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On the Structure of Octonion Regular Functions

Janne Kauhanen* and Heikki Orelma

Abstract. In this paper, we study octonion regular functions and the structural differences between regular functions in octonion, quaternion, and Clifford analyses.

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1. Introduction

In our recent papers [5,6], we started to study octonion algebraic methods in analysis. This work is a continuation of our studies in this fascinating field. Over the years, many results of octonion analysis have been published and studied since the fundamental paper of Dentoni and Sce [2] was published. One thing which has remained unclear to us is what is octonion analysis all about? A consensus has been that octonion, quaternion and Clifford analyses are similar from a theoretical point of view and, maybe for this reason, octonion analysis has received less attention. Our aim is to prove that octonion analysis and Clifford analysis are different theories from the perpective of regular functions. Thus, octonion analysis is a completely independent research topic.

We start by recalling preliminaries of octonions and Clifford numbers and their connections via triality. We define our fundamental function classes, i.e., left-, right- and bi-regular functions. We give chararacterizations for function classes in biaxial quaternion analysis and in Clifford analysis. The classical Riesz system of Stein and Weiss is used as a familiar reference to clearly see the differences.

The topic of this paper is highly technical, but we have tried to write everything as simply as possible. Hopefully we have succeeded in this job.

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*Corresponding author.

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Many questions remain open and the reader may find a lot of open research problems between the lines. We expect to answer some of these questions when the saga continues.

2. Preliminaries: Octonion and Clifford Algebras

In this algebraic part of the paper, we first recall briefly the basic definitions and notations related to octonion and Clifford algebras. In the second place, we study their connections in detail. In the whole paper, our principle is to consider the standard orthonormal basis $\{e_0, e_1, \ldots, e_7\}$ for $\mathbb{R}^8$ equipped with multiplications which lead to nonisomorphic algebras. We will denote the octonion product by $e_i \circ e_j$, and the Clifford product by $e_i e_j$.

2.1. Octonions

The algebra of octonions $\mathbb{O}$ is the non-commutative and non-associative 8-dimensional algebra with the basis $\{1, e_1, \ldots, e_7\}$, where e_0 is denoted by 1 (the identity) and omitted whenever clear from the context, and multiplication given by the following table.

$\circ$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

Let us point out that there are several ways to define an octonion product such that $e_0 = 1$. Our choice is historically made, justified by tradition, and, for this reason, we may call it the canonical one. However, for instance, Lounesto uses a different multiplication table in his famous book [8].

For $1 \leq i, j \leq 7$ we have

$$e_i \circ e_i = e_i^2 = -1, \qquad \text{and} \qquad e_i \circ e_j = -e_j \circ e_i \quad \text{if } i \neq j.$$

An element $x \in \mathbb{O}$ may be represented in the forms

$$\begin{aligned} x &= x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7 \\ &= x_0 + \underline{x} \\ &= (x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3) + (x_4 + x_5 e_1 + x_6 e_2 + x_7 e_3) \circ e_4 \\ &= u + v \circ e_4 \\ &= (u_0 + \underline{u}) + (v_0 + \underline{v}) \circ e_4. \end{aligned}$$

Here, $x_0, \ldots, x_7 \in \mathbb{R}$, x_0 is the *real part*, $\underline{x}$ is the *vector part*, and u and $v \in \mathbb{H}$ are quaternions. The last form is called the *quaternion form* of an octonion.

The *conjugate* of x is denoted and defined by $\overline{x} = x_0 - \underline{x}$. Furthermore, the product of two octonions can be written as

$$\begin{aligned} x \circ y = (x_0 + \underline{x}) \circ (y_0 + \underline{y}) &= \sum_{i,j=0}^{7} x_i y_j e_i \circ e_j = \sum_{i=0}^{7} x_i y_i e_i^2 + \sum_{\substack{i,j=0 \\ i \neq j}}^{7} x_i y_j e_i \circ e_j \\ &= x_0 y_0 - \sum_{i=1}^{7} x_i y_i + x_0 \sum_{i=1}^{7} y_i e_i + y_0 \sum_{i=1}^{7} x_i e_i + \sum_{\substack{i,j=1 \\ i \neq j}}^{7} x_i y_j e_i \circ e_j \\ &= x_0 y_0 - \underline{x} \cdot \underline{y} + x_0 \underline{y} + y_0 \underline{x} + \underline{x} \times \underline{y}, \end{aligned} \tag{1}$$

where $\underline{x} \cdot \underline{y}$ is the *dot product* and $\underline{x} \times \underline{y}$ the *cross product* of vectors $\underline{x}$ and $\underline{y}$ in $\mathbb{R}^7$.

Denote the quaternion forms of the octonions x and y by

$$\begin{aligned} x &= (u_0 + \underline{u}) + (v_0 + \underline{v}) \circ e_4, \\ y &= (a_0 + \underline{a}) + (b_0 + \underline{b}) \circ e_4. \end{aligned} \tag{2}$$

In Lemma 2.3 we will write the cross product $\underline{x} \times \underline{y}$ of octonion vector parts $\underline{x}$ and $\underline{y}$ using the classical 3-dimensional cross products of the vector parts $\underline{u}$, $\underline{v}$, $\underline{a}$, and $\underline{b}$ of quaternions (see, e.g., [3,8,10]).

Lemma 2.1. ([5, Lemma 2.10]) *Let $u, v \in \mathbb{H}$. Then*

$$\begin{aligned} e_4 \circ u &= \overline{u} \circ e_4, \\ e_4 \circ (u \circ e_4) &= -\overline{u}, \\ (u \circ e_4) \circ e_4 &= -u, \\ u \circ (v \circ e_4) &= (v \circ u) \circ e_4, \\ (u \circ e_4) \circ v &= (u \circ \overline{v}) \circ e_4, \\ (u \circ e_4) \circ (v \circ e_4) &= -\overline{v} \circ u. \end{aligned}$$

Lemma 2.2. *If $x, y \in \mathbb{O}$ are written as in (2), then*

$$\begin{aligned} \underline{u} \times e_4 &= \underline{u} \circ e_4, \\ e_4 \times \underline{a} &= -\underline{a} \circ e_4, \\ \underline{u} \times (\underline{b} \circ e_4) &= -(\underline{u} \times \underline{b}) \circ e_4 - (\underline{u} \cdot \underline{b}) e_4, \\ e_4 \times (\underline{b} \circ e_4) &= \underline{b}, \\ (\underline{v} \circ e_4) \times \underline{a} &= -(\underline{v} \times \underline{a}) \circ e_4 + (\underline{a} \cdot \underline{v}) e_4, \\ (\underline{v} \circ e_4) \times e_4 &= -\underline{v}, \\ (\underline{v} \circ e_4) \times (\underline{b} \circ e_4) &= -\underline{v} \times \underline{b}. \end{aligned}$$

