen parameters $(\mathcal{G}_1, \mathcal{G}_2, K)$ and $\kappa > 0$ solves the Robust Output Regulation Problem for an impedance energy-preserving port-Hamiltonian system of even order.

*Theorem 1:* Let the control system be as described in Section IV. There is a controller of the form (21) such that for every $\kappa > 0$ there exists an $\epsilon_\kappa > 0$ such that for every $0 < \epsilon \leq \epsilon_\kappa$ the Robust Output Regulation Problem is solved.

*Proof:* Let us begin the proof from the stabilization of the plant (18). We denote temporarily $Kz(t) = r(t)$, and hence, the input for the plant (18) is of the form $u(t) = r(t) - \kappa y(t)$. Since the plant is an impedance energy-preserving port-Hamiltonian system of even order, such an input exponentially stabilizes the plant due to Lemma 3. Thus, there is an operator $A_s : D(A_s) \to X$ with $D(A_s) = D(\mathcal{A}) \cap \mathcal{N}(\mathcal{B} + \kappa\mathcal{C})$ and $A_sx = \mathcal{A}x$ for $x \in D(A_s)$ that generates an exponentially stable $C_0$-semigroup.

Now that the plant is exponentially stabilized, we will choose the controller parameters $(\mathcal{G}_1, \mathcal{G}_2, K)$ such that the controller exponentially stabilizes the closed-loop system and solves the robust output regulation problem. We can utilize the controller parameter choices made in [6] where a robust regulating controller was constructed for an exponentially stable system. Essentially the controller parameters are chosen in such a way that the controller incorporates a $p$-copy of the internal model of the exosystem and satisfies the $\mathcal{G}$-conditions.

Following [6] and [10] we define $Z = Y^q$ and choose the controller parameters as

$$\mathcal{G}_1 = \mathrm{diag}\,(i\omega_1 I_Y, i\omega_2 I_Y, \ldots, i\omega_q I_Y) \in \mathcal{L}(Z), \tag{25a}$$
$$K = \epsilon K_0 = \epsilon \left[K_0^1, K_0^2, \ldots, K_0^q\right] \in \mathcal{L}(Z, U), \tag{25b}$$
$$\mathcal{G}_2 = (\mathcal{G}_2^k)_{k=1}^q = (-(P_\kappa(i\omega_k)K_0^k)^*)_{k=1}^q = \begin{bmatrix} -(P_\kappa(i\omega_1)K_0^1)^* \\ \vdots \\ -(P_\kappa(i\omega_q)K_0^q)^* \end{bmatrix} \in \mathcal{L}(Y, Z), \tag{25c}$$

where $P_\kappa(i\omega_k) = P(i\omega_k)(I + \kappa P(i\omega_k))^{-1}$ is the transfer function of the stabilized plant [11]. As we assumed that $P(i\omega_k)$ is surjective for every $k \in \{1, 2, \ldots, q\}$, it follows that $P_\kappa(i\omega_k)$ is surjective as well for every $k$.

Since the surjectivity assumption of the transfer function holds, we choose the components $K_0^k$ of $K_0$ such that the operators $P_\kappa(i\omega_k)K_0^k$ are invertible, e.g., by choosing $K_0^k = P_\kappa(i\omega_k)^\dagger$ (the Moore-Penrose pseudoinverse of $P_\kappa(i\omega_k)$), in which case we have $\mathcal{G}_2^k = -I_Y$ for all $k \in \{1, 2, \ldots, q\}$ [10].

It has been shown in [6] that, if the plant is exponentially stable, there exists an $\epsilon^* > 0$ such that the closed-loop system is exponentially stable for every $0 < \epsilon \leq \epsilon^*$ and that the proposed controller solves the robust output regulation problem. Since we choose the controller parameters according to [6] and exponentially stabilized the plant, it follows from the results of [6] that, when the plant is exponentially stabilized with output feedback $u(t) = Kz(t) - \kappa y(t)$, where $\kappa > 0$, there exists an $\epsilon_\kappa > 0$ such that for every $0 < \epsilon \leq \epsilon_\kappa$ the closed-loop system is exponentially stable and the robust output regulation problem is solved. ■

## VII. Example

As an example we study Schrödinger equation on the spatial interval $\zeta \in [0,1]$ considered in [1], given by

$$\frac{\partial}{\partial t}w(\zeta,t) = i\frac{\partial^2}{\partial \zeta^2}w(\zeta,t), \quad t \geq 0, \tag{26a}$$

which is a second-order port-Hamiltonian system with $P_2 = i$, $P_1 = P_0 = 0$, $\mathcal{H}(\zeta) = 1$ and state $x(\zeta,t) = w(\zeta,t)$ [1]. The inputs are given by

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} x'(0,t) \\ x(1,t) \end{bmatrix} + w(t), \tag{26b}$$

and the outputs are given by

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} ix(0,t) \\ ix'(1,t) \end{bmatrix} \tag{26c}$$

Using the boundary port variables $f_\partial$ and $e_\partial$ the inputs and outputs can be written as

$$\begin{aligned} u(t) = W_B \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} &= \frac{1}{\sqrt{2}}\begin{bmatrix} i & 0 & 0 & 1 \\ 0 & i & 1 & 0 \end{bmatrix}\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}, \\ y(t) = W_C \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} &= \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & i & 0 \\ 1 & 0 & 0 & i \end{bmatrix}\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}. \end{aligned} \tag{27}$$

As the matrices $W_B$ and $W_C$ satisfy equations (15a)–(15b), the system (26) is an impedance energy-preserving port-Hamiltonian system of order two, and thus, we may use Theorem 1 to construct a robust regulating controller for the system.

