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Menon-type identities again: A note on a paper by Li, Kim and Qiao

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Abstract

We give common generalizations of the Menon-type identities by Sivaramakrishnan (1969) and Li, Kim, Qiao (2019). Our general identities involve arithmetic functions of several variables, and also contain, as special cases, identities for gcd-sum type functions. We point out a new Menon-type identity concerning the lcm function. We present a simple character free approach for the proof.

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1 Introduction

Menon's classical identity [13] states that for every $n \in \mathbb{N} := \{1, 2, \ldots\}$,

$$M(n) := \sum_{\substack{a=1 \\ (a,n)=1}}^{n} (a-1, n) = \varphi(n)\tau(n), \tag{1.1}$$

where (a, n) stands for the greatest common divisor of a and n, $\varphi(n)$ is Euler's totient function and $\tau(n) = \sum_{d|n} 1$ is the divisor function.

Menon [13] proved this identity by three distinct methods, the first one being based on the Cauchy-Frobenius-Burnside lemma on group actions. This method was used later by Sury [16], Tóth [18], Li and Kim [10], and other authors to derive different generalizations and analogs of (1.1). Number theoretic methods were also applied in several papers to deduce various Menon-type identities. See, e.g., [4, 6, 7, 9, 11, 12, 20, 21, 22, 24, 25].

It is less known and is not considered in the above mentioned papers the following old generalization, due to Sivaramakrishnan [15], in a slightly different form:

$$M(m,n,t) := \sum_{\substack{a=1\\(a,m)=1}}^{t} (a-1,n) = \frac{t\,\varphi(m)\tau(n)}{m} \prod_{p^\nu || n_1} \left(1 - \frac{\nu}{(\nu+1)p}\right), \tag{1.2}$$

where $m, n, t \in \mathbb{N}$ such that $m \mid t$, $n \mid t$ and $n_1 = \max\{d \in \mathbb{N} : d \mid n, (d,m) = 1\}$. If $m = n = t$, then $M(n,n,n) = M(n)$, that is, (1.2) reduces to (1.1). However, if $n \mid m$ and $t = m$, then it follows from (1.2) that

$$\sum_{\substack{a=1\\(a,m)=1}}^{m} (a-1,n) = \varphi(m)\tau(n),$$

which was recently obtained by Jafari and Madadi [8, Cor. 2.2], using group theoretic arguments, without referring to the paper [15]. It was pointed out by Sivaramakrishnan [15] that if $t = [m,n]$, the least common multiple of m and n, then $M(m,n,[m,n])$ is a multiplicative function of two variables.

In a quite recent paper, Li, Kim and Qiao [12, Th. 2.5] proved that for any integers $n \geq 1$, $k \geq 0$, $\ell \geq 1$ one has

$$\sum_{\substack{1\le a,b_1,\ldots,b_k\le n\\(a,n)=1}} (a^\ell - 1, b_1, \ldots, b_k, n) = \varphi(n)(\mathrm{id}_k * C^{(\ell)})(n), \tag{1.3}$$

where $*$ denotes the Dirichlet convolution, $\mathrm{id}_k(n) = n^k$ and $C^{(\ell)}(n)$ is the number of solutions of the congruence $x^\ell \equiv 1 \pmod{n}$ with $(x,n) = 1$. Note that the condition $(x,n) = 1$ can be omitted here. For the proof they used properties of characters of finite abelian groups. The case $k = 0$ recovers certain identities given by the second author [18]. If $k = 0$ and $\ell = 1$, then (1.3) reduces to (1.1).

The sum $M(n)$ is related to the gcd-sum function, also known as Pillai's arithmetical function, given by

$$G(n) := \sum_{a=1}^{n} (a,n) = n \sum_{d \mid n} \frac{\varphi(d)}{d} \quad (n \in \mathbb{N}). \tag{1.4}$$

A large number of different generalizations and analogs of the function $G(n)$ is presented in the literature. See, e.g., [5, 17].

It is the goal of this paper to give common generalizations of the identities (1.2) and (1.3), and to present a simple character free approach for the proof. Our general identity, included in Theorem 3.1, involves arithmetic functions of several variables, and also contains, as a special case, identities for gcd-sum type functions, such as identity (1.4). The identity of Theorem 3.7 is concerning arithmetic functions of a single variable.