Proof. On the one hand, the first two equalities are direct consequences of (1). On the other hand, Lemma 2.1 implies

$$\begin{aligned} e_i \circ (e_j \circ e_4) &= (e_j \circ e_i) \circ e_4, \\ e_4 \circ (e_j \circ e_4) &= e_j, \\ (e_i \circ e_4) \circ e_j &= -(e_i \circ e_j) \circ e_4, \end{aligned}$$

$$\begin{aligned}(e_i \circ e_4) \circ e_4 &= -e_i,\\ (e_i \circ e_4) \circ (e_j \circ e_4) &= e_j \circ e_i\end{aligned}$$

for $1 \leq i, j \leq 3$. Then

$$\begin{aligned}
\underline{u} \times (\underline{b} \circ e_4) &= \sum_{i,j=1}^{3} u_i b_j e_i \circ (e_j \circ e_4) = \left(\sum_{i,j=1}^{3} u_i b_j e_j \circ e_i \right) \circ e_4\\
&= \left(-\sum_{\substack{i,j=1\\ i\neq j}}^{3} u_i b_j e_i \circ e_j - \sum_{i=1}^{3} u_i b_i \right) \circ e_4 = -(\underline{u} \times \underline{b}) \circ e_4 - (\underline{u} \cdot \underline{b}) e_4,\\
e_4 \times (\underline{b} \circ e_4) &= \sum_{j=1}^{3} b_j e_4 \circ (e_j \circ e_4) = \sum_{j=1}^{3} b_j e_j = \underline{b},\\
(\underline{v} \circ e_4) \times \underline{a} &= \sum_{i,j=1}^{3} v_i a_j (e_i \circ e_4) \circ e_j = \left(-\sum_{i,j=1}^{3} v_i a_j e_i \circ e_j \right) \circ e_4\\
&= \left(-\sum_{\substack{i,j=1\\ i\neq j}}^{3} v_i a_j e_i \circ e_j + \sum_{i=1}^{3} v_i a_i \right) \circ e_4 = -(\underline{v} \times \underline{a}) \circ e_4 + (\underline{a} \cdot \underline{v}) e_4,\\
(\underline{v} \circ e_4) \times e_4 &= \sum_{i=1}^{3} v_i (e_i \circ e_4) \circ e_4 = -\sum_{i=1}^{3} v_i e_i = -\underline{v},
\end{aligned}$$

and

$$\begin{aligned}
(\underline{v} \circ e_4) \times (\underline{b} \circ e_4) &= \sum_{\substack{i,j=1\\ i\neq j}}^{3} v_i b_j (e_i \circ e_4) \circ (e_j \circ e_4) = \sum_{\substack{i,j=1\\ i\neq j}}^{3} v_i b_j e_j \circ e_i\\
&= -\sum_{\substack{i,j=1\\ i\neq j}}^{3} v_i b_j e_i \circ e_j\\
&= -\underline{v} \times \underline{b}.
\end{aligned}$$

□

Lemma 2.3. *Consider the quaternion representations of the vectors* $\underline{x}$ *and* $\underline{y} \in \mathbb{O}$ *as in* (2)*. Then the cross product in quaternion form is*

$$\begin{aligned}
\underline{x} \times \underline{y} = v_0 \underline{b} - \underline{v} b_0 + \underline{u} \times \underline{a} - \underline{v} \times \underline{b} \quad & \in \mathrm{span}\{e_1, e_2, e_3\}\\
+ (\underline{v} \cdot \underline{a} - \underline{u} \cdot \underline{b}) e_4 \quad & \in \mathrm{span}\{e_4\}\\
+ (\underline{u} b_0 - v_0 \underline{a} - \underline{u} \times \underline{b} - \underline{v} \times \underline{a}) \circ e_4 \quad & \in \mathrm{span}\{e_5, e_6, e_7\}
\end{aligned}$$

Proof. By Lemma 2.2, we obtain

$$\begin{aligned}
\underline{x} \times \underline{y} = \underline{u} \times \underline{a} &+ \underline{u} \times (b_0 e_4) + \underline{u} \times (\underline{b} \circ e_4)\\
&+ (v_0 e_4) \times \underline{a} + (v_0 e_4) \times (b_0 e_4) + (v_0 e_4) \times (\underline{b} \circ e_4)\\
&+ (\underline{v} \circ e_4) \times \underline{a} + (\underline{v} \circ e_4) \times (b_0 e_4) + (\underline{v} \circ e_4) \times (\underline{b} \circ e_4)
\end{aligned}$$

$$
\begin{aligned}
&= \underline{u} \times \underline{a} + \underline{u} b_0 \circ e_4 - (\underline{u} \times \underline{b}) \circ e_4 \\
&\quad - v_0 \underline{a} \circ e_4 + 0 + v_0 \underline{b} \\
&\quad - (\underline{v} \times \underline{a}) \circ e_4 - \underline{v} b_0 - \underline{v} \times \underline{b} \\
&= \underline{u} \times \underline{a} - \underline{v} \times \underline{b} + v_0 \underline{b} - \underline{v} b_0 \\
&\quad + (\underline{v} \cdot \underline{a} - \underline{u} \cdot \underline{b}) e_4 \\
&\quad + (\underline{u} b_0 - v_0 \underline{a} - \underline{u} \times \underline{b} - \underline{v} \times \underline{a}) \circ e_4. \qquad \square
\end{aligned}
$$

Corollary 2.4. *If $x, y \in \mathbb{O}$ are written as in (2), then*

$$
\begin{aligned}
x \circ y = {} & u_0 a_0 - v_0 b_0 - \underline{u} \cdot \underline{a} - \underline{v} \cdot \underline{b} && \in \mathbb{R} \\
& + u_0 \underline{a} + a_0 \underline{u} + v_0 \underline{b} - \underline{v} b_0 + \underline{u} \times \underline{a} - \underline{v} \times \underline{b} && \in \operatorname{span}\{e_1, e_2, e_3\} \\
& + (u_0 b_0 + a_0 v_0 + \underline{v} \cdot \underline{a} - \underline{u} \cdot \underline{b}) e_4 && \in \operatorname{span}\{e_4\} \\
& + (u_0 \underline{b} + a_0 \underline{v} + \underline{u} b_0 - v_0 \underline{a} - \underline{u} \times \underline{b} - \underline{v} \times \underline{a}) \circ e_4 && \in \operatorname{span}\{e_5, e_6, e_7\}
\end{aligned}
$$

2.2. The Clifford Algebra $\mathcal{C}\ell_{0,7}$ and Triality

Since the dimension of the underlying vector space of the octonions and Clifford paravectors is 8, they behave similarly as vector spaces. Moreover, we may ask if there is a connection between the octonion product and the Clifford product? The answer is given by Pertti Lounesto in his book [8]. We will recall his ideas here in detail. Let us recall the basic definitions and properties of Clifford algebras.

We continue working with the basis $\{e_0, e_1, \dots, e_7\}$ for $\mathbb{R}^8$. The Clifford product is defined by

$$
e_i e_j + e_j e_i = -2\delta_{ij}, \; i, j = 1, \dots, 7,
$$

where δ_{ij} is the Kronecker delta symbol. Here, $e_0 = 1$. Then, similarly to the case of octonions, $e_0^2 = 1$, and $e_j^2 = -1$ for all $j = 1, \dots, 7$. The Clifford product $e_i e_j$ is not necessarily a vector or a scalar. This product generates an associative algebra, called the Clifford algebra, denoted by $\mathcal{C}\ell_{0,7}$. The dimension of this Clifford algebra is 2^7, and an element $a \in \mathcal{C}\ell_{0,7}$ may be represented as a sum

$$
a = \sum_{j=0}^{7} [a]_j
$$

of a scalar part $[a]_0$, generated by 1, a 1-vector part $[a]_1$, generated by e_j's, a 2-vector part $[a]_2$, generated by the products $e_i e_j$, where $1 \le i < j \le 7$, etc. Clifford numbers of the form $[a]_1$ are called *vectors* and those of the form $[a]_{0,1} = [a]_0 + [a]_1$ are called *paravectors*. The set of paravectors may be identified with $\mathbb{R}^8$.

