On local triangle algebras

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Abstract

In this paper, we have introduced the notion of local and semilocal triangle algebras and propose the theorems that characterize these algebraic structures. Additionally, we have established the new properties of these algebraic structures and discussed the relations between local triangle algebras and some interval valued residuated lattice (IVRL)-filters, such as n-fold IVRL-extended integral filters and IVRL-extended maximal filters. The obtained results proved that the MTL-triangle algebra is a subdirect product of local triangle algebras. Moreover, a correlation was observed between the set of the dense elements and local triangle algebras. Finally, semilocal triangle algebras were introduced and assessed in detail, and an association was observed between the semilocal triangle algebras and quotient triangle algebras.

Keywords: **Residuated lattice, triangle algebra, IVRL-filter, local triangle algebra, semilocal triangle algebra.**

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1. Introduction

There is uncertainty regarding every sphere of daily life. Conventional mathematical tools do to suffice to manage all practical problems, and controversies are constantly revealed in major fields such as social sciences, engineering, and economics. In 1965, Zadeh introduced the fuzzy set theory to address such uncertainties where traditional tools commonly fail. Later, the fuzzy logic became popular and has been exploited in computer sciences to deal with uncertain information.

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In recent years, the interest in the fuzzy logic has grown rapidly. The algebraic structures that are involved in the structures of truth values have been introduced and axiomatized. Residuated lattices originated in 1969 [2, 7, 9] when Goguen [4] studied residuated lattices as the algebras of inexact concepts. In the 1970-s, Gaines and Pavelka were the first to observe the usefulness of residuated lattices in the cintext of fuzzy logic [3, 9].

H. Ono considered residuated lattices as an algebraic structure of substructural logic [8]. Furthermore, P. Hajek introduced the notion of BL-algebras as residuated lattices with two more conditions (divisibility and prelinearity) to prove the completeness of the BL-logic as a highly valued logic [5]. Therefore, residuated lattices allow the study of all these algebras with a common language. In particular, deductive systems of residuated lattices have a bijective counterpart in substructural logics, namely the sets of logic formulas that are closed with respect to Modus Ponens. Thus, all the information about deductive systems in an algebra can be interpreted as knowledge of provable formulas in the corresponding logic. In literature, deductive systems are also called filters (not to be confused with lattice filters).

L. P. Belluce et al. introduced the notion of local MV-algebras [1], while E. Turunen et al. studied the concept of local BL-algebras [12]. By definition, BL-algebras become local if they have a unique maximal deductive system, thereby generalizing the correspondent concept for MV-algebras. On the other hand, E. Turunen et al. evaluated local BL-algebras similar to Belluce et al., analyzing local MV-algebras. Using this context, these researchers proved some of the basic properties of BL-algebras. In addition, S. Hoo characterized semilocal MV-algebras [6], while E. Turunen introduced semilocal BL-algebras by initially showing that BL-algebras indefinitely generate natural algebras and characterizing semilocal BL-algebras [13].

Van Gasse et al. introduced the class of triangle algebras as a variety of residuated lattices equipped with approximation operators ν and μ, as well as a third angular point u are different from 0 and 1. According to Theorem 26, researchers have claimed that these algebras serve as an equational representation of interval-valued residuated lattices (IVRLs) [17]. Based on the definition and properties of triangle algebras, researchers have defined triangle logic (TL), demonstrating that this logic is sound and complete with respect to the variety of triangle algebras [17]. The theory of triangle algebras has been enriched with the filter theory, and researchers have introduced the notion of IVRL-filters in triangle algebras, defining the Boolean and prime IVRL-filters and reporting their remarkable properties [16].

Triangle algebras play key role in the study of fuzzy logics and the associated algebraic structures. Moreover, filter theory, i.e. studies of deductive systems of triangular algebras is essentially involved in the study of these algebras. Indeed, from a logic perspective, various filters have a natural interpretation as sets of provable formulas, while no studies have been focused on local triangle algebras so far. This issue motivated us to investigate the notion of local and semilocal triangle algebras.

In triangle algebras, ν and μ are important, which were used in our research to define local and semilocal triangle algebras, which play a pivotal role in the recognition of such algebraic structures. Local triangle algebras behave differently, and we attempted to state and prove the Propositions and theorems that determine the properties of these structure. Furthermore, we demonstrated that triangle algebra A is local iff $ord(\nu x) < \infty$ or $ord(\neg \nu x) < \infty$. It was proven that MTL-triangle algebras are a subdirect product of local triangle algebras. In this regard, F is considered to be the n-fold IVRL-extended integral filter iff A/F is local. The correlation between semilocal triangle algebras and quotient triangle algebras was also discussed, and a classification was proposed for triangle algebras accordingly.

In Section 2 of the article, some of the definitions and properties of residuated lattices and triangle algebras have been discussed. In Section 3, we have defined local triangle algebras, while proposing further characterizations for these algebras. In addition, we have determined the correlations between local triangle algebras and some types of IVRL-filters. In Section 4, triangle algebras have been considered semilocal if they only contains many finite different IVRL-extended maximal filters, and some properties have also been denoted for this algebra.

2. Preliminaries

Definition 2.1. *[17] A residuated lattice is an algebra $(L, \vee, \wedge, *, \to, 0, 1)$ with four binary operations and two constants 0,1 such that:*

- $(L, \vee, \wedge, 0, 1)$ *is a bounded lattice,*
- *operation $*$ is commutative and associative, with 1 as neutral element, and*
- $x * y \leq z$ *iff* $x \leq y \to z$*, for all x, y and z in L.*

The ordering $\leq$ and negation $\neg$ in a residuated lattice $(L, \vee, \wedge, *, \to, 0, 1)$ are defined as follows, for all x and y in L: $x \leq y$ iff $x \to y = 1$, and $\neg x = x \to 0$, $x^n = \underbrace{x * \cdots * x}_{n-times}$.

Lemma 2.1. *[10, 16] Let $(L, \vee, \wedge, *, \rightarrow, 0, 1)$ be a residuated lattice. Then the following properties are valid, for all x, y and z in L:*

(1) $x \vee y \leq (x \rightarrow y) \rightarrow y$ *(in particular $x \leq \neg\neg x$),*

(2) $x * \bigvee_{i \in I} y_i = \bigvee_{i \in I}(x * y_i)$,

(3) $(\bigvee_{i \in I} y_i) \rightarrow x = \bigwedge_{i \in I}(y_i \rightarrow x)$,

(4) $(x \rightarrow y) * (y \rightarrow z) \leqslant (x \rightarrow z)$,

(5) *If $x \leqslant y$, then $x * z \leqslant y * z, z \rightarrow x \leqslant z \rightarrow y$ and $y \rightarrow z \leqslant x \rightarrow z$,*

(6) $(y \rightarrow z) \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$,

(7) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) = (x * y) \rightarrow z$,

(8) $\bigvee_{i \in I}(y_i \rightarrow x) \leq (\bigwedge_{i \in I} y_i) \rightarrow x$,

(9) $x \rightarrow y \leqslant (y \rightarrow z) \rightarrow (x \rightarrow z)$,

(10) $\neg x \wedge \neg y \leq \neg(x \vee y)$.