Let us first consider the transfer function of the system (26), given by

$$P(s) = \begin{bmatrix} -\frac{\tanh(i\sqrt{is})}{\sqrt{is}} & \frac{i}{\cosh(i\sqrt{is})} \\ \frac{i}{\cosh(i\sqrt{is})} & -\sqrt{is}\tanh(i\sqrt{is}) \end{bmatrix}. \tag{28}$$

The transfer function is surjective for every $s \neq 0$ and $s \neq -i\left(\frac{(2m+1)\pi}{2}\right)^2$ where $m \in \mathbb{N}$, and thus, we cannot track signals including those frequencies.

Let the exosystem be given by $S = \operatorname{diag}(-4i\pi^2, -i\pi^2)$ and $E = F = I$. If we choose the output feedback parameter as $\kappa = 1$, the transfer function for the stabilized plant is given by $P_\kappa(s) = P(s)(I + P(s))^{-1}$, and thus, for the eigenvalues of the signal generator $S$ we have

$$P_\kappa(-4i\pi^2) = \frac{1}{2}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = P_\kappa(-i\pi^2)^*. \tag{29}$$

Thus, if we choose $K_0^k = P_\kappa(i\omega_k)^{-1}$, the controller parameters are given by

$$\begin{aligned} \mathcal{G}_1 &= \operatorname{diag}(-i4\pi^2, -i4\pi^2, -i\pi^2, -i\pi^2), \\ \mathcal{G}_2 &= \begin{bmatrix} -I_Y \\ -I_Y \end{bmatrix}, \\ K &= \epsilon\left[P_\kappa(-4i\pi^2)^{-1}, P_\kappa(-i\pi^2)^{-1}\right], \end{aligned} \tag{30}$$

and based on Theorem 1 there now exists an $\epsilon_\kappa > 0$ such that for every $0 < \epsilon \leq \epsilon_\kappa$ the closed-loop system is exponentially stable and the robust output regulation problem is solved for the system (26).

## VIII. Conclusions

We presented a simple robust regulating controller for an unstable, impedance energy-preserving port-Hamiltonian system of even order. By deriving a new condition for exponential stability of even-order port-Hamiltonian systems we were able to stabilize the system, which allowed us to utilize the theory of robust output regulation for exponentially stable system. Thus, we constructed a simple controller for an unstable system that exponentially stabilizes the original plant and solves the Robust Output Regulation Problem for the stabilized plant.

## References

[1] Augner, B. and Jacob, B. Stability and Stabilization of Infinite-dimensional Linear Port-Hamiltonian Systems. Evolution Equations and Control Theory 3(2): 207–229, 2014.

[2] Curtain, R. F. and Zwart, H. An Introduction to Infinite-Dimensional Linear Systems Theory. Texts in applied Mathematics. Springer-Verlag, 1995.

[3] Davison, E. J. Multivariable Tuning Regulators: The Feedforward and Robust Control of a General Servomechanism Problem. IEEE Trans. Automat. Control, 21(1): 35–47, 1976.

[4] Le Gorrec, Y., Zwart, H. and Maschke, B. Dirac structures and boundary control systems associated with skew-symmetric differential operators. SIAM J. Control Optim., 44(2): 1864–1892, 2005.

[5] Hämäläinen, T. and Pohjolainen, S. A Finite-Dimensional Robust Controller for Systems in the CD-Algebra. IEEE Trans. Automat. Control, 45(3): 421–431, 2000.

[6] Hämäläinen, T. and Pohjolainen, S. Robust Regulation for Exponentially Stable Boundary Control Systems in Hilbert Space. Proc. of the 8th IEEE Int. Conf. on Methods and Models on Automation and Control, September 2–5. pp. 171–178, 2002.

[7] Jacob, B. and Zwart, H. Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces, volume 223 of Operator Theory: Advances and Applications. Birkhäuser, Basel, Switzerland. 2012.

[8] Paunonen, L. and Pohjolainen, S. Internal model theory for distributed parameter systems. SIAM J. Control Optim. 48(7): 4753–4775, 2010.

[9] Paunonen, L. and Pohjolainen, S. The Internal model principle for systems with unbounded control and observation. SIAM J. Control Optim. 52(6): 3967–4000, 2014.

[10] Paunonen, L. Controller Design for Robust Output Regulation of Regular Linear Systems. IEEE Trans. Automat. Control. (to appear. http://dx.doi.org/10.1109/TAC.2015.2509439)

[11] Weiss, G. Regular linear systems with feedback. Mathematics of Control Signals and Systems. 7: 23–57, 1994.

[12] Villegas, J., Zwart, H., Le Gorrec, Y., Maschke, B. and van der Schaft, A. Stability and stabilization of a class of boundary control systems. Proc. of the 44th IEEE Conference on Decision and Control and the European Control Conference, December 12–15. pp. 3850–3855, 2005.