The Clifford product of two paravectors x and y can be written as

$$
\begin{aligned}
xy = (x_0 + \underline{x})(y_0 + \underline{y}) &= \sum_{i,j=0}^{7} x_i y_j e_i e_j \\
&= \sum_{i=0}^{7} x_i y_i e_i^2 + \sum_{\substack{i,j=0 \\ i \ne j}}^{7} x_i y_j e_i e_j
\end{aligned}
$$

$$= x_0y_0 - \sum_{i=1}^{7} x_iy_i + x_0\sum_{i=1}^{7} y_ie_i + y_0\sum_{i=1}^{7} x_ie_i + \sum_{\substack{i,j=1\\ i\neq j}}^{7} x_iy_je_ie_j$$

$$= x_0y_0 - \underline{x}\cdot\underline{y} + x_0\underline{y} + y_0\underline{x} + \underline{x}\wedge\underline{y}, \tag{3}$$

where $\underline{x}\wedge\underline{y}$ is the *wedge product* of vectors $\underline{x}$ and $\underline{y}$. In particular, $\underline{x}\underline{y} = \underline{x}\wedge\underline{y} - \underline{x}\cdot\underline{y}$.

The reader can see that formally the octonion and the Clifford products are similar, and a reasonable question is "how they are connected?". We would like to construct the octonion product using the Clifford algebra $\mathcal{C}\ell_{0,7}$. Let us consider the octonion product of the basis elements e_i and e_j, where $1 \leq i, j \leq 7$, $i \neq j$:

$$e_i \circ e_j = e_k.$$

Then $1 \leq k \leq 7$, and $i \neq k \neq j$. The corresponding Clifford product e_ie_j may be mapped to e_k by multiplying it by the trivector $e_je_ie_k$, i.e.,

$$(e_ie_j)(e_je_ie_k) = e_i^2e_j^2e_k = e_k.$$

Using the same trivector, e_je_i is mapped to $-e_k$. If $\underline{a}$ and $\underline{b}$ are vectors, then

$$\underline{a}\,\underline{b}(e_je_ie_k) = (a_ib_j - a_jb_i)e_k + [\underline{a}\,\underline{b}(e_je_ie_k)]_3 + [\underline{a}\,\underline{b}(e_je_ie_k)]_5.$$

Picking the 1-vector part

$$[\underline{a}\,\underline{b}(e_je_ie_k)]_1 = (a_ib_j - a_jb_i)e_k,$$

we get a part of the kth component of the octonion product $\underline{a}\circ\underline{b}$. Using this idea, we may express the octonion product $a\circ b$ as the paravector part of the Clifford product $ab(1-W)$, where W is a suitable 3-vector.

Lemma 2.5. ([8, Sec 23.3], [12, Lemma 4.1]) *Define*

$$W = e_{123} + e_{145} + e_{176} + e_{246} + e_{257} + e_{347} + e_{365}.$$

Let $a = a_0 + \underline{a}$ and $b = b_0 + \underline{b}$ be paravectors. Then

$$a \circ b = [ab(1-W)]_{0,1}$$

and in particular, $\underline{a}\times\underline{b} = -[(\underline{a}\wedge\underline{b})W]_1$.

Lounesto states Lemmas 2.5 and 2.7 without proofs at pages 303–304 in [8]. He uses a different multiplication table of octonions, and therefore the seven basis vectors e_{ijk} have different indices i, j, and k in his trivector W. Venäläinen gives a proof for Lemma 2.5 in her licentiate thesis [12]. For the convenience of the reader, we give a proof of Lemma 2.5 here.

Proof. To begin with, we have

$$[ab(1-W)]_{0,1} = [ab]_{0,1} - [abW]_1 = a_0b_0 - \underline{a}\cdot\underline{b} + a_0\underline{b} + b_0\underline{a} - [abW]_1.$$

By (1) and (3), it is enough to show that $\underline{a}\times\underline{b} = -[(\underline{a}\wedge\underline{b})W]_1$. Consider the triplets

$$\nu = 123, 145, 176, 246, 257, 347, 365.$$

The product $e_ie_je_\nu$ is a vector only if the pair of indices ij belongs to the triplet ν. Since the cross and the wedge products

$$\underline{a}\times\underline{b}=\sum_{\substack{i,j=1\\ i<j}}^{7}(a_ib_j-a_jb_i)e_i\circ e_j \quad \text{and} \quad \underline{a}\wedge\underline{b}=\sum_{\substack{i,j=1\\ i<j}}^{7}(a_ib_j-a_jb_i)e_ie_j$$

have the same coefficients, and each pair ij, $1\le i<j\le 7$, is contained in exatly one of the triplets ν, say ν_0, it is enough to check that $e_ie_je_{\nu_0}=-e_i\circ e_j$ for all such pairs ij. □

A straightforward computation shows:

Lemma 2.6. *The trivector* $W=e_{123}+e_{145}+e_{176}+e_{246}+e_{257}+e_{347}+e_{365}$ *is invertible with*

$$W^{-1}=\frac{1}{7}(W-6e_{12\cdots 7}).$$

In the above, we identified octonions $\mathbb{O}$ with the 8-dimensional paravectors. The dimension 8 plays a special role in the theory of spin groups, since Spin(8) has the so called *exceptional automorphisms*. This feature is called *triality*, and the first time it was noticed was in the book of Study [11]. For modern references, see [4,8,10]. The triality means that in addition to paravectors, we may identify octonions with the spinor spaces $S^\pm$. The spinor spaces may be realized by

$$S^\pm=\mathcal{C}\ell_{0,7}\,I^\pm,$$

where $I^\pm$ is a primitive idempotent,

$$I^\pm=\frac{1}{16}(1+We_{12\cdots 7})(1\pm e_{12\cdots 7}), \tag{4}$$

see [1,8]. A straightforward computation shows that the octonion product in spinor spaces may be determined as follows.

Lemma 2.7. [8, Sec 23.3] *For paravectors* a *and* b*, we have*

$$a\circ b=16[abI^-]_{0,1}. \tag{5}$$

3. Octonion Analysis

In this section, we recall basic facts of octonion analysis, i.e., the theory of Cauchy–Riemann operators in the octonionic setting. After that, we carefully study the general structure of the null solutions of these operators and define four different classes of regular functions: left-, right-, B-, and R-regular functions. R-regular functions are just solutions of the classical Riesz system. We use the Riesz system here as a familiar reference to better understand the structure of octonion regular functions. In Clifford analysis the corresponding function classes are equal. This structural difference is a fundamental difference between octonion and Clifford analyses.

3.1. Cauchy–Riemann Operators

A function $f\colon \mathbb{R}^8 \to \mathbb{O}$ is of the form $f = f_0 + f_1e_1 + \cdots + f_7e_7 = f_0 + \underline{f}$, where $f_j\colon \mathbb{R}^8 \to \mathbb{R}$. We define the *Cauchy–Riemann operator*

$$D_x = \partial_{x_0} + e_1 \circ \partial_{x_1} + \cdots + e_7 \circ \partial_{x_7}.$$

Its vector part,

$$D_{\underline{x}} = e_1 \circ \partial_{x_1} + \cdots + e_7 \circ \partial_{x_7}$$

is called the *Dirac operator*. If the coordinate functions of f have partial derivatives, then D_x operates on f from the left and from the right as

$$D_x f = \sum_{i,j=0}^{7} e_i \circ e_j \partial_{x_i} f_j \quad \text{and} \quad f D_x = \sum_{i,j=0}^{7} e_j \circ e_i \partial_{x_i} f_j.$$

Decomposition (1) gives (see [7])

$$D_x f = \partial_{x_0} f_0 - D_{\underline{x}} \cdot \underline{f} + \partial_{x_0}\underline{f} + D_{\underline{x}} f_0 + D_{\underline{x}} \times \underline{f} \quad \text{and} \tag{6}$$

$$f D_x = \partial_{x_0} f_0 - D_{\underline{x}} \cdot \underline{f} + \partial_{x_0}\underline{f} + D_{\underline{x}} f_0 - D_{\underline{x}} \times \underline{f}, \tag{7}$$

where $\partial_{x_0} f_0 - D_{\underline{x}} \cdot \underline{f}$ is the *divergence* of f and $D_{\underline{x}} \times \underline{f}$ is the *rotor* of $\underline{f}$.