Definition 2.2. *[17] Given a lattice $(A, \vee, \wedge)$, its triangularization $\mathbb{T}(A)$ is the structure $\mathbb{T}(A) = (Int(A), \vee, \wedge)$ defined by*

$\bullet Int(A) = \{[x_1, x_2] : (x_1, x_2) \in A^2$ *and* $x_1 \leq x_2\}$,

$\bullet [x_1, x_2] \wedge [y_1, y_2] = [x_1 \wedge y_1, x_2 \wedge y_2]$,

$\bullet [x_1, x_2] \vee [y_1, y_2] = [x_1 \vee y_1, x_2 \vee y_2]$.

The set $D_A = \{[x, x] : x \in L\}$ is called the diagonal of $\mathbb{T}(A)$.

Definition 2.3. *[17] An interval-valued residuated lattice (IVRL) is a residuated lattice $(Int(A), \vee, \wedge, \odot, \rightarrow_{\odot}, [0, 0], [1, 1])$ on the triangularization $\mathbb{T}(A)$ of a bounded lattice A, in which the diagonal D_A is closed under $\odot$ and $\rightarrow_{\odot}$, i.e. $[x, x] \odot [y, y] \in D_A$ and $[x, x] \rightarrow_{\odot} [y, y] \in D_A$, for all x, y in A. When we add $[0, 1]$ as a constant, and p_v and p_h (defined by $p_v([x_1, x_2]) = [x_1, x_1]$ and $p_h([x_1, x_2]) = [x_2, x_2]$, for all $[x_1, x_2]$ in $Int(A)$) as unary operators, the structure $(Int(L), \vee, \wedge, \rightarrow, *, p_v, p_h, [0, 0], [0, 1], [1, 1])$ is called an extended IVRL.*

Definition 2.4. *[17] A triangle algebra is a structure $(A, \vee, \wedge, *, \rightarrow, \nu, \mu, 0, u, 1)$ in which $(A, \vee, \wedge, *, \rightarrow, 0, 1)$ is a residuated lattice, ν and μ are unary operations on A, u a constant, and satisfying the* *following conditions:*

$(T.1)$ $\nu x \leq x$, $(T.1^{'})$ $x \leq \mu x$,

$(T.2)$ $\nu x \leq \nu\nu x$, $(T.2^{'})$ $\mu\mu x \leq \mu x$,

$(T.3)$ $\nu(x \wedge y) = \nu x \wedge \nu y$, $(T.3^{'})$ $\mu(x \wedge y) = \mu x \wedge \mu y$,

$(T.4)$ $\nu(x \vee y) = \nu x \vee \nu y$, $(T.4^{'})$ $\mu(x \vee y) = \mu x \vee \mu y$,

$(T.5)$ $\nu u = 0$, $(T.5^{'})$ $\mu u = 1$,

$(T.6)$ $\nu\mu x = \mu x$, $(T.6^{'})$ $\mu\nu x = \nu x$,

$(T.7)$ $\nu(x \to y) \leq \nu x \to \nu y$,

$(T.8)$ $(\nu x \leftrightarrow \nu y) * (\mu x \leftrightarrow \mu y) \leq (x \leftrightarrow y)$,

$(T.9)$ $\nu x \to \nu y \leq \nu(\nu x \to \nu y)$.

In a triangle algebra $(A, \vee, \wedge, *, \to, \nu, \mu, 0, u, 1)$, the operator ν (necessity) and μ (possibility) are modal operators, and u (uncertainty, $u \neq 0, u \neq 1$) is a new constant. It turns out that triangle algebras are the equational representations of interval-valued residuated lattices (IVRLs).

Theorem 2.1. *[17] There is a one-to-one correspondence between the class of IVRLs and the class of triangle algebras. Every extended IVRL is a triangle algebra and conversely, every triangle algebra is isomorphic to an extended IVRL.*

From now on $(A, \vee, \wedge, \to, *, \nu, \mu, 0, u, 1)$ or simply A is a triangle algebra unless otherwise specified.

Proposition 2.1. *[17] Suppose $(A, \vee, \wedge, \to, 0, 1)$ is a residuated lattice such that $\neg$ is involutive ($\neg\neg x = x$ for all $x \in A$). If there exists an element u in A such that $\neg u = u$, if ν is a unary operator on A that satisfies $T.1$- $T.6$, $T.8$, $T.9$ and if $(\nu x \leftrightarrow \nu y) * (\nu\neg x \leftrightarrow \nu\neg y) \leq x \leftrightarrow y$, then $(A, \vee, \wedge, \to, *, \nu, \mu, 0, u, 1)$ is a triangle algebra if we define $\mu x = \neg\nu\neg x$.*

Proposition 2.2. *[15] In a triangle algebra $(A, \vee, \wedge, \to, *, \nu, \mu, 0, u, 1)$, the following identities and inequalities hold, for every x, y and z in A:*

(i) $\nu(x * y) = \nu x * \nu y$.

(ii) $\mu(x * y) \leq \mu x * \mu y$.

Definition 2.5. *[21] Let $\mathcal{A} = (A, \vee, \wedge, *, \to, \nu, \mu, 0, u, 1)$ and $\mathcal{B} = (B, \sqcup, \sqcap, \odot, \Rightarrow, \overline{\nu}, \overline{\mu}, 0, \overline{u}, 1)$ be any two triangle algebras. If the mapping $h : A \longrightarrow B$ satisfies for $a, b \in A$*

$$h(a \vee b) = h(a) \sqcup h(b), \qquad h(a \wedge b) = h(a) \sqcap h(b),$$

$$h(a * b) = h(a) \odot h(b), \qquad h(a \to b) = h(a) \Rightarrow h(b),$$

$$h(\nu a) = \overline{\nu} h(a), \qquad h(\mu a) = \overline{\mu} h(a).$$

Then h is called a homomorphism.

Definition 2.6. *[19] A triangle algebra A is called an MTL-triangle algebra if $(a \to b) \vee (b \to a) = 1$ (prelinearity), for all $a, b \in A$.*

Definition 2.7. *[16] An IVRL-filter of triangle algebra A is a non-empty subset F of A satisfying:*

$(F.1)$ if $x \in F, y \in A$ and $x \leq y$, then $y \in F$,

$(F.2)$ if $x, y \in F$, then $x * y \in F$,

$(F.3)$ if $x \in F$, then $\nu x \in F$.

Definition 2.8. *[22] An alternative definition for an IVRL-filter F (called deductive system) of a triangle algebra $(A, \vee, \wedge, *, \to, \nu, \mu, 0, u, 1)$ is the following:*

$(F.1')$ $1 \in F$,

$(F.2')$ for all x and y in A: if $x \in F$ and $x \to y \in F$, then $y \in F$.

$(F.3')$ if $x \in F$, then $\nu x \in F$.

For all $x, y \in A$, we write $x \equiv_F y$ iff $x \to y$ and $y \to x$ are both in F.