We point out the following new Menon-type identity, which is another special case of our results. See Theorem 4.1 and Corollary 4.2. If $n \in \mathbb{N}$, then

$$\sum_{\substack{1\le a,b\le n\\(a,n)=(b,n)=1}} [(a-1,n),(b-1,n)] = \varphi(n)^2 \prod_{p^\nu || n} \left(1 + 2\nu - \frac{p^\nu - 1}{p^{\nu-1}(p-1)^2}\right). \tag{1.5}$$

Note that identity (1.2) was generalized by Sita Ramaiah [14, Th. 9.1] in another way, namely in terms of regular convolutions. Our results can further be generalized to regular convolutions and to k-reduced residue systems. For the sake of brevity, we do not present the details. For appropriate material we refer to [2, 3, 14].

2 Preliminaries

2.1 Arithmetic functions of several variables

Let $f, g : \mathbb{N}^k \to \mathbb{C}$ be arithmetic functions of k variables. Their Dirichlet convolution is defined as

$$(f *_k g)(n_1, \ldots, n_k) = \sum_{d_1|n_1, \ldots, d_k|n_k} f(d_1, \ldots, d_k) g(n_1/d_1, \ldots, n_k/d_k).$$

In the case $k = 1$ we write simply $f *_1 g = f * g$. The identity under $*_k$ is

$$\delta_k(n_1, \ldots, n_k) = \delta(n_1) \cdots \delta(n_k),$$

where $\delta(1) = 1$ and $\delta(n) = 0$ for $n \neq 1$. An arithmetic function f of k variables possesses an inverse under $*_k$ if and only if $f(1, \ldots, 1) \neq 0$. Let $\zeta_k(n_1, \ldots, n_k)$ be defined as $\zeta_k(n_1, \ldots, n_k) = 1$ for all $n_1, \ldots, n_k \in \mathbb{N}$. Its Dirichlet inverse is the Möbius function μ_k of k variables given as

$$\mu_k(n_1, \ldots, n_k) = \mu(n_1) \cdots \mu(n_k),$$

where μ is the classical Möbius function of one variable.

Let g be an arithmetic function of one variable. Then the principal function $\mathrm{Pr}_k(g)$ associated with g is the arithmetic function of k variables defined as

$$\mathrm{Pr}_k(g)(n_1, \ldots, n_k) = \begin{cases} g(n), & \text{if } n_1 = \cdots = n_k = n, \\ 0, & \text{otherwise.} \end{cases}$$

(See Vaidyanathaswamy [23].) Let f be the arithmetic function of k variables defined by

$$f(n_1, \ldots, n_k) = g((n_1, \ldots, n_k)),$$

having the gcd on the right-hand side. Then

$$f(n_1, \ldots, n_k) = \sum_{d|(n_1, \ldots, n_k)} (\mu * g)(d) = \sum_{d_1|n_1, \ldots, d_k|n_k} \mathrm{Pr}_k(\mu * g)(d_1, \ldots, d_k),$$

that is,

$$f = \mathrm{Pr}_k(\mu * g) *_k \zeta_k,$$

which means that

$$\mathrm{Pr}_k(\mu * g) = \mu_k *_k f. \tag{2.1}$$

An arithmetic function f of k variables is said to be multiplicative if $f(1, \ldots, 1) = 1$ and

$$f(m_1 n_1, \ldots, m_k n_k) = f(m_1, \ldots, m_k) f(n_1, \ldots, n_k)$$

for all positive integers $m_1, \ldots, m_k$ and $n_1, \ldots, n_k$ with $(m_1 \cdots m_k, n_1 \cdots n_k) = 1$. For example, the gcd function $(n_1, \ldots, n_k)$ and the lcm function $[n_1, \ldots, n_k]$ are multiplicative. If f and g are multiplicative functions of k variables, then their Dirichlet convolution $f *_k g$ is also multiplicative. See [19, 23].

2.2 Number of solutions of congruences

For a given polynomial $P \in \mathbb{Z}[x]$ let $N_P(n)$ denote the number of solutions $x \pmod{n}$ of the congruence $P(x) \equiv 0 \pmod{n}$ and let $\widehat{N}_P(n)$ be the number of solutions $x \pmod{n}$ such that $(x,n) = 1$. Furthermore, for a fixed integer $s \in \mathbb{N}$, let $\widehat{N}_P(n,s)$ be the number of solutions $x \pmod{n}$ such that $(x,n,s) = 1$.