If $D_x f = 0$ (resp., $f D_x = 0$), then f is called *left* (resp., *right*) *regular.*

In Clifford analysis one studies functions $f\colon \mathbb{R}^8 \to \mathcal{C}\ell_{0,7}$. We define the Cauchy–Riemann operator similarly as in octonion analysis:

$$\partial_x = \partial_{x_0} + e_1\partial_{x_1} + \cdots + e_7\partial_{x_7} = \partial_{x_0} + \partial_{\underline{x}}.$$

Functions satisfying $\partial_x f = 0$ (resp., $f\partial_x = 0$) on $\mathbb{R}^8$ are called *left* (resp., *right*) *monogenic.* In this paper we only need to consider paravector valued functions

$$f = f_0 + f_1e_1 + \cdots + f_7e_7.$$

3.2. Left-, Right-, B- and R-Regular Functions

Comparing the real and vector parts in (6) and (7) yields the following well known results.

Proposition 3.1. *A function $f\colon \mathbb{R}^8 \to \mathbb{O}$ is left regular if and only if it satisfies the* Moisil–Teodorescu type system

$$\begin{aligned} \partial_{x_0} f_0 - D_{\underline{x}} \cdot \underline{f} &= 0, \\ \partial_{x_0}\underline{f} + D_{\underline{x}} f_0 + D_{\underline{x}} \times \underline{f} &= 0, \end{aligned} \tag{8}$$

whose componentwise form is

$$\begin{aligned}
&\partial_{x_0} f_0 - \partial_{x_1} f_1 - \ldots - \partial_{x_7} f_7 = 0,\\
&\quad \partial_{x_0} f_1 + \partial_{x_1} f_0 + \partial_{x_2} f_3 - \partial_{x_3} f_2 + \partial_{x_4} f_5 - \partial_{x_5} f_4 - \partial_{x_6} f_7 + \partial_{x_7} f_6 = 0,\\
&\quad \partial_{x_0} f_2 + \partial_{x_2} f_0 - \partial_{x_1} f_3 + \partial_{x_3} f_1 + \partial_{x_4} f_6 - \partial_{x_6} f_4 + \partial_{x_5} f_7 - \partial_{x_7} f_5 = 0,\\
&\quad \partial_{x_0} f_3 + \partial_{x_3} f_0 + \partial_{x_1} f_2 - \partial_{x_2} f_1 + \partial_{x_4} f_7 - \partial_{x_7} f_4 - \partial_{x_5} f_6 + \partial_{x_6} f_5 = 0,\\
&\quad \partial_{x_0} f_4 + \partial_{x_4} f_0 - \partial_{x_1} f_5 + \partial_{x_5} f_1 - \partial_{x_2} f_6 + \partial_{x_6} f_2 - \partial_{x_3} f_7 + \partial_{x_7} f_3 = 0,\\
&\quad \partial_{x_0} f_5 + \partial_{x_5} f_0 + \partial_{x_1} f_4 - \partial_{x_4} f_1 - \partial_{x_2} f_7 + \partial_{x_7} f_2 + \partial_{x_3} f_6 - \partial_{x_6} f_3 = 0,\\
&\quad \partial_{x_0} f_6 + \partial_{x_6} f_0 + \partial_{x_1} f_7 - \partial_{x_7} f_1 + \partial_{x_2} f_4 - \partial_{x_4} f_2 - \partial_{x_3} f_5 + \partial_{x_5} f_3 = 0,\\
&\quad \partial_{x_0} f_7 + \partial_{x_7} f_0 - \partial_{x_1} f_6 + \partial_{x_6} f_1 + \partial_{x_2} f_5 - \partial_{x_5} f_2 + \partial_{x_3} f_4 - \partial_{x_4} f_3 = 0.
\end{aligned} \tag{9}$$

We will denote the space of left regular functions by $\mathcal{M}^{(\ell)}$, and similary, right regular functions by $\mathcal{M}^{(r)}$.

Proposition 3.2. *A function $f\colon \mathbb{R}^8 \to \mathbb{O}$ is both left and right regular if and only if it satisfies the system*

$$\begin{aligned}
\partial_{x_0} f_0 - D_{\underline{x}} \cdot \underline{f} &= 0,\\
\partial_{x_0} \underline{f} + D_{\underline{x}} f_0 &= 0,\\
D_{\underline{x}} \times \underline{f} &= 0,
\end{aligned} \tag{10}$$

whose componentwise form is

$$\begin{aligned}
&\partial_{x_0} f_0 - \partial_{x_1} f_1 - \ldots - \partial_{x_7} f_7 = 0,\\
&\partial_{x_0} f_i + \partial_{x_i} f_0 = 0, \quad i = 1, \ldots, 7,\\
&\partial_{x_2} f_3 - \partial_{x_3} f_2 + \partial_{x_4} f_5 - \partial_{x_5} f_4 - \partial_{x_6} f_7 + \partial_{x_7} f_6 = 0,\\
&- \partial_{x_1} f_3 + \partial_{x_3} f_1 + \partial_{x_4} f_6 - \partial_{x_6} f_4 + \partial_{x_5} f_7 - \partial_{x_7} f_5 = 0,\\
&\partial_{x_1} f_2 - \partial_{x_2} f_1 + \partial_{x_4} f_7 - \partial_{x_7} f_4 - \partial_{x_5} f_6 + \partial_{x_6} f_5 = 0,\\
&- \partial_{x_1} f_5 + \partial_{x_5} f_1 - \partial_{x_2} f_6 + \partial_{x_6} f_2 - \partial_{x_3} f_7 + \partial_{x_7} f_3 = 0,\\
&\partial_{x_1} f_4 - \partial_{x_4} f_1 - \partial_{x_2} f_7 + \partial_{x_7} f_2 + \partial_{x_3} f_6 - \partial_{x_6} f_3 = 0,\\
&\partial_{x_1} f_7 - \partial_{x_7} f_1 + \partial_{x_2} f_4 - \partial_{x_4} f_2 - \partial_{x_3} f_5 + \partial_{x_5} f_3 = 0,\\
&- \partial_{x_1} f_6 + \partial_{x_6} f_1 + \partial_{x_2} f_5 - \partial_{x_5} f_2 + \partial_{x_3} f_4 - \partial_{x_4} f_3 = 0.
\end{aligned} \tag{11}$$

We will call functions satisfying (11) *B-regular*, and denote the space of such functions by $\mathcal{M}_B$. Naturally

$$\mathcal{M}_B = \mathcal{M}^{(\ell)} \cap \mathcal{M}^{(r)}.$$

The fundamental difference between octonion and Clifford analyses is that in Clifford analysis the paravector valued null solutions to the Cauchy–Riemann operator satisfy the Riesz system and are at the same time left and right monogenic, which is not true in octonion analysis. The following well-known proposition follows from the definitions, similarly as in octonion analysis, by comparing the scalar parts, 1-vector parts, and 2-vector parts.

Proposition 3.3. *Suppose* $f\colon \mathbb{R}^8 \to \mathcal{C}\ell_{0,7}$ *is a paravector valued function. Then* $\partial_x f = 0$ *if and only if* $f\partial_x = 0$*, and this is equivalent to* f *satisfying the* Riesz-system

$$\begin{aligned} \partial_{x_0} f_0 - \partial_{\underline{x}} \cdot \underline{f} = 0, \\ \partial_{x_0} \underline{f} + \partial_{\underline{x}} f_0 = 0, \\ \partial_{\underline{x}} \wedge \underline{f} = 0, \end{aligned} \tag{12}$$

whose componentwise form is

$$\begin{aligned} \partial_{x_0} f_0 - \partial_{x_1} f_1 - \ldots - \partial_{x_7} f_7 = 0, \\ \partial_{x_0} f_i + \partial_{x_i} f_0 = 0, \quad i = 1, \ldots, 7, \\ \partial_{x_i} f_j - \partial_{x_j} f_i = 0, \quad i, j = 1, \ldots, 7, \ i \neq j, \end{aligned} \tag{13}$$

Functions satisfying (13) are called *R-regular*, and the space of such functions is denoted by $\mathcal{M}_R$.