$\equiv_F$ is always a congruence relation [16]. Note that $(F.3)$ is a necessary condition for this statement. Indeed, if $\equiv_F$ is a congruence relation on a triangle algebra $\mathcal{A} = (A, \vee, \wedge, *, \to, \nu, \mu, 0, u, 1)$ and $x \in F$, then $x \equiv_F 1$ and therefore $\nu x \equiv_F \nu 1 = 1$, which is equivalent with $\nu x \in F$.

Definition 2.9. *[21] Let $S \subseteq A$, a nonempty subset of A, $a \in A$. Then $[S) = \{x \in A \mid s_1 * ... * s_n \leq \nu x$, for some $n \geq 1$ and $s_1, ..., s_n \in S\}$.*

Definition 2.10. *[17] The set of exact elements $E(A)$ of a triangle algebra A is $\{x \in A | \nu x = x\}$.*

Proposition 2.3. *[16] Let A be a triangle algebra, $(E(A), \vee, \wedge, *, \to, 0, 1)$ be its subalgebra of exact elements and $F \subseteq A$. Then F is a filter of the triangle algebra A if and only if $(F.3^{'})$ holds and $F \bigcap E(A)$ is a filter of the residuated lattice $E(A)$.*

Proposition 2.3 suggests two different ways to define special kinds of IVRL-filters of triangle algebras. The first is to impose a property on a filter of the subalgebra of exact elements and extend this filter to the whole triangle algebra, using $(F.3^{'})$. We call these IVRL-extended filters. For example, an IVRL-extended prime filter of triangle algebra $(A, \vee, \wedge, *, \rightarrow, \nu, \mu, 0, u, 1)$ is a subset F of A such that $F \cap E(A)$ is a prime filter of $E(A)$ and $x \in F$ if and only if $\nu x \in F \cap E(A)$.

The second way is to impose a property on the whole IVRL-filter. For example, a prime IVRL-filter of a triangle algebra $(A, \vee, \wedge, *, \rightarrow, \nu, \mu, 0, u, 1)$ is an IVRL-filter of A such that F is a prime filter of $(A, \vee, \wedge, *, \rightarrow, 0, 1)$ [16].

Definition 2.11. *[16, 18, 21] Let A be a triangle algebra.*

- *A proper IVRL-filter M is an IVRL-extended maximal filter of A iff for all $x \in A$, $x \notin M$ there exist $m \in M$, $n \geq 1$ such that $m * \nu x^n = 0$.*

- *The intersection of all IVRL-extended maximal filters of a triangle algebra A is called the radical of A and is denoted by $Rad(A)$.*

- *An IVRL-filter extended prime filter of A is a filter F of A such that $\nu x \rightarrow \nu y \in F$ or $\nu y \rightarrow \nu x \in F$, for all $x, y \in A$.*

- *A proper IVRL-filter F is called n-fold IVRL-extended integral filter if for all, $x, y \in A$, $\neg(\nu x * \nu y) \in F$ implies $\neg(\nu x^n) \in F$ or $\neg(\nu y^n) \in F$, for some n.*

Definition 2.12. *[21] The set of dense elements of a triangle algebra A is defined as $D_s(A) = \{a \in A : \neg \nu a = 0\}$. Its restriction to an IVRL-filter F is defined as $D_s(F) = \{a \in F : \neg \nu a = 0\}$.*

Definition 2.13. *[21] The order of $x \in A$, denoted by $ord(x)$, is the smallest $n \in \mathbb{N}$ such that $x^n = 0$. If there is no such n, then $ord(x) = \infty$.*

Definition 2.14. *A triangle algebra A is called a linear triangle algebra if $x \leq y$ or $y \leq x$, for all $x, y \in A$.*

Lemma 2.2. *Let A be an MTL-triangle algebra and $a \in A$, $a \neq 1$. Then there is an IVRL-extended prime filter F of A not containing a.*

Proof. There are IVRL-filters not containing a, for example, $F_0 = \{1\}$. We shall show that if F is any IVRL-filter not containing a and $x, y \in A$ are such that $\nu x \rightarrow \nu y \notin F$ and $\nu y \rightarrow \nu x \notin F$, then there is an IVRL-filter $F' \supseteq F$ such that $a \notin F'$ and $\nu x \rightarrow \nu y \in F'$ or $\nu y \rightarrow \nu x \in F'$. Note that the least IVRL-filter F' containing F as a subset and z as an element is $F' = \{\nu u \mid (\exists v \in F)(\exists n \in \mathbb{N})(v * z^n \leq u)\}$. Clearly, if $F'' \supseteq F$ is an IVRL-filter and $z \in F$ then for each $v \in F$ and $n \in \mathbb{N}$, $\nu(v * z^n) \in F''$, on the other hand, F' itself is an IVRL-filter since it is obviously closed under $*, \nu$ and contains with each z all $z' \geq z$. Thus assume $\nu x \to \nu y \notin F, \nu y \to \nu x \notin F$ and F_1, F_2 be the smallest IVRL-filters containing F, as a subset and $\nu x \to \nu y, \nu y \to \nu x$ respectively as an element. We claim that $a \notin F_1$ or $a \notin F_2$. Assume $a \in F_1$ and $a \in F_2$. Then for some $v \in F$ and $n \in \mathbb{N}, v * (\nu x \to \nu y)^n \leq a$ and $v * (\nu y \to \nu x)^n \leq a$. Thus $a \geq v * (\nu x \to \nu y)^n \vee v * (\nu y \to \nu x)^n = v * ((\nu x \to \nu y)^n \vee (\nu y \to \nu x)^n) = v * 1 = v$. Hence $a \in F$, which is a contradiction. So $a \notin F_1$ or $a \notin F_2$. Now, if A is countable, then we may arrange all pairs $(x, y) \in A^2$ into a sequence $\{(x_n, y_n) \mid n \in \mathbb{N}\}$. Let $F_0 = \{1\}$ and having constructed F_n such that $a \notin F_n$. We take $F_{n+1} \supseteq F_n$ such that $a \notin F_{n+1}$. If possible we take F_{n+1} such that $(\nu x_n \to \nu y_n) \in F_{n+1}$. If not, we take that with $(\nu y_n \to \nu x_n) \in F_{n+1}$. So $\cup_n F_n$ is IVRL-extended prime filter. If A is uncountable, then one has to use the axiom of choice and work similarly with a transfinite sequence of IVRL-filters. $\square$

Remark 2.1. *[21] Let A be an MTL-triangle algebra. Then F is an IVRL-extended prime filter of A iff $\nu x \vee \nu y \in F$ implies $\nu x \in F$ or $\nu y \in F$, for all $x, y \in A$.*

Lemma 2.3. *An MTL-triangle algebra is linear iff any proper IVRL-filter of A is IVRL-extended prime filter of A.*

Proof. If A is linear and F is proper IVRL-filter of A, then for all $x, y \in A$, $x \vee y = x$ or $x \vee y = y$. Thus $x \vee y \in F$ iff $\nu(x \vee y) \in F$ iff $\nu x \vee \nu y \in F$ iff $\nu x \in F$ or $\nu y \in F$.