The functions $N_P(n)$, $\widehat{N}_P(n)$ and $\widehat{N}_P(n,s)$ are multiplicative in n, which are direct consequences of the Chinese remainder theorem.

It is easy to see that if $P(x) = a_0 + a_1 x + \cdots + a_k x^k$ and $(a_0, n) = 1$, then $N_P(n) = \widehat{N}_P(n) = \widehat{N}_P(n,s)$. This applies, in particular, to $P(x) = -1 + x^\ell$. See the Introduction regarding the notation $C^{(\ell)}(n)$, used by Li, Kim and Qiao [12].

2.3 Lemma

We will need the next lemma.

Lemma 2.1. *Let $d, r, s \in \mathbb{N}$, $x \in \mathbb{Z}$ such that $d \mid r$, $s \mid r$. Then*

$$\sum_{\substack{1 \le a \le r \\ (a,s)=1 \\ a \equiv x \pmod{d}}} 1 = \begin{cases} \dfrac{r}{d} \displaystyle\prod_{\substack{p \mid s \\ p \nmid d}} \left(1 - \frac{1}{p}\right), & \text{if } (d,s,x) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

In the special case $r = s$ this is known in the literature, usually proved by the inclusion-exclusion principle. See, e.g., [1, Th. 5.32]. Also see [7, Lemma] for a generalization in terms of regular convolutions. Here we use a different approach, similar to the proof of [14, Th. 9.1] and to the proofs of our previous papers [21, 22].

Proof of Lemma 2.1. Let A denote the given sum. If $(d,s,x) \ne 1$, then the sum A is empty and equal to zero. Indeed, if we assume $p \mid (d,s,x)$ for some prime p, then $p \mid a \equiv x \pmod{d}$. Hence $p \mid (a,s) = 1$, a contradiction.

Assume now that $(d,s,x) = 1$. By using the property of the Möbius function, the given sum can be written as

$$A = \sum_{\substack{1 \le a \le r \\ a \equiv x \pmod{d}}} \sum_{\delta \mid (a,s)} \mu(\delta) = \sum_{\delta \mid s} \mu(\delta) \sum_{\substack{j=1 \\ \delta j \equiv x \pmod{d}}}^{r/\delta} 1. \tag{2.2}$$

Let $\delta \mid s$ be fixed. The linear congruence $\delta j \equiv x \pmod{d}$ has solutions in j if and only if $(\delta, d) \mid x$. Here $(\delta, d) \mid \delta$ and $\delta \mid s$, hence $(\delta, d) \mid s$. Also, $(\delta, d) \mid d$. If $(\delta, d) \mid x$ holds, then $(\delta, d) \mid (d,s,x) = 1$, therefore $(\delta, d) = 1$. We deduce that the above congruence has

$$N = \frac{r}{d\delta}$$

solutions $\pmod{r/\delta}$ and the last sum in (2.2) is N. This gives

$$A = \frac{r}{d} \sum_{\substack{\delta \mid s \\ (\delta,d)=1}} \frac{\mu(\delta)}{\delta} = \frac{r}{d} \prod_{\substack{p \mid s \\ p \nmid d}} \left(1 - \frac{1}{p}\right).$$

□

3 Main results

Assume that
(1) $k, \ell \geq 0$ are fixed integers, not both zero;
(2) $m_i, r_i, s_i, n_j, t_j \in \mathbb{N}$ are integers such that $m_i \mid r_i$, $s_i \mid r_i$, $n_j \mid t_j$ $(1 \leq i \leq k, 1 \leq j \leq \ell)$;
(3) $f : \mathbb{N}^{k+\ell} \to \mathbb{C}$ is an arbitrary arithmetic function of $k + \ell$ variables;
(4) $P_i, Q_j \in \mathbb{Z}[x]$ are arbitrary polynomials $(1 \leq i \leq k, 1 \leq j \leq \ell)$.
Consider the sum

$$S := \sum_{\substack{1\leq a_i\leq r_i \\ (a_i,s_i)=1 \\ 1\leq i\leq k}} \sum_{\substack{1\leq b_j\leq t_j \\ 1\leq j\leq \ell}} f((P_1(a_1), m_1), \ldots, (P_k(a_k), m_k), (Q_1(b_1), n_1), \ldots, (Q_\ell(b_\ell), n_\ell)),$$

where $(P_i(a_i), m_i)$ and $(Q_j(b_j), n_j)$ represent the gcd's of the corresponding values $(1 \leq i \leq k$, $1 \leq j \leq \ell)$.