To convince the reader about the existence of these function classes, we recall the following classical method from Clifford analysis.

Remark 3.4. (Cauchy–Kovalevskaya extension) If $f\colon \Omega \to \mathbb{R}$ is a real analytic function defined on an open $\Omega \subset \mathbb{R}^7 \cong \mathbb{O} \cap \{x_0 = 0\}$ we may construct its Cauchy–Kovalevskaya extension analogously to Clifford analysis (see [1]) by defining

$$\mathrm{CK}[f](x) = e^{-x_0 D_{\underline{x}}} f(\underline{x}).$$

It is easy to see that since f is real valued, $D_x \mathrm{CK}[f] = \mathrm{CK}[f] D_x = 0$, i.e., $\mathrm{CK}[f] \in \mathcal{M}_B$. Since $\mathbb{O}$ is an alternative division algebra, that is $x(xy) = x^2 y$ for all $x, y \in \mathbb{O}$, the Cauchy–Kovalevskaya extension may be extended to octonion valued real analytic functions. A necessary condition for $\mathrm{CK}[f] \in \mathcal{M}^{(\ell)}$ is that f is an octonion valued real analytic function with $\underline{f} \neq 0$. This condition is not sufficient since, e.g.,

$$\mathrm{CK}[\underline{x}](x) = 7x_0 + \underline{x}$$

belongs to $\mathcal{M}_B$.

We may conclude that although Clifford and octonion analyses have formally very similar definitions, the corresponding function spaces are different.

Proposition 3.5. $\mathcal{M}_R \subsetneq \mathcal{M}_B \subsetneq \mathcal{M}^{(\ell)}$

Proof. Two inclusions follow from Propositions 3.1–3.3. The examples showing that the inclusions are strict, respectively, are: if $f = x_2 e_1 - x_7 e_4$, then $D_x f = f D_x = 0$, but $\partial_{x_2} f_1 - \partial_{x_1} f_2 = 1 \neq 0$, and if $f = x_1 - x_2 e_3$, then $D_x f = 0$, but $f D_x = 2e_1 \neq 0$. □

This result is crucial in understanding the fundamental character of octonion analysis and the structural differences between octonion, quaternion, and Clifford analyses.

Remark 3.6. (Quaternion analysis) If we make the corresponding definitions for quaternion regular function classes by considering the Cauchy–Riemann operator $D_x = \partial_{x_0} + e_1 \circ \partial_{x_1} + e_2 \circ \partial_{x_2} + e_3 \circ \partial_{x_3}$ acting on quaternion valued functions $f = f_1 + f_1 e_1 + f_2 e_2 + f_3 e_3$, then, by comparing (11) and (13), we observe immediately that

$$\mathcal{M}_R = \mathcal{M}_B \subsetneq \mathcal{M}^{(\ell)}.$$

Remark 3.7. (Clifford analysis) If we make the corresponding definitions for paravector valued monogenic functions, then, by Proposition 3.3,

$$\mathcal{M}_R = \mathcal{M}_B = \mathcal{M}^{(\ell)}.$$

4. Function Classes in Biaxial Quaternion Analysis

In the preceding section we gave characterizations for left-, B-, and R-regular functions using componentwise and vector forms. In this section we write the three systems of Sect. 3.2 in quaternion forms. The use of the quaternion forms of the function and the Cauchy–Riemann operator is called the *biaxial quaternion analysis.*

Consider the Cauchy–Riemann operator D_x and the function $f\colon \mathbb{R}^8 \to \mathbb{O}$ in the quaternion forms:

$$\begin{aligned} D_x &= \partial_{u_0} + \partial_{\underline{u}} + (\partial_{v_0} + \partial_{\underline{v}}) \circ e_4, \\ f &= g_0 + \underline{g} + (h_0 + \underline{h}) \circ e_4. \end{aligned}$$

According to Corollary 2.4, we can write

$$\begin{aligned} D_x f = & \partial_{u_0} g_0 - \partial_{v_0} h_0 - \partial_{\underline{u}} \cdot \underline{g} - \partial_{\underline{v}} \cdot \underline{h} \\ & + \partial_{u_0} \underline{g} + \partial_{\underline{u}} g_0 + \partial_{v_0} \underline{h} - \partial_{\underline{v}} h_0 + \partial_{\underline{u}} \times \underline{g} - \partial_{\underline{v}} \times \underline{h} \\ & + (\partial_{u_0} h_0 + \partial_{v_0} g_0 + \partial_{\underline{v}} \cdot \underline{g} - \partial_{\underline{u}} \cdot \underline{h}) e_4 \\ & + (\partial_{u_0} \underline{h} + \partial_{\underline{v}} g_0 + \partial_{\underline{u}} h_0 - \partial_{v_0} \underline{g} - \partial_{\underline{u}} \times \underline{h} - \partial_{\underline{v}} \times \underline{g}) \circ e_4. \end{aligned}$$

This implies the quaternion forms (14) and (15) of, respectively, the Moisil–Teodorescu type system (8) and the system (10).

Proposition 4.1. *$f\colon \mathbb{R}^8 \to \mathbb{O}$ is left regular if and only if it satisfies the system*

$$\begin{aligned} & \partial_{u_0} h_0 + \partial_{v_0} g_0 + \partial_{\underline{v}} \cdot \underline{g} - \partial_{\underline{u}} \cdot \underline{h} = 0, \\ & \partial_{u_0} g_0 - \partial_{v_0} h_0 - \partial_{\underline{u}} \cdot \underline{g} - \partial_{\underline{v}} \cdot \underline{h} = 0, \\ & \partial_{u_0} \underline{g} + \partial_{\underline{u}} g_0 + \partial_{v_0} \underline{h} - \partial_{\underline{v}} h_0 + \partial_{\underline{u}} \times \underline{g} - \partial_{\underline{v}} \times \underline{h} = 0, \\ & \partial_{u_0} \underline{h} + \partial_{\underline{v}} g_0 + \partial_{\underline{u}} h_0 - \partial_{v_0} \underline{g} - \partial_{\underline{u}} \times \underline{h} - \partial_{\underline{v}} \times \underline{g} = 0. \end{aligned} \tag{14}$$

Proposition 4.2. *$f\colon \mathbb{R}^8 \to \mathbb{O}$ is left and right regular if and only if it satisfies the system*

$$\begin{aligned}
&\partial_{u_0} h_0 + \partial_{v_0} g_0 + \partial_{\underline{v}} \cdot \underline{g} - \partial_{\underline{u}} \cdot \underline{h} = 0, \\
&\partial_{u_0} g_0 - \partial_{v_0} h_0 - \partial_{\underline{u}} \cdot \underline{g} - \partial_{\underline{v}} \cdot \underline{h} = 0, \\
&\partial_{u_0} \underline{g} + \partial_{\underline{u}} g_0 + \partial_{v_0} \underline{h} - \partial_{\underline{v}} h_0 = 0, \\
&\partial_{u_0} \underline{h} + \partial_{\underline{v}} g_0 + \partial_{\underline{u}} h_0 - \partial_{v_0} \underline{g} = 0, \\
&\partial_{\underline{u}} \times \underline{g} - \partial_{\underline{v}} \times \underline{h} = 0, \\
&\partial_{\underline{u}} \times \underline{h} + \partial_{\underline{v}} \times \underline{g} = 0.
\end{aligned} \tag{15}$$

One example of the use of biaxial quaternion analysis is the proof of the following vector calculus identity in the octonionic case.