Conversely, let assume, any proper IVRL-filter of A is IVRL-extended prime filter of A. Then in particular $\{1\}$ is an IVRL-extended prime filter. Since for any $x, y \in A$, $(x \to y) \vee (y \to x) \in \{1\}$, we get $(x \to y) \in \{1\}$ or $(y \to x) \in \{1\}$, so $y \leq x$ or $x \leq y$. $\square$

Lemma 2.4. *Each triangle algebra is a subalgebra of the direct product on a set of linearly ordered triangle algebras.*

Proof. Let $\mathcal{S}$ be the set of all IVRL-extended prime filters on A. For $F \in \mathcal{S}$, let $A_F = A/F$ and $A^* = \prod_{F \in \mathcal{S}} A_F$. Then A^* is the direct product of linearly ordered triangle algebra $\{A_F \mid F \in \mathcal{S}\}$ of A^*. For $x \in A$, let $i(x)$ be the element $\{[x]_F \mid F \in \mathcal{S}\}$ of A^*. Clearly this map preserves operations, it remain to show that it is one to one. If $x, y \in A$ and $x \neq y$, then $x \nleq y$ or $y \nleq x$. Assume $\nu x \to \nu y \neq 1$ in A. By Lemma 2.2 let F be an IVRL-extended prime filter on A not containing $\nu x \to \nu y$. Then in A/F, $[x]_F \nleq [y]_F$, hence $[x]_F \neq [y]_F$, and so $i(x) \neq i(y)$. $\square$

Proposition 2.4. *Let $A_1, ..., A_k$ be triangle algebras and $A = A_1 \times ... \times A_k$. Then $Fil(A) = Fil(A_1) \times ... \times Fil(A_k)$ (where $Fil(A)$ is the set of all IVRL-filters of A).*

Proof. $F_i \in Fil(A_i)$ for $i = 1, ..., k$, then $F_1 \times ... \times F_k$ is an IVRL-filter of A. Conversely, if F is an IVRL-filter of A, then for $i = 1, ..., k$, $F_i = \pi_i(F)$ is an IVRL-filter of A_i and $F = F_1 \times ... \times F_k$. So the proof is complete. □

3. Local triangle algebra

Definition 3.1. *A triangle algebra A is said to be local iff has exactly one IVRL-extended maximal filter.*

Example 3.1. *Let $A = \{0, u, 1\}$. We define operators $\nu, \mu, *, \to$ as follows:*

x	νx
0	0
u	0
1	1

x	μx
0	0
u	1
1	1

$*$	0	u	1
0	0	0	0
u	0	u	u
1	0	u	1

$\to$	0	u	1
0	1	1	1
u	0	1	1
1	0	u	1

$(A, \vee, \wedge, *, \to, \nu, \mu, 0, u, 1)$ is a triangle algebra. It is clear that, $F = \{1\}$ is the only IVRL-extended maximal filter of A. So A is local triangle algebra.

Example 3.2. *Let $X = \{0, a, b, 1\}$, where $0 \leq a \leq 1, 0 \leq b \leq 1$. Define $\odot$ and $\Rightarrow$ on X as follows:*

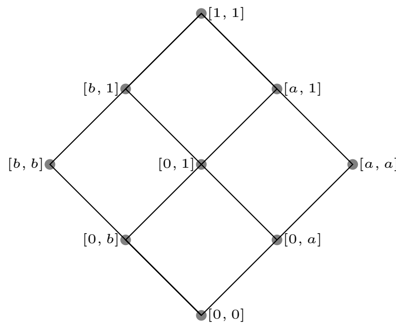
$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\Rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Then $(X, \odot, \Rightarrow, \vee, \wedge)$ is a residuated lattice. We have $Int(X) = A = \{[0,0], [0,a], [0,b], [a,a], [b,b], [0,1], [a,1], [b,1], [1,1]\}$, if we define ν, μ, $*$ and $\to$ on A as follows:

$\nu[x_1, x_2] = [x_1, x_1]$, $\mu[x_1, x_2] = [x_2, x_2]$, $[x_1, x_2] * [y_1, y_2] = [x_1 \odot y_1, x_2 \odot y_2]$, $[x_1, x_2] \to [y_1, y_2] = [(x_1 \Rightarrow y_1) \wedge (x_2 \Rightarrow y_2), x_2 \Rightarrow y_2]$.

Then $(A, \vee, \wedge, *, \to, \nu, \mu, [0,0], [0,1], [1,1])$ is a triangle algebra with $[0,0]$ as the smallest and $[1,1]$ as the greatest element. Clearly, $F_1 = \{[a,a], [a,1], [1,1]\}, F_2 = \{[b,b], [b,1], [1,1]\}$ are IVRL-extended maximal filters of A. So A is not a local triangle algebra.



Definition 3.2. *Let A be a triangle algebra. Then we define*

$$D(A) = \{x \in A \mid \nu x^n \neq 0, \textit{ for all } n \in \mathbb{N}\}.$$

Theorem 3.1. *Let A be a triangle algebra. Then the following are equivalent:*

(i) $D(A)$ is an IVRL-filter,

(ii) $[D(A))$ is a proper IVRL-filter,

(iii) A is local,

(iv) the unique IVRL-extended maximal filter of A is $D(A)$,

(v) if $\nu x^n, \nu y^n \neq 0$ for all $n \geq 1$, then $\nu x^n * \nu y^n \neq 0$, where $x, y \in A$.

Proof. $(i \Rightarrow ii)$ Let $D(A)$ be an IVRL-filter. Then it is easy to see that $[D(A)) = D(A)$ and this IVRL-filter is proper since $0 \notin D(A)$, so (ii) holds.

$(ii \Rightarrow i)$ if (ii) hold and $x, x \to y \in D(A) \subseteq [D(A))$, then for all $n \geq 1$ $\nu x^n, \nu(x \to y)^n \neq 0$ hence $0 \neq \nu x^n * \nu(x \to y)^n = \nu[x * (x \to y)]^n \leq \nu y^n$, thus $y \in D(A)$. If $x \in D(A)$, then $\nu x^n \neq 0$ and $\nu\nu x^n = \nu x^n \neq 0$. So $\nu x \in D(A)$. Therefore (i) and (ii) are equivalent.

$(i \Leftrightarrow v)$ Since $\nu 1^n = 1, \nu\nu x = \nu x$ and $\nu(x * y)^n = \nu x^n * \nu y^n \leq \nu x^n, \nu y^n$ it is obvious that a necessary and sufficient condition for (i) to hold is (v).

$(iv \Rightarrow iii)$ It is trivial.

$(i \Rightarrow iv)$ Let F' be an IVRL-filter such that $x \in F', x \notin D(A)$, for some $x \in A$. Then $\nu x^n = 0$, for some $n \in \mathbb{N}$. Hence F' is not proper. So $D(A)$ contains all the proper IVRL-filters of A and so (iv) holds.