Theorem 3.1. *Under the above assumptions (1)-(4) we have*

$$S = r_1 \cdots r_k t_1 \cdots t_\ell \sum_{\substack{d_i|m_i \\ 1\leq i\leq k}} \sum_{\substack{e_j|n_j \\ 1\leq j\leq \ell}} \frac{(\mu_{k+\ell} *_{k+\ell} f)(d_1, \ldots, d_k, e_1, \ldots, e_\ell)}{d_1 \cdots d_k e_1 \cdots e_\ell}$$

$$\times \left(\prod_{1\leq i\leq k} \widehat{N}_{P_i}(d_i, s_i)\beta(s_i, d_i) \right) \left(\prod_{1\leq j\leq \ell} N_{Q_j}(e_j) \right),$$

where $\widehat{N}_{P_i}(d_i, s_i)$ and $N_{Q_j}(e_j)$ $(1 \leq i \leq k, 1 \leq j \leq \ell)$ are defined in Section 2.2, and

$$\beta(s_i, d_i) = \prod_{\substack{p|s_i \\ p\nmid d_i}} \left(1 - \frac{1}{p}\right).$$

Corollary 3.2. *If $\ell = 0$, then Theorem 3.1 gives the pure Menon-type identity*

$$S = r_1 \cdots r_k \sum_{\substack{d_i|m_i \\ 1\leq i\leq k}} \frac{(\mu_k *_k f)(d_1, \ldots, d_k)}{d_1 \cdots d_k} \left(\prod_{1\leq i\leq k} \widehat{N}_{P_i}(d_i, s_i)\beta(s_i, d_i) \right), \tag{3.1}$$

and if $k = 0$, it gives the pure gcd-sum identity

$$S = t_1 \cdots t_\ell \sum_{\substack{e_j|n_j \\ 1\leq j\leq \ell}} \frac{(\mu_\ell *_\ell f)(e_1, \ldots, e_\ell)}{e_1 \cdots e_\ell} \left(\prod_{1\leq j\leq \ell} N_{Q_j}(e_j) \right).$$

If $k = 1$, $f(n) = n$ $(n \in \mathbb{N})$ and $P(x) = x - 1$, then identity (3.1) reduces to

$$\sum_{\substack{a=1 \\ (a,s)=1}}^{r} (a - 1, m) = r \sum_{d|m} \frac{\varphi(d)}{d} \prod_{\substack{p|s \\ p\nmid d}} \left(1 - \frac{1}{p}\right)$$

$$= \frac{r\,\varphi(s)\tau(m)}{s} \prod_{p^\nu || m_1} \left(1 - \frac{\nu}{(\nu+1)p}\right),$$

where $m \mid r$, $s \mid r$ and $m_1 = \max\{d \in \mathbb{N} : d \mid m, (d,s) = 1\}$, which is identity (1.2) (with the corresponding change of notations).

Remark 3.1. Haukkanen and Wang [7] considered systems of polynomials in several variables and a different constraint, namely $(a_1, \ldots, a_k, n) = 1$ in the first sum defining S.

Proof of Theorem 3.1. It is an immediate consequence of the definition of the function μ_k that

$$f(n_1, \ldots, n_k) = \sum_{d_1 \mid n_1, \ldots, d_k \mid n_k} (\mu_k *_k f)(d_1, \ldots, d_k). \tag{3.2}$$

By using (3.2) we have

$$S = \sum_{\substack{1 \le a_i \le r_i \\ (a_i, s_i) = 1 \\ 1 \le i \le k}} \sum_{\substack{1 \le b_j \le t_j \\ 1 \le j \le \ell}} \sum_{\substack{d_i \mid (P_i(a_i), m_i) \\ 1 \le i \le k}} \sum_{\substack{e_j \mid (Q_j(b_j), n_j) \\ 1 \le j \le \ell}} (\mu_{k+\ell} *_{k+\ell} f)(d_1, \ldots, d_k, e_1, \ldots, e_\ell)$$

$$= \sum_{\substack{d_i \mid m_i \\ 1 \le i \le k}} \sum_{\substack{e_j \mid n_j \\ 1 \le j \le \ell}} (\mu_{k+\ell} *_{k+\ell} f)(d_1, \ldots, d_k, e_1, \ldots, e_\ell)$$

$$\times \Big(\prod_{1 \le i \le k} \sum_{\substack{1 \le a_i \le r_i \\ (a_i, s_i) = 1 \\ P_i(a_i) \equiv 0 \pmod{d_i}}} 1 \Big) \Big(\prod_{1 \le j \le \ell} \sum_{\substack{1 \le b_j \le t_j \\ Q_j(b_j) \equiv 0 \pmod{e_j}}} 1 \Big).$$