Lemma 4.3. *Let the coordinates of* $f\colon \mathbb{R}^8 \to \mathbb{O}$ *and* $g\colon \mathbb{R}^8 \to \mathbb{O}$ *have partial derivatives. Then*

$$D_{\underline{x}} \cdot (\underline{f} \times \underline{g}) = (D_{\underline{x}} \times \underline{f}) \cdot \underline{g} - \underline{f} \cdot (D_{\underline{x}} \times \underline{g}).$$

Proof. We use quaternion decompositions

$$\begin{aligned}
D_{\underline{x}} &= \partial_{\underline{u}} + \partial_{v_0} e_4 + \partial_{\underline{v}} \circ e_4, \\
\underline{f} &= \underline{f_1} + F_0 e_4 + \underline{F_1} \circ e_4, \\
\underline{g} &= \underline{g_1} + G_0 e_4 + \underline{G_1} \circ e_4.
\end{aligned}$$

On the left-hand side we apply Lemma 2.3 to the cross product $\underline{f} \times \underline{g}$, and use the classical vector calculus identity

$$\nabla \cdot (\underline{u} \times \underline{v}) = (\nabla \times \underline{u}) \cdot \underline{v} - \underline{u} \cdot (\nabla \times \underline{v})$$

for $\underline{u}, \underline{v}\colon \mathbb{R}^3 \to \mathbb{R}^3$:

$$\begin{aligned}
D_{\underline{x}} \cdot (\underline{f} \times \underline{g}) &= \big(\partial_{\underline{u}} + \partial_{v_0} e_4 + \partial_{\underline{v}} \circ e_4\big) \cdot \big(F_0 \underline{G_1} - \underline{F_1} G_0 + \underline{f_1} \times \underline{g_1} - \underline{F_1} \times \underline{G_1} \\
&\quad + (\underline{F_1} \cdot \underline{g_1} - \underline{f_1} \cdot \underline{G_1}) e_4 \\
&\quad + (\underline{f_1} G_0 - F_0 \underline{g_1} - \underline{f_1} \times \underline{G_1} - \underline{F_1} \times \underline{g_1}) \circ e_4\big) \\
&= \partial_{\underline{u}} \cdot (F_0 \underline{G_1}) - \partial_{\underline{u}} \cdot (\underline{F_1} G_0) + \partial_{\underline{u}} \cdot (\underline{f_1} \times \underline{g_1}) - \partial_{\underline{u}} \cdot (\underline{F_1} \times \underline{G_1}) \\
&\quad + \partial_{v_0} (\underline{F_1} \cdot \underline{g_1}) - \partial_{v_0} (\underline{f_1} \cdot \underline{G_1}) \\
&\quad + \partial_{\underline{v}} \cdot (\underline{f_1} G_0) - \partial_{\underline{v}} \cdot (F_0 \underline{g_1}) - \partial_{\underline{v}} \cdot (\underline{f_1} \times \underline{G_1}) - \partial_{\underline{v}} \cdot (\underline{F_1} \times \underline{g_1}) \\
&= (\partial_{\underline{u}} F_0) \cdot \underline{G_1} + F_0 (\partial_{\underline{u}} \cdot \underline{G_1}) - (\partial_{\underline{u}} \cdot \underline{F_1}) G_0 - \underline{F_1} \cdot (\partial_{\underline{u}} G_0) \\
&\quad + (\partial_{\underline{u}} \times \underline{f_1}) \cdot \underline{g_1} - \underline{f_1} \cdot (\partial_{\underline{u}} \times \underline{g_1}) - (\partial_{\underline{u}} \times \underline{F_1}) \cdot \underline{G_1} + \underline{F_1} \cdot (\partial_{\underline{u}} \times \underline{G_1}) \\
&\quad + (\partial_{v_0} \underline{F_1}) \cdot \underline{g_1} + \underline{F_1} \cdot (\partial_{v_0} \underline{g_1}) - (\partial_{v_0} \underline{f_1}) \cdot \underline{G_1} - \underline{f_1} \cdot (\partial_{v_0} \underline{G_1}) \\
&\quad + (\partial_{\underline{v}} \cdot \underline{f_1}) G_0 + \underline{f_1} \cdot (\partial_{\underline{v}} G_0) - (\partial_{\underline{v}} F_0) \cdot \underline{g_1} - F_0 (\partial_{\underline{v}} \cdot \underline{g_1}) \\
&\quad - (\partial_{\underline{v}} \times \underline{f_1}) \cdot \underline{G_1} + \underline{f_1} \cdot (\partial_{\underline{v}} \times \underline{G_1}) - (\partial_{\underline{v}} \times \underline{F_1}) \cdot \underline{g_1} + \underline{F_1} \cdot (\partial_{\underline{v}} \times \underline{g_1}).
\end{aligned}$$

On the right-hand side we apply Lemma 2.3 to the rotors $D_{\underline{x}} \times \underline{f}$ and $D_{\underline{x}} \times \underline{g}$:

$$\begin{aligned}
(D_{\underline{x}} \times \underline{f}) \cdot \underline{g} &= \big(\partial_{v_0} \underline{F_1} - \partial_{\underline{v}} F_0 + \partial_{\underline{u}} \times \underline{f_1} - \partial_{\underline{v}} \times \underline{F_1} \\
&\quad + (\partial_{\underline{v}} \cdot \underline{f_1} - \partial_{\underline{u}} \cdot \underline{F_1}) e_4 \\
&\quad + (\partial_{\underline{u}} F_0 - \partial_{v_0} \underline{f_1} - \partial_{\underline{u}} \times \underline{F_1} - \partial_{\underline{v}} \times \underline{f_1}) \circ e_4\big) \cdot \\
&\quad \big(\underline{g_1} + G_0 e_4 + \underline{G_1} \circ e_4\big) \\
&= (\partial_{v_0} \underline{F_1}) \cdot \underline{g_1} - (\partial_{\underline{v}} F_0) \cdot \underline{g_1} + (\partial_{\underline{u}} \times \underline{f_1}) \cdot \underline{g_1} - (\partial_{\underline{v}} \times \underline{F_1}) \cdot \underline{g_1}
\end{aligned}$$

$$+ (\partial_{\underline{v}} \cdot \underline{f_1})G_0 - (\partial_{\underline{u}} \cdot \underline{F_1})G_0$$
$$+ (\partial_{\underline{u}} F_0) \cdot \underline{G_1} - (\partial_{v_0} \underline{f_1}) \cdot \underline{G_1} - (\partial_{\underline{u}} \times \underline{F_1}) \cdot \underline{G_1} - (\partial_{\underline{v}} \times \underline{f_1}) \cdot \underline{G_1},$$

and

$$\begin{aligned}
\underline{f} \cdot (D_{\underline{x}} \times \underline{g}) = \big(&\underline{f_1} + F_0 e_4 + \underline{F_1} \circ e_4\big) \cdot \\
&\big(\partial_{v_0} \underline{G_1} - \partial_{\underline{v}} G_0 + \partial_{\underline{u}} \times \underline{g_1} - \partial_{\underline{v}} \times \underline{G_1} \\
&+ (\partial_{\underline{v}} \cdot \underline{g_1} - \partial_{\underline{u}} \cdot \underline{G_1}) e_4 \\
&+ (\partial_{\underline{u}} G_0 - \partial_{v_0} \underline{g_1} - \partial_{\underline{u}} \times \underline{G_1} - \partial_{\underline{v}} \times \underline{g_1}) \circ e_4\big) \\
= &\underline{f_1} \cdot (\partial_{v_0} \underline{G_1}) - \underline{f_1} \cdot (\partial_{\underline{v}} G_0) + \underline{f_1} \cdot (\partial_{\underline{u}} \times \underline{g_1}) - \underline{f_1} \cdot (\partial_{\underline{v}} \times \underline{G_1}) \\
&+ F_0(\partial_{\underline{v}} \cdot \underline{g_1}) - F_0(\partial_{\underline{u}} \cdot \underline{G_1}) \\
&+ \underline{F_1} \cdot (\partial_{\underline{u}} G_0) - \underline{F_1} \cdot (\partial_{v_0} \underline{g_1}) - \underline{F_1} \cdot (\partial_{\underline{u}} \times \underline{G_1}) - \underline{F_1} \cdot (\partial_{\underline{v}} \times \underline{g_1}).
\end{aligned}$$