$(iii \Rightarrow iv, i)$ Let A be a local triangle algebra and M_0 be the unique IVRL-extended maximal filter of A. Then any element $x \in D(A)$ generates a proper IVRL-filter $D_x = \{\nu x^n \mid n \geq 0\}$, which can be extended to an IVRL-extended maximal filter M_x. But $M_x = M_0$. Thus for all

$x \in D(A), x \in M_0$ and so $D(A) \subseteq M_0$. Since M_0 is proper, $M_0 \subseteq D(A)$. Hence $M_0 = D(A)$, therefore (iii) implies (iv) and (i). $\square$

By Proposition 2.2 and Theorem 3.1, we have:

Corollary 3.1. *Let A be a local triangle algebra. If $\nu x^n, \nu y^n \neq 0$ (and so $\mu x^n, \mu y^n \neq 0$) for all $n \geq 1$, then $\mu x^n * \mu y^n \neq 0$, where $x, y \in A$.*

Proposition 3.1. *A triangle algebra A is local iff $ord(\nu x) < \infty$ or $ord(\neg \nu x) < \infty$, for all $x \in A$.*

Proof. Let A be a local triangle algebra but $\nu x^n > 0$ and $(\neg \nu x)^n > 0$, for some $x \in A$ and for all $n \in \mathbb{N}$. Then $\neg x, x \in [D(A))$, so $0 = x * \neg x \in [D(A))$, which contradicts to Theorem 3.1, (ii).

Conversely, let $0 \in [D(A))$. Then for some $x_1, ..., x_n \in D(A)$, we have $\nu x_1 * ... * \nu x_n \leq 0$, so $\nu x_1 * ... \nu x_{n-1} \leq \neg \nu x_n$. Since $ord(\nu x_n) = \infty, ord(\neg \nu x_n) = k_n < \infty$. Thus

$$\nu x_1^{k_n} * ... * \nu x_{n-1}^{k_n} \leq (\neg \nu x_n)^{k_n} = 0,$$

whence

$$\nu x_1^{k_n} * ... * \nu x_{n-2}^{k_n} \leq \neg(\nu x_{n-1}^{k_n}).$$

Clearly, $ord(\nu x_{n-1}^{k_n}) = \infty$. Hence $ord(\neg(\nu x_{n-1}^{k_n})) = k_{n-1} < \infty$, and so

$$\nu x_1^{k_n k_{n-1}} * ... * \nu x_{n-2}^{k_n k_{n-1}} \leq (\neg(\nu x_{n-1}^{k_n}))^{k_{n-1}} = 0.$$

By continuing n times this procedure, we arrive into contradiction $\nu x_1^{k_n \cdots k_2} = 0$. Therefore $0 \notin [D(A))$, so A is local. $\square$

Theorem 3.2. *F is n-fold IVRL-extended integral filter iff A/F is local.*

Proof. Assume that A/F is local and $\neg(\nu x * \nu y) = \nu y \to \neg \nu x \in F$. Then $(\nu y)/F \to (\neg \nu x)/F = (\nu y \to \neg \nu x)/F = 1/F$, so $(\nu y)/F \leq (\neg \nu x)/F$. Let $\neg(\nu x^n) \notin F$, for all n. Then $\neg(\nu x^n)/F \neq 1/F$, thus $(\nu x^n)/F \neq 0/F$. Since A/F is local, $(\neg \nu x)^k/F = 0/F$, for some k. Also, $(\nu y^k)/F \leq (\neg \nu x)^k/F = 0/F$, whence $\neg(\nu y^k)/F = 1/F$ i.e $\neg(\nu y^k) \in F$. And so F is n-fold IVRL-extended integral filter.

Conversely, let F be an n-fold IVRL-extended integral filter. Since $\neg(\nu x * \neg \nu x) = 1 \in F$, for all $x \in A$, we have $\neg(\nu x^n) \in F$ or $\neg((\neg \nu x)^n) \in F$ for some n, i.e $\neg(\nu x^n)/F = 1/F$ or

$\neg((\neg\nu x)^n)/F = 1/F$. Therefore $(\nu x^n)/F = 0/F$ or $((\neg\nu x)^n)/F = 0/F$. Thus A/F is local triangle algebra. $\square$

Proposition 3.2. *Let F be an IVRL-extended maximal filter of A and $\neg\nu x \notin F$, for all $0 \neq x \in A$. Then A/F is a local triangle algebra.*

Proof. If $\neg\nu x \notin F$, for all $0 \neq x \in A$ and F is an IVRL-extended maximal filter, then $\neg\neg\nu x \in F$. Hence $\neg\nu x/F = 0/F$. Thus A/F is a local triangle algebra. $\square$

In the following example we show that the converse of the above proposition is not true in general.

Example 3.3. *Consider $L^I = [0,1]$ and define $*$ and $\to$ on L^I as follows:*

$x * y = min(x,y)$ *and* $x \to y = \begin{cases} 1 & x \leq y \\ y & y < x \end{cases}$, *then $(L^I, \vee, \wedge, *, \to, 0, 1)$ is a residuated lattice. Now, we define*

$$[x_1, x_2] \odot [y_1, y_2] = [x_1 * y_1, x_2 * y_2],$$

$$[x_1, x_2] \Rightarrow [y_1, y_2] = [(x_1 \to y_1) \wedge (x_2 \to y_2), x_2 \to y_2].$$

The structure $(L^I \times L^I, \vee, \wedge, \odot, \Rightarrow, [0,0], [1,1])$ is a residuated lattice too. If we define

$$\nu[x_1, x_2] = [x_1, x_1], \mu[x_1, x_2] = [x_2, x_2], u = [0,1].$$

then $(L^I \times L^I, \vee, \wedge, \odot, \Rightarrow, \nu, \mu, [0,0], [0,1], [1,1])$ is a local triangle algebra. If $F = \{[1,1]\}$, then F is not an IVRL-extended maximal filter of $L^I \times L^I$. Clearly, $L^I \times L^I/F = L^I \times L^I/\{[1,1]\} \equiv L^I \times L^I$ is a local triangle algebra.

Proposition 3.3. *Every IVRL-extended prime filter of A is an n-fold IVRL-extended integral filter.*

Proof. If F is an IVRL-extended prime filter of A, then $\nu x \to \nu y \in F$ or $\nu y \to \nu x \in F$, for all $x, y \in A$. Let $\nu x \to \nu y \in F$, $\neg(\nu x * \nu y) \in F$. Then

$$(\nu x \to \nu y) * \neg(\nu x * \nu y) = (\nu x \to \nu y) * (\nu y \to \neg\nu x) \leq (\nu x \to \neg\nu x) = \neg(\nu x^2) \in F.$$

Similarly, if $(\nu y \to \nu x), \neg(\nu x * \nu y) \in F$, then $\neg(\nu y^2) \in F$. Therefore F is an n-fold IVRL-extended integral filter of A. $\square$

Lemma 3.1. *Let F be an IVRL-filter of A. Then A/F is linearly ordered iff F is an IVRL-extended prime filter of A.*

Proof. Let F be an IVRL-extended prime filter and $x, y \in A$. Then $\nu x \to \nu y \in F$ or $\nu y \to \nu x \in F$. So $[x]_F \leq [y]_F$ or $[y]_F \leq [x]_F$. Thus A/F is linearly ordered.