Now we use Lemma 2.1 to evaluate the sum

$$B_i := \sum_{\substack{1 \le a_i \le r_i \\ (a_i, s_i) = 1 \\ P_i(a_i) \equiv 0 \pmod{d_i}}} 1.$$

For any x such that $(x, d_i, s_i) = 1$ we have

$$\sum_{\substack{1 \le a_i \le r_i \\ (a_i, s_i) = 1 \\ a_i \equiv x \pmod{d_i}}} 1 = \frac{r_i}{d_i} \beta(s_i, d_i),$$

and there are $\widehat{N}_{P_i}(d_i, s_i)$ such values of $x \pmod{d_i}$. Hence,

$$B_i = \frac{r_i}{d_i} \beta(s_i, d_i) \widehat{N}_{P_i}(d_i, s_i).$$

We also have

$$\sum_{\substack{1 \le b_j \le t_j \\ Q_j(b_j) \equiv 0 \pmod{e_j}}} 1 = \frac{t_j}{e_j} N_{Q_j}(e_j).$$

Notice that here r_i/d_i and t_j/e_j are integers for any i, j.
Putting together this gives

$$S = \sum_{\substack{d_i | m_i \\ 1 \le i \le k}} \sum_{\substack{e_j | n_j \\ 1 \le j \le \ell}} (\mu_{k+\ell} *_{k+\ell} f)(d_1, \ldots, d_k, e_1, \ldots, e_\ell)$$

$$\times \Big(\prod_{1 \le i \le k} \widehat{N}_{P_i}(d_i, s_i) \frac{r_i}{d_i} \beta(s_i, d_i) \Big) \Big(\prod_{1 \le j \le \ell} \frac{t_j}{e_j} N_{Q_j}(e_j) \Big)$$

$$= r_1 \cdots r_k t_1 \cdots t_\ell \sum_{\substack{d_i | m_i \\ 1 \le i \le k}} \sum_{\substack{e_j | n_j \\ 1 \le j \le \ell}} \frac{(\mu_{k+\ell} *_{k+\ell} f)(d_1, \ldots, d_k, e_1, \ldots, e_\ell)}{d_1 \cdots d_k e_1 \cdots e_\ell}$$

$$\times \Big(\prod_{1 \le i \le k} \widehat{N}_{P_i}(d_i, s_i) \beta(s_i, d_i) \Big) \Big(\prod_{1 \le j \le \ell} N_{Q_j}(e_j) \Big).$$

□

Corollary 3.3. *Assume that $m_i \mid s_i$ and $s_i \mid r_i$ for any i with $1 \le i \le k$ Then*

$$S = r_1 \frac{\varphi(s_1)}{s_1} \cdots r_k \frac{\varphi(s_k)}{s_k} t_1 \cdots t_\ell \sum_{\substack{d_i | m_i \\ 1 \le i \le k}} \sum_{\substack{e_j | n_j \\ 1 \le j \le \ell}} \frac{(\mu_{k+\ell} *_{k+\ell} f)(d_1, \ldots, d_k, e_1, \ldots, e_\ell)}{\varphi(d_1) \cdots \varphi(d_k) e_1 \cdots e_\ell}$$

$$\times \Big(\prod_{1 \le i \le k} \widehat{N}_{P_i}(d_i) \Big) \Big(\prod_{1 \le j \le \ell} N_{Q_j}(e_j) \Big).$$

Proof. Apply Theorem 3.1. Since $d_i \mid m_i$, we have $d_i \mid s_i$. Hence $\widehat{N}_{P_i}(d_i, s_i) = \widehat{N}_{P_i}(d_i)$ and

$$\beta(s_i, d_i) = \prod_{\substack{p | s_i \\ p \nmid d_i}} \left(1 - \frac{1}{p} \right) = \frac{\varphi(s_i)/s_i}{\varphi(d_i)/d_i}.$$

□

Corollary 3.4. *Assume that $m_i = r_i = s_i$ and $n_j = t_j$ for any i, j $(1 \le i \le k,\ 1 \le j \le \ell)$. Then*

$$S = \varphi(m_1) \cdots \varphi(m_k) n_1 \cdots n_\ell \sum_{\substack{d_i | m_i \\ 1 \le i \le k}} \sum_{\substack{e_j | n_j \\ 1 \le j \le \ell}} \frac{(\mu_{k+\ell} *_{k+\ell} f)(d_1, \ldots, d_k, e_1, \ldots, e_\ell)}{\varphi(d_1) \cdots \varphi(d_k) e_1 \cdots e_\ell} \tag{3.3}$$

$$\times \Big(\prod_{1 \le i \le k} \widehat{N}_{P_i}(d_i) \Big) \Big(\prod_{1 \le j \le \ell} N_{Q_j}(e_j) \Big).$$

Theorem 3.5. *Assume conditions (1)-(4). Furthermore, let $r_i = [m_i, s_i]$, $t_j = n_j$ $(1 \le i \le k,$ $1 \le j \le \ell)$ and let f be a multiplicative function of $k + \ell$ variables. Then the sum*

$$S = S(m_1, \ldots, m_k, s_1, \ldots, s_k, n_1, \ldots, n_\ell)$$

represents a multiplicative function of $2k + \ell$ variables.

Proof. Note that

$$\beta(s_i, d_i) = \sum_{\substack{\delta | s_i \\ (\delta, d_i)=1}} \frac{\mu(\delta)}{\delta} = \sum_{\delta | s_i} \frac{\mu(\delta)}{\delta} h(\delta, d_i), \tag{3.4}$$

where the function of two variables

$$h(\delta, d_i) = \sum_{\substack{c|\delta \\ c|d_i}} \mu(c)$$

is multiplicative, being the convolution of multiplicative functions. Therefore, $\beta(s_i, d_i)$, given by the convolution (3.4) is also multiplicative.

We conclude that S, given in Theorem 3.1 as a convolution of $2k+\ell$ variables of multiplicative functions, is multiplicative, as well. $\square$

Corollary 3.6. *Assume that $m_i = r_i = s_i$ and $n_j = t_j$ for any i, j and f is multiplicative, viewed as a function of $k + \ell$ variables. Then S given by (3.3) is also multiplicative in $m_1, \ldots, m_k, n_1, \ldots, n_\ell$, as a function of $k + \ell$ variables.*

Remark 3.2. Note that in his original paper Menon [13, Lemma] proved that if f is a multiplicative arithmetic function of r variables and $P_i \in \mathbb{Z}[x]$ are polynomials, then the function

$$F(n) := \sum_{a=1}^{n} f((P_1(a), n), \ldots, (P_r(a), n))$$

is multiplicative in the single variable n. Here $F(n)$ is not a special case our sum S, but it can be treated in a similar way. By using (3.2) one obtains the formula

$$F(n) = n \sum_{d_1|n, \ldots, d_r|n} \frac{(\mu_r *_r f)(d_1, \ldots, d_r)}{[d_1, \ldots, d_r]} N(d_1, \ldots, d_r), \tag{3.5}$$

valid for any function f of r variables, where $N(d_1, \ldots, d_r)$ is the number of solutions (mod $[d_1, \ldots, d_r]$) of the simultaneous congruences $P_1(x) \equiv 0 \pmod{d_1}$, ..., $P_r(x) \equiv 0 \pmod{d_r}$. Note that $N(d_1, \ldots, d_r)$ is a multiplicative function of r variables. If f is multiplicative, then the convolution representation (3.5) shows that F is also multiplicative.

In what follows assume that
(1') $k, \ell \geq 0$ are fixed integers, not both zero;
(2') $n, r_i, s_i, t_j \in \mathbb{N}$ are integers such that $n \mid r_i$, $s_i \mid r_i$, $n \mid t_j$ ($1 \leq i \leq k$, $1 \leq j \leq \ell$);
(3') $g : \mathbb{N} \to \mathbb{C}$ is an arbitrary arithmetic function;
(4') $P_i, Q_j \in \mathbb{Z}[x]$ are arbitrary polynomials ($1 \leq i \leq k$, $1 \leq j \leq \ell$).
Consider the sum

$$T := \sum_{\substack{1 \leq a_i \leq r_i \\ (a_i, s_i)=1 \\ 1 \leq i \leq k}} \sum_{\substack{1 \leq b_j \leq t_j \\ 1 \leq j \leq \ell}} g((P_1(a_1), \ldots, P_k(a_k), Q(b_1), \ldots, Q_\ell(b_\ell), n)),$$

with the gcd on the right hand side.