□

Remark 4.4. (The set of Regular functions is not a module) In quaternion analysis $\partial_u g = 0$ implies $\partial_u(g \circ a) = 0$ for all $a \in \mathbb{H}$ (see Lemma 4.5). The same does not hold in octonion analysis. For example, define $g\colon \mathbb{H} \to \mathbb{H}$, $g(x) = x_1 - x_2 e_3$. Then $D_x g = e_1 - e_2 e_3 = 0$, but $D_x(g \circ e_4) = D_x(x_1 e_4 - x_2 e_7) = e_1 e_4 - e_2 e_7 = 2e_5$.

For quaternion functions we have the product rules (16) and (17) for the Cauchy–Riemann operator. Remark 4.4 suggests that we do not have any kind of a non-trivial product rule for octonion valued functions. In practice, one way to compute $D_x(fg)$ for octonion valued functions is to use biaxial quaternion analysis, and then to apply (16)–(24).

Lemma 4.5. *([3, Thm 1.3.2]) Let the coordinates of $f\colon \mathbb{H} \to \mathbb{H}$ and $g\colon \mathbb{H} \to \mathbb{H}$ have partial derivatives. Then*

$$\partial_{\underline{u}}(f \circ g) = (\partial_{\underline{u}} f) \circ g + \overline{f} \circ (\partial_{\underline{u}} g) - 2(\underline{f} \cdot \partial_{\underline{u}})g \tag{16}$$

and

$$(f \circ g)\partial_{\underline{u}} = (f\partial_{\underline{u}}) \circ \overline{g} + f \circ (g\partial_{\underline{u}}) - 2(\underline{g} \cdot \partial_{\underline{u}})f. \tag{17}$$

Here, $(\underline{f} \cdot \partial_{\underline{u}})g = \sum_{i=1}^{3} f_i \partial_{x_i} g.$

Corollary 4.6. *Let the coordinates of $f\colon \mathbb{H} \to \mathbb{H}$ and $g\colon \mathbb{H} \to \mathbb{H}$ have partial derivatives. Then*

$$\partial_{\underline{u}}((f \circ e_4) \circ g) = [(f\partial_{\underline{u}}) \circ g + f \circ (\overline{g}\partial_{\underline{u}}) + 2(\underline{g} \cdot \partial_{\underline{u}})f] \circ e_4 \tag{18}$$
$$\partial_{\underline{u}}(f \circ (g \circ e_4)) = [(g\partial_{\underline{u}}) \circ \overline{f} + g \circ (f\partial_{\underline{u}}) - 2(\underline{f} \cdot \partial_{\underline{u}})g] \circ e_4 \tag{19}$$
$$\partial_{\underline{u}}((f \circ e_4) \circ (g \circ e_4)) = -(\partial_{\underline{u}}\overline{g}) \circ f - g \circ (\partial_{\underline{u}} f) - 2(\underline{g} \cdot \partial_{\underline{u}})f \tag{20}$$
$$(\partial_{\underline{v}} \circ e_4)(f \circ g) = [(\partial_{\underline{v}}\overline{g}) \circ \overline{f} + g \circ (\partial_{\underline{v}}\overline{f}) + 2(\underline{g} \cdot \partial_{\underline{v}})\overline{f}] \circ e_4 \tag{21}$$
$$(\partial_{\underline{v}} \circ e_4)((f \circ e_4) \circ g) = -(g\partial_{\underline{v}}) \circ f - g(\overline{f}\partial_{\underline{v}}) - 2(\underline{f} \cdot \partial_{\underline{v}})g \tag{22}$$
$$(\partial_{\underline{v}} \circ e_4)(f \circ (g \circ e_4)) = -(\overline{f}\partial_{\underline{v}}) \circ g - \overline{f} \circ (\overline{g}\partial_{\underline{v}}) - 2(\underline{g} \cdot \partial_{\underline{v}})\overline{f} \tag{23}$$
$$(\partial_{\underline{v}} \circ e_4)((f \circ e_4) \circ (g \circ e_4)) = [-(\partial_{\underline{v}}\overline{f})g - f(\partial_{\underline{v}} g) - 2(\underline{f} \cdot \partial_{\underline{v}})g] \circ e_4 \tag{24}$$

Proof. Apply Lemmas 2.1 and 4.5, and use the fact $\overline{fg} = \overline{g}\,\overline{f}$. □

5. Function Classes in Clifford Analysis

In this last section, we study the classes of left-, B-, and R-regular functions using Clifford analysis. We begin with the following algebraic lemma.

Lemma 5.1. *Let I be the primitive idempotent*

$$I = \frac{1}{16}(1 + We_{12\cdots 7})(1 - e_{12\cdots 7}),$$

where $W = e_{123} + e_{145} + e_{176} + e_{246} + e_{257} + e_{347} + e_{365}$, and let $a = a_0 + \underline{a}$ and $b = b_0 + \underline{b} \in \mathcal{C}\ell_{0,7}$ be paravectors. Then

$$16[abI]_0 = a_0b_0 - \underline{a}\cdot\underline{b}, \tag{25}$$

$$16[abI]_1 = a_0\underline{b} + \underline{a}b_0 - [(\underline{a}\wedge\underline{b})W]_1, \tag{26}$$

$$16[abI]_2 = \underline{a}\wedge\underline{b} - [(a_0\underline{b} + \underline{a}b_0)W]_2 + [(\underline{a}\wedge\underline{b})We_{1\cdots 7}]_2, \tag{27}$$

$$16[abI]_3 = -(a_0b_0 - \underline{a}\cdot\underline{b})W + [(a_0\underline{b} + \underline{a}b_0)We_{1\cdots 7}]_3 - [(\underline{a}\wedge\underline{b})W]_3, \tag{28}$$

$$16[abI]_4 = (a_0b_0 - \underline{a}\cdot\underline{b})We_{1\cdots 7} - [(a_0\underline{b} + \underline{a}b_0)W]_4 + [(\underline{a}\wedge\underline{b})We_{1\cdots 7}]_4, \tag{29}$$

$$16[abI]_5 = [(a_0\underline{b} + \underline{a}b_0)We_{1\cdots 7}]_5 - [(\underline{a}\wedge\underline{b})W]_5 - (\underline{a}\wedge\underline{b})e_{1\cdots 7}, \tag{30}$$

$$16[abI]_6 = -(a_0\underline{b} + \underline{a}b_0)e_{1\cdots 7} + [(\underline{a}\wedge\underline{b})We_{1\cdots 7}]_6, \tag{31}$$

$$16[abI]_7 = -(a_0b_0 - \underline{a}\cdot\underline{b})e_{1\cdots 7}, \tag{32}$$

and

$$[abI]_k = 0 \;\Leftrightarrow\; [abI]_{7-k} = 0, \quad k = 0, 1, \ldots, 7. \tag{33}$$

If $[abI]_0 = 0$, then the conditions $[abI]_j = 0$, $j = 2, 3, 4, 5$, are pairwise equivalent. In particular, if $[abI]_{0,1,2} = 0$, then $abI = 0$.