Conversely, if A/F is linearly ordered and $x, y \in A$, then either $[y]_F \leq [x]_F$ and so $\nu y \to \nu x \in F$ or $[x]_F \leq [y]_F$ and so $\nu x \to \nu y \in F$. Hence F is an IVRL-extended prime filter. □

Proposition 3.4. *A is a local triangle algebra iff every proper IVRL-filter of A is an n-fold IVRL-extended integral filter.*

Proof. Let A be a local triangle algebra and F be a proper IVRL-filter of A. Then $D(A)$ is the unique IVRL-extended maximal filter containing F, thus $D(A)/F$ is the unique IVRL-extended maximal filter of triangle algebra A/F, i.e. A/F is local and F is n-fold IVRL-extended integral filter by Theorem 3.2.

Conversely, if any proper IVRL-filter of A is n-fold IVRL-extended integral filter, then in particular, $\{1\}$ is n-fold IVRL-extended integral filter. Hence $A \equiv A/\{1\}$ is local by Theorem 3.2. □

Theorem 3.3. *Every MTL-triangle algebra is a subdirect product of local triangle algebras.*

Proof. Trivially every linear triangle algebra is local. By Lemma 2.4, the proof is complete. □

If $\nu(x \to y) = \nu x \to \nu y$ and $x * (x \to y) = x \wedge y$, for all $x, y \in A$, then by Theorem 3.1, we have:

Proposition 3.5. (i) *Let A be a local triangle algebra. Then $D_s(A) \subseteq D(A)$.*
(ii) *If $D_s(A) = A \setminus \{0\}$, then A is a local triangle algebra.*

Proof. (i) Clearly, $1 \in D_s(A)$. If $x, x \to y \in D_s(A)$, then $0 = \neg \nu x = \nu x \to 0 = \nu x \to \neg \nu(x \to y) = \nu x \to \neg(\nu x \to \nu y) = \neg(\nu x * (\nu x \to \nu y))$. Since $(\nu x * (\nu x \to \nu y)) \leq \nu y$, we have $\neg \nu y \leq \neg((\nu x * (\nu x \to \nu y))) = 0$. Hence $y \in D_s(A)$. Since $\neg \nu \nu x = \neg \nu x = 0$, then $\nu x \in D_s(A)$. Therefore $D_s(A)$ is an IVRL-filter. Clearly $0 \notin D_s(A)$, so $D_s(A)$ is proper. Whence $D_s(A)$ can be extended to $D(A)$.
(ii) Since $D_s(A)$ is an IVRL-filter, $D_s(A)$ is the unique IVRL-extended maximal filter of A. □

It is worth to note that the converse of (ii) in above proposition does not hold, in general. For this we give the following example.

Example 3.4. *[19] Consider $L^I = [0,1]$ and define $*, \to$ on L^I as follows:*

$$x * y = max(0, x + y - 1), x \to y = min(1, 1 - x + y).$$

Then $(L^I, \vee, \wedge, *, \to, 0, 1)$ is a residuated lattice. Now we define

$[x_1, x_2] \odot [y_1, y_2] = [x_1 * y_1, x_2 * y_2]$,

$[x_1, x_2] \Rightarrow [y_1, y_2] = [(x_1 \to y_1) \wedge (x_2 \to y_2), x_2 \to y_2]$.

The structure $(L^I \times L^I, \vee, \wedge, \odot, \Rightarrow, [0,0], [1,1])$ is a residuated lattice too. If we define

$$\nu[x_1, x_2] = [x_1, x_1], \mu[x_1, x_2] = [x_2, x_2], u = [0,1].$$

then $(L^I \times L^I, \vee, \wedge, \odot, \Rightarrow, \nu, \mu, [0,0], [0,1], [1,1])$ is a triangle algebra. It is clear that $L^I \times L^I$ is a local triangle algebra, but $x = [0.5, 0.7] \notin D_s(L^I \times L^I)$. So $D_s(L^I \times L^I) \neq L^I \times L^I \setminus \{0\}$.

Proposition 3.6. *The following conditions are equivalent:*
(i) $\neg\nu(y \to x) = \neg\nu(x \to y)$, for all $0 \neq x, y \in A$
(ii) $D_s(A) = A \setminus \{0\}$.

Proof. Let $\neg\nu(y \to x) = \neg\nu(x \to y)$, for all $0 \neq x, y \in A$. For $y = 1$, we have $\neg\nu x = 0$, for all $0 \neq x \in A$. Hence $D_s(A) = A \setminus \{0\}$.

Conversely, assume (ii) holds and let $x, y \neq 0$, Then $x, y \in D_s(A)$. Since $x \leq y \to x$, also $y \to x \in D_s(A)$, so $\neg(\nu(y \to x)) = 0$. Similarly $\neg(\nu(x \to y)) = 0$, and so (i).

□

Under the conditions Proposition 2.1 and by Proposition 3.6, we have (in this case the negation is assumed to be involutive):

Proposition 3.7. *The following conditions are equivalent:*
(i) $\mu\neg(y \to x) = \mu\neg(x \to y)$, for all $0 \neq x, y \in A$
(ii) $D_s(A) = A \setminus \{0\}$.

Corollary 3.2. *Let $\neg\nu(y \to x) = \neg\nu(x \to y)$, for all $0 \neq x, y \in A$. Then A is a local triangle algebra.*

Theorem 3.4. *Let $D_s(A) = A \setminus \{0\}$. Then*

(i) A/F is local triangle algebra, for every IVRL-filter of A,

(ii) $A/D_s(A)$ is local triangle algebra.

Proof. (i) Let F be a proper IVRL-filter of A. For all $0/F \neq x/F, y/F \in A/F$,

$$\begin{aligned}\neg(\nu x/F \to \nu y/F) &= \neg((\nu x \to \nu y)/F)\\ &= \neg(\nu x \to \nu y)/F\\ &= \neg(\nu y \to \nu x)/F\\ &= \neg(\nu y/F \to \nu x/F).\end{aligned}$$

Thus A/F is local triangle algebra.

(ii) Since $D_s(A)$ is an IVRL-filter of A, by (i), $A/D_s(A)$ is local triangle algebra. □

In the following example we show that the converse of above theorem is not true, in general.