We have the following result.

Theorem 3.7. *Assume conditions (1')-(4'). Then*

$$T = r_1 \cdots r_k t_1 \cdots t_\ell \sum_{d \mid n} \frac{(\mu * g)(d)}{d^{k+\ell}} \left(\prod_{1 \le i \le k} \widehat{N}_{P_i}(d, s_i) \beta(s_i, d) \right) \left(\prod_{1 \le j \le \ell} N_{Q_j}(d) \right),$$

where

$$\beta(s_i, d) = \prod_{\substack{p \mid s_i \\ p \nmid d}} \left(1 - \frac{1}{p}\right)$$

Proof. Apply Theorem 3.1 in the case when $m_i = n_j = n$ ($1 \le i \le k$, $1 \le j \le \ell$) and

$$f(x_1, \ldots, x_k, y_1, \ldots, y_\ell) = g((x_1, \ldots, x_k, y_1, \ldots, y_\ell)).$$

Then

$$\begin{aligned} &f((P_1(a_1), m_1), \ldots, (P_k(a_k), m_k), (Q(b_1), n_1), \ldots, (Q_\ell(b_\ell), n_\ell)) \\ &\quad = g((P_1(a_1), \ldots, P_k(a_k), Q(b_1), \ldots, Q_\ell(b_\ell), n)). \end{aligned}$$

From (2.1) we obtain

$$(\mu_{k+\ell} *_{k+\ell} f)(x_1, \ldots, x_k, y_1, \ldots, y_\ell) = \begin{cases} (\mu * g)(n), & \text{if } x_1 = \cdots = x_k = y_1 = \ldots = y_\ell = n, \\ 0, & \text{otherwise.} \end{cases}$$

□

In the special case $g(n) = n$, $Q_j(x) = x$, $r_i = s_i = t_j = n$ ($1 \le i \le k$, $1 \le j \le \ell$) we obtain from Theorem 3.7 the next result.

Corollary 3.8.

$$\sum_{\substack{1 \le a_i \le n \\ (a_i, n) = 1 \\ 1 \le i \le k}} \sum_{\substack{1 \le b_j \le n \\ 1 \le j \le \ell}} (P_1(a_1), \ldots, P_k(a_k), b_1, \ldots, b_\ell, n) = \varphi(n)^k (\mathrm{id}_\ell * G_k)(n),$$

where

$$G_k(n) = \varphi(n)^{1-k} \prod_{1 \le i \le k} \widehat{N}_{P_i}(n).$$

If $P_i(x) = x^{q_i} - 1$ ($1 \le i \le k$), then we obtain

Corollary 3.9. *If $q_i \in \mathbb{N}$ ($1 \le i \le k$), then*

$$\sum_{\substack{1 \le a_i \le n \\ (a_i, n) = 1 \\ 1 \le i \le k}} \sum_{\substack{1 \le b_j \le n \\ 1 \le j \le \ell}} (a_1^{q_1} - 1, \ldots, a_k^{q_k} - 1, b_1, \ldots, b_\ell, n) = \varphi(n)^k (\mathrm{id}_\ell * H_k)(n),$$

where

$$H_k(n) = \varphi(n)^{1-k} \prod_{1 \le i \le k} C^{(q_i)}(n), \tag{3.6}$$

$C^{(q_i)}(n)$ being the number of solutions of the congruence $x^{q_i} \equiv 1 \pmod{n}$.

For $k = 1$, (3.6) reduces to identity (1.3) by Li, Kim and Qiao [12].