Proof. Write the real part and the 1- and 2-vector parts of ab using (3), and expand the definition (4) of I using the fact $e_{12\cdots 7}^2 = 1$:

$$ab = (a_0b_0 - \underline{a}\cdot\underline{b}) + (a_0\underline{b} + \underline{a}b_0) + \underline{a}\wedge\underline{b},$$
$$16I = 1 - W + We_{12\cdots 7} - e_{12\cdots 7}.$$

Here, W is a 3-vector and $We_{12\cdots 7}$ is a 4-vector. Then, for example, $\underline{a}W$ only contains 2- and 4-vector parts, and therefore $[\underline{a}W]_3 = 0$. This kind of reasoning implies (25)–(32).

Now, (33) follows from the facts that for any $c \in \mathcal{C}\ell_{0,7}$,

$$c = 0 \Leftrightarrow ce_{12\cdots 7} = 0, \quad \text{and}$$
$$[c]_k e_{12\cdots 7} = [ce_{12\cdots 7}]_{7-k}, \quad k = 0, 1, \ldots, 7.$$

To prove the last claim, it is now enough to show that in the case $[abI]_0 = 0$, $[abI]_2 = 0$ if and only if $[abI]_3 = 0$. This can be seen by computing

$$\begin{aligned}16[abI]_2 = {} & (a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2 + a_4b_5 - a_5b_4 \\ & - a_6b_7 + a_7b_6)(e_{23} + e_{45} - e_{67}) \\ & + (a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1 + a_4b_6 \\ & + a_5b_7 - a_6b_4 - a_7b_5)(-e_{13} + e_{46} + e_{57}) \\ & + (a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0 \\ & + a_4b_7 - a_5b_6 + a_6b_5 - a_7b_4)(e_{12} + e_{47} - e_{56})\end{aligned}$$

$$
\begin{aligned}
&+ (a_0b_4 - a_1b_5 - a_2b_6 - a_3b_7\\
&+ a_4b_0 + a_5b_1 + a_6b_2 + a_7b_3)(-e_{15} - e_{26} - e_{37})\\
&+ (a_0b_5 + a_1b_4 - a_2b_7 + a_3b_6\\
&- a_4b_1 + a_5b_0 - a_6b_3 + a_7b_2)(e_{14} - e_{27}e_{36})\\
&+ (a_0b_6 + a_1b_7 + a_2b_4 - a_3b_5\\
&- a_4b_2 + a_5b_3 + a_6b_0 - a_7b_1)(e_{17} + e_{24} - e_{35})\\
&+ (a_0b_7 - a_1b_6 + a_2b_5 + a_3b_4 - a_4b_3\\
&- a_5b_2 + a_6b_1 + a_7b_0)(-e_{16} + e_{25} + e_{34}),
\end{aligned}
$$

and in the case $[abI]_0 = 0$,

$$
\begin{aligned}
16[abI]_3 = &(a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2 + a_4b_5 - a_5b_4\\
&- a_6b_7 + a_7b_6)(e_{247} - e_{256} - e_{346} - e_{357})\\
&+ (a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1 + a_4b_6\\
&+ a_5b_7 - a_6b_4 - a_7b_5)(-e_{147} + e_{156} + e_{345} - e_{367})\\
&+ (a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0\\
&+ a_4b_7 - a_5b_6 + a_6b_5 - a_7b_4)(e_{146} + e_{157} - e_{245} + e_{267})\\
&+ (a_0b_4 - a_1b_5 - a_2b_6 - a_3b_7 + a_4b_0\\
&+ a_5b_1 + a_6b_2 + a_7b_3)(e_{127} - e_{136} + e_{235} - e_{567})\\
&+ (a_0b_5 + a_1b_4 - a_2b_7 + a_3b_6 - a_4b_1\\
&+ a_5b_0 - a_6b_3 + a_7b_2)(-e_{126} - e_{137} - e_{234} + e_{467})\\
&+ (a_0b_6 + a_1b_7 + a_2b_4 - a_3b_5 - a_4b_2\\
&+ a_5b_3 + a_6b_0 - a_7b_1)(e_{125} + e_{134} - e_{237} - e_{457})\\
&+ (a_0b_7 - a_1b_6 + a_2b_5 + a_3b_4 - a_4b_3\\
&- a_5b_2 + a_6b_1 + a_7b_0)(-e_{124} + e_{135} + e_{236} + e_{456}).
\end{aligned}
$$

□

We infer that left-, B-, and R-regularity can be studied by considering paravector-spinor valued functions fI.

Theorem 5.2. *Suppose that $f\colon \mathbb{R}^8 \to \mathbb{R}^8$ is a paravector valued function such that the coordinate functions have partial derivatives.*

(a) *f is left-regular if and only if*

$$[\partial_x fI]_j = 0 \text{ for } j = 0, 1. \tag{34}$$

(b) *f is B-regular if and only if*

$$[\partial_x fI]_j = 0 \text{ for } j = 0, 1, \quad \text{and} \quad [\partial_x fW]_1 = 0. \tag{35}$$

Proof. (a) follows using Lemma 2.7:

$$D_x f = 16[\partial_x fI]_0 + 16[\partial_x fI]_1.$$

(b) From (3) and (26), we obtain

$$\begin{aligned}[\partial_x f W]_1 &= [(\partial_{x_0} f_0 - \partial_{\underline{x}} \cdot \underline{f})W]_1 + [(\partial_{x_0} \underline{f} + \partial_{\underline{x}} f_0)W]_1 + [(\partial_{\underline{x}} \wedge \underline{f})W]_1 \\ &= [(\partial_{\underline{x}} \wedge \underline{f})W]_1 \\ &= -16[\partial_x f I]_1 + \partial_{x_0} \underline{f} + D_{\underline{x}} f_0.\end{aligned}$$

Since $D_{\underline{x}} \times \underline{f} = -[(\partial_{\underline{x}} \wedge \underline{f})W]_1$ (Lemma 2.5), the claim now follows from (a) and Propositions 3.1–3.2.

□

Remark 5.3. If $\partial_x f = 0$, then (trivially) $[\partial_x f I]_j = 0$ for all $j = 0, 1, \ldots, 7$. The converse does not hold. This follows from the fact that the equation $aI = 0$ does not have a unique solution $a = 0$ in the Clifford algebra. Hence, paravector spinor valued solutions of the Cauchy–Riemann equations form a bigger function class, and the class of R-regular solutions is

$$\begin{aligned}\mathcal{M}_R \subsetneq \{f : \partial_x f I = 0\} &= \{f : [\partial_x f I]_j = 0,\ j = 0, 1, \ldots, 7\} \\ &= \{f : [\partial_x f I]_j = 0,\ j = 0, 1, 2\}.\end{aligned}$$

The equality of the latter two function classes follows from Lemma 5.1. An example showing that the inclusion is strict: if $f = x_2 e_1 - x_7 e_4$, then $\partial_x f = e_4 e_7 - e_1 e_2$, but $[\partial_x f I]_j = 0$ for $j = 0, 1, 2$.

6. Conclusion

The key idea of this paper is to study differences between octonion and Clifford analyses. This leads us to observe the fundamental difference between octonion regular and Clifford monogenic functions. The structure of octonion regular functions is studied by comparing left-, right-, B-, and R-regular functions. The existence of these classes is a consequence of different algebraic properties of the algebras. In the heart of octonion analysis is the study of the properties of these function classes and their relations, which distinguishes it essentially from Clifford analysis.

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Janne Kauhanen
Mathematics
Tampere University
33014 Tampere
Finland
e-mail: `janne.kauhanen@tuni.fi`

Heikki Orelma
Civil Engineering
Tampere University
33014 Tampere
Finland
e-mail: `heikki.orelma@tuni.fi`

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