Example 3.5. *In Example 3.1, we have $D_s(A) = \{1\} \neq A \setminus \{0\}$. Let $F = \{1\}$. Then $A/F = \{1\} = A/D_s(A) = A$ is local triangle algebra.*

4. Semilocal triangle algebras

Definition 4.1. *Local triangle algebra A is called locally finite if $ord(x) < \infty$, for all $x \in A \setminus \{1\}$.*

Definition 4.2. *A triangle algebra A whose only proper IVRL-filter is the set $\{1\}$, is called semisimple triangle algebra.*

Let L, K be two locally finite triangle algebras. Then a product triangle algebra $L \times K$ contains two disjoint descending chains of IVRL-filters (unless element 1, since $1 \in F$, for any IVRL-filter F of triangle algebra K), namely $L \times K \supseteq L \times \{1\} \supseteq \{(1,1)\}$ and $L \times K \supseteq L \times \{1\}$. Clearly, $L \times K$ is a semisimple triangle algebra and the two IVRL-extended maximal filters $\{1\} \times K$ and $L \times \{1\}$ are disjoint. Also, n locally finite triangle algebras $L_1, L_2, ..., L_n$, a product triangle algebra $\prod_{i=1}^{n} L_i$ is semisimple, contains $2^n - 1$ proper IVRL-filters and n disjoint IVRL-extended maximal filters $M_i = L_1 \times ... \times \{1\} \times ... \times L_n$, $i = 1, ..., n$ and any strict descending chain of IVRL-filters is finite. In particular, we have

Proposition 4.1. *Let A be a triangle algebra and $M_1, ..., M_n$, n IVRL-extended maximal filters of A. Then the product triangle algebra $\prod_{i=1}^{n} A/M_i$ is semisimple, contains $2^n - 1$ proper IVRL-filters and n disjoint IVRL-extended maximal filters. Also, every strict descending chain of IVRL-filters of $\prod_{i=1}^{n} A/M_i$ is finite.*

Definition 4.3. *Let F, G be two proper IVRL-filters of A. Then we call F and G relatively prime if $[F \cup G) = A$.*

Example 4.1. *In Example 3.2, $F_1 = \{[a, a], [a, 1], [1, 1]\}$ and $F_2 = \{[b, b], [b, 1], [1, 1]\}$ are relatively prime.*

Proposition 4.2. *Let F and G be two relatively prime IVRL-filters of A. Then there is an element $x \in A$ such that $x \equiv_F 1$ and $x \equiv_G 0$.*

Proof. Since $0 \in A = [F \bigcup G)$, there are $x \in F$, $y \in G$ such that $x * y = 0$. Clearly, $x \equiv_F 1$. Since $y \leq \neg x, \neg x \in G$. So $x \equiv_G 0$. □

Proposition 4.3. *Let $F_1, ..., F_m$ be IVRL-filters of A such that F_i, F_j are relatively prime IVRL-filters of A, for all $i, j = 1, ..., m$ and $i \neq j$. Then there is $x \in A$ such that $x \equiv_{F_i} x_i$ for $i = 1, ..., m$.*

Proof. First, let $m = 2$. Since $[F_1 \cup F_2) = A$, By Proposition 4.2, there exist $f_{12} \in F_1$ and $f_{21} \in F_2$ such that $f_{12} * f_{21} = 0$. By Lemma 2.1, we have $f_{12} \leq \neg f_{21}$. Then $\neg f_{21} \in F_1$, and hence $f_{21} \equiv_{F_1} 0$. Since $f_{12} \leq \neg f_{21}$, we get $\neg\neg f_{21} \leq \neg f_{12}$, also we have, $f_{21} \leq \neg\neg f_{21}$. Thus $f_{21} \leq \neg f_{12}$ and so $\neg f_{12} \in F_2$. Hence $f_{12} \equiv_{F_2} 0$. Let $x = (f_{12} * x_1) \vee (f_{21} * x_2)$, where $x_1, x_2 \in A$. By Lemma 2.1, we have

$$\begin{aligned} x/F_1 &= (f_{12}/F_1 * x_1/F_1) \vee (f_{21}/F_1 * x_2/F_1) \\ &= (1/F_1 * x_1/F_1) \vee (0/F_1 * x_2/F_1) \\ &= x_1/F_1. \end{aligned}$$

So $x \equiv_{F_1} x_1$. Similarly, $x \equiv_{F_2} x_2$. Now let m be arbitrary, for $i, j = 1, ..., m$ and $i \neq j$, there exist $f_{ij} \in F_i$ and $f_{ji} \in F_j$ such that $f_{ij} * f_{ji} = 0$. Considering $x = \vee_{i=1}^{m} (f_{i1} * ... * f_{i,i-1} * f_{i,i+1} * ... * f_{i,m} * x_i)$ and reasoning as above we see that $x \equiv_{F_i} x_i$, for $i = 1, ..., m$. □

Theorem 4.1. *Let A be a triangle algebra and $F_1, ..., F_n$ be n disjoint IVRL-extended maximal filters of A. Then a mapping $g : A \longrightarrow \mathcal{A} = \prod_{i=1}^{n} A/F_i$ defined by $g(a) = (a/F_1, ..., a/F_n)$, for all $a \in A$, is a surjective triangle homomorphism such that $g(a) = 1_{\mathcal{A}}$ iff $a \equiv_{F_i} 1$, for all $i = 1, ..., n$. Hence $A/\cap_{i=1}^{n} F_i$ is isomorphic to $\prod_{i=1}^{n} A/F_i$.*

Proof. Since g is a product of the natural triangle homomorphisms $g_i : A \longrightarrow A/F_i$ such that $g(a) = a/F_i$, $i = 1, ..., n$, g is a triangle homomorphism. Now, we prove that g is surjective. Let $a' = (a_1/F_1, ..., a_n/F_n) \in \mathcal{A}$, for all $a_i \in a/F_i$, $i = 1, ..., n$ are representatives of the corresponding equivalence classes. Then $a_i \in F_i$ i.e. $a_i \equiv_{F_i} 1$. We construct an element a'' such that $g(a'') = a'$. By Proposition 4.2, for all $i, j \in \{1, ..., n\}$, $i \neq j$, there exists an element $x_{ij} \in A$ such that $x_{ij} \equiv_{F_i} 1, x_{ij} \equiv_{F_j} 0$. Set

$$\begin{gathered} r_1 = x_{12} * ... * x_{1n} \\ r_2 = x_{21} * ... * x_{2n} \\ . \\ . \\ r_n = x_{n1} * ... * x_{n,n-1}. \end{gathered}$$

Then for all $i = 1, ..., n, i \neq j, r_i \in F_i$ and $r_i \leq x_{ij}$. Thus $\neg x_{ij} \leq \neg r_i$, for all $\neg x_{ij} \in F_j$. So $\neg r_i \in F_j$. Hence $r_i \equiv_{F_i} 1, r_i \equiv_{F_j} 0$. We set, $a'' = \neg[\neg(a_1 * r_1) * ... * \neg(a_n * r_n)]$ and show that $g(a'') = a'$. First we show $(a_i \to a'') * (a'' \to a_i) \in F_i$, for all $i \in \{1, ..., n\}$. Indeed, we have $a_i * r_i \leq a_i \to a'' = (a_i * [\neg(a_1 * r_1) * ... * \neg(a_n * r_n)]) \to 0$ iff $(a_i * r_i) * ([a_i * (\neg(a_1 * r_1) * ... * \neg(a_n * r_n))]) \leq 0$. Since $(a_i * r_i) * ([a_i * (\neg(a_1 * r_1) * ... * \neg(a_n * r_n))] \leq (a_i * r_i) * \neg(a_i * r_i) = 0$, $(a_i \to a'') \in F_i$. Since $a_i \leq a'' \to a_i$ and $a_i \in F_i$, $a'' \to a_i \in F_i$. So $(a_i \to a'') * (a'' \to a_i) \in F_i$. Clearly, $g(a) = (1/F_1, ..., 1/F_n) = 1_{\mathcal{A}}$ iff $a \equiv_{F_i} 1$, for all $i = 1, ..., n$. Thus $a \in \cap_{i=1}^{n} F_i$. □

Definition 4.4. *A triangle algebra A is said to be semilocal if it contain only finite IVRL-extended maximal filter.*

Remark 4.1. *Clearly, every local triangle algebra is semilocal.*

In the following example we show that the converse of above remark is not true, in general.