Several other special cases can be discussed. For example, let $\ell = 0$. By formula (3.3) we have

$$V(n_1,\ldots,n_k) := \sum_{\substack{1\le a_i\le n_i\\(a_i,n_i)=1\\1\le i\le k}} f((P_1(a_1),n_1),\ldots,(P_k(a_k),n_k)) \tag{3.7}$$

$$= \varphi(n_1)\cdots\varphi(n_k) \sum_{\substack{d_i\mid n_i\\1\le i\le k}} \frac{(\mu_k *_k f)(d_1,\ldots,d_k)}{\varphi(d_1)\cdots\varphi(d_k)}\Big(\prod_{1\le i\le k} \widehat{N}_{P_i}(d_i)\Big).$$

If $f:\mathbb{N}\to\mathbb{C}$ is multiplicative, then $V(n_1,\ldots,n_k)$ is multiplicative, as well, by Corollary 3.6. For prime powers $p^{\nu_1},\ldots,p^{\nu_k}$ the values $V(p^{\nu_1},\ldots,p^{\nu_k})$ can be computed in the case of special functions f and special polynomials P_i.

We confine ourselves with the case of the lcm function $f(n_1,\ldots,n_k)=[n_1,\ldots,n_k]$ and the polynomials $P_i(x)=x-1$ ($1\le i\le k$), included in the next section.

4 A special case

In this section we consider the function

$$W(n_1,\ldots,n_k) := \sum_{\substack{1\le a_i\le n_i\\(a_i,n_i)=1\\1\le i\le k}} [(a_1-1,n_1),\ldots,(a_k-1,n_k)].$$

Theorem 4.1. *For any $n_1,\ldots,n_k\in\mathbb{N}$,*

$$W(n_1,\ldots,n_k)=\varphi(n_1)\cdots\varphi(n_k)h(n_1,\ldots,n_k),$$

where the function h is multiplicative, symmetric in the variables and for any prime powers $p^{\nu_1},\ldots,p^{\nu_k}$ such that $\nu_1\ge\cdots\ge\nu_t\ge 1$, $\nu_{t+1}=\cdots=\nu_k=0$,

$$h(p^{\nu_1},\ldots,p^{\nu_k})$$

$$=1+(\nu_1+\cdots+\nu_t)+\sum_{j=1}^{t-1}\frac{(-1)^j p^j}{(p-1)^j(p^j-1)}\Bigg(\binom{t}{j+1}-\sum_{\substack{M\subseteq\{1,\ldots,t\}\\ \#M=j+1}}\frac{1}{p^{j\nu_{\max M}}}\Bigg).$$

Proof. According to (3.7) we have

$$W(n_1,\ldots,n_k)=\varphi(n_1)\cdots\varphi(n_k)\sum_{\substack{d_i\mid n_i\\1\le i\le k}}\frac{(\mu_k *_k f)(d_1,\ldots,d_k)}{\varphi(d_1)\cdots\varphi(d_k)},$$

where $f(n_1,\ldots,n_k)=[n_1,\ldots,n_k]$.

Here $W(n_1,\ldots,n_k)$ is multiplicative and we compute the values $W(p^{\nu_1},\ldots,p^{\nu_k})$. Let $g=\mu_k *_k f$, that is,

$$g(n_1,\ldots,n_k)=\sum_{d_1\mid n_1,\ldots,d_k\mid n_k}\mu(d_1)\cdots\mu(d_k)\,[n_1/d_1,\ldots,n_k/d_k]\,.$$

Then g is multiplicative and for any prime powers $p^{\nu_1},\dots,p^{\nu_k}$ $(\nu_1,\dots,\nu_k\ge 0)$,

$$g(p^{\nu_1},\dots,p^{\nu_k}) = \sum_{d_1,\dots,d_k\in\{1,p\}} \mu(d_1)\cdots\mu(d_k)\,[p^{\nu_1}/d_1,\dots,p^{\nu_k}/d_k]\,.$$

Assume that there is $j\ge 1$ such that $\nu_1=\nu_2=\cdots=\nu_j=\nu>\nu_{j+1}\ge\nu_{j+2}\ge\cdots\ge\nu_m\ge 1$, $\nu_{m+1}=\cdots=\nu_k=0$. Then we have for any $d_1,\dots,d_m\in\{1,p\}$, $d_{m+1},\dots,d_k=1$,

$$[p^{\nu_1}/d_1,\dots,p^{\nu_k}/d_k] = \begin{cases} p^{\nu-1}, & \text{if } d_1=\cdots=d_j=p,\\ p^{\nu}, & \text{otherwise,}\end{cases}$$

and

$$g(p^{\nu_1},\dots,p^{\nu_k}) = \Big(p^{\nu}\