Example 4.2. *In Example 3.2, Clearly A has two IVRL-extended maximal filters. So A is semilocal triangle algebra but A is not local triangle algebra.*

By Theorem 3.2 and Remark 4.1 we have:

Corollary 4.1. *If F is an n-fold IVRL-extended integral filter of A, then A/F is a semilocal triangle algebras.*

In the following example we show that the converse of above corollary is not true.

Example 4.3. *In Example 3.2, let* $F = \{[1, 1]\}$*. Then* $A/\{[1, 1]\} \equiv A$*, so* $A/\{[1, 1]\}$ *is semilocal triangle algebras. But* F *is not an* n*-fold IVRL-extended integral filter of* A*, since* $\neg(\nu[a, 1]*\nu[b, b]) \in F$ *but* $\neg(\nu[a, 1]) = [b, b] \notin F$ *and* $\neg(\nu[b, b]) = [a, a] \notin F$*.*

Theorem 4.2. *A is a semilocal triangle algebra iff any proper descending chain of IVRL-filters in $A/F(A)$ is finite, where $F(A) = \cap\{F \mid F$ is an IVRL-extended maximal filter of $A\}$.*

Proof. Let $F_1, ..., F_n$ be the n disjoint IVRL-extended maximal filters of A. Then $F(A) = \cap_{i=1}^{n} F_i$ and by Proposition 4.1, $A/F(A)$ is isomorphic to $\prod_{i=1}^{n} A/F_i$. By Proposition 4.1, in $\prod_{i=1}^{n} A/F_i$ any properly descending chain of IVRL-filters is finite. If A contains infinitely many IVRL-extended maximal filters $F_1, F_2, ...,$ then $F_1 \supseteq F_1 \bigcap F_2 \supseteq ...$ is an infinite properly descending chain of IVRL-filters generating an infinite properly descending chain of IVRL-filters generating an infinite properly descending chain $F_1/F(A) \supseteq (F_1 \bigcap F_2)/F(A) \supseteq ...$ of IVRL-filters into $A/F(A)$. The proof is complete. □

Theorem 4.3. *Let A be a triangle algebra. The following is equivalent:*

(i) A is a semilocal triangle algebra,

(ii) $A/Rad(A)$ is isomorphic to a direct product of finitely many semisimple linear triangle algebra,

(iii) $A/Rad(A)$ has finitely many IVRL-filters.

Proof. $(i \Rightarrow ii)$ Let A be a semilocal triangle algebra and $\{M_1, ...M_k\}$ be the set of all IVRL-extended maximal filters of A. Then $Rad(A) = M_1 \cap ... \cap M_k$. So each A/M_i is semisimple linear triangle algebra. We define the map $\varphi : A/Rad(A) \longrightarrow A/M_1 \times ... \times A/M_k$ by $\varphi(x/Rad(A)) = (x/M_1, ..., x/M_k)$. Then φ is clearly a homomorphism. We show that φ is an isomorphism. Let $(x/M_1, ..., x/M_k) \in A/M_1 \times ... \times A/M_k$. Since $[M_i \cup M_j) = A$ for $i, j = 1, ..., k$ and $i \neq j$ by Proposition 4.3, there exists $x \in A$ such that $x/M_i = x_i/M_i$, for all $i = 1, ..., k$. Thus $(x_1/M_1, ...x_k/M_k) = (x/M_1, ...x/M_k) = \varphi(x/Rad(A))$ and so φ is surjective. Now, we have to show φ is injective. Suppose that $\varphi(x/Rad(A)) = \varphi(y/Rad(A))$, for all $x, y \in A$. Hence $x/M_i = y/M_i$ for all $i = 1, ..., k$. Thus $x \to y \in M_i$ and $y \to x \in M_i$, for $i = 1, ..., k$, that is, $x \to y \in Rad(A)$ and $y \to x \in Rad(A)$. So $x/Rad(A) = y/Rad(A)$. It is proved that φ is an isomorphism.

$(ii \Rightarrow iii)$ Let $A/Rad(A) \equiv A_1 \times ... \times A_k$, where A_i are semisimple linear triangle algebra for $i = 1, ..., k$. By Lemma 2.4, $\mid Fil(A/Rad(A)) \mid = \mid Fil(A_1) \times ... \times Fil(A_k) \mid$. Since $Fil(A_i)$ has two elements for every $i = 1, ..., k$, we have $Fil(A/Rad(A)) = 2^k$. Thus $A/Rad(A)$ has finite many IVRL-filters.

$(iii \Rightarrow i)$ Let A have infinitely many IVRL-extended maximal filters F_n, $n \in \mathbb{N}$. Obviously, all $F_n/Rad(A)$ are IVRL-filters of $A/Rad(A)$. So we have

$$F/Rad(A) = F'/Rad(A) \Rightarrow F = F', \tag{1}$$

where F, F' are IVRL-extended maximal filters of A. Let $F/Rad(A) = F'/Rad(A)$ and let $x \in F$. Then $x/Rad(A) \in F'/Rad(A)$ and so $x/Rad(A) = y/Rad(A)$ for some $y \in F'$. So $y \to x \in Rad(A) \subseteq F'$. Therefore $(y \to x) * y \in F'$. Thus $x \in F'$ and so $F \subseteq F'$. Similarly, $F' \subseteq F$, and we obtain $F = F'$. Hence 1 holds. From 1 it follows that $A/Rad(A)$ has infinitely many IVRL-filters $F_n/Rad(A)$, which is impossible. $\square$

Conclusion and future work

The notions of triangle algebras and interval valued residuated lattices have been defined by Van Gasse et al., who proved that there is a one-to-one correspondence between the classes of IVRLs and triangle algebras [17]. The same authors defined filters in triangle algebras, suggesting two different ways to define the specific types of these filters, proposing remarkable findings [16].

In this study, we investigated several important properties of local and semilocal triangle algebras. The special set $D(A)$ was defined, and the correlation between the set and local triangle algebras was determined, while their key properties were also summarized. Furthermore, the correlations between these algebras and some IVRL-filters were assessed. Finally, semilocal triangle algebras were introduced and studied in detail, and the important properties of these structures were presented.

In our future work, we will continue our study of algebraic properties of this special sets on triangle algebras, with the view to identify a classification for these structures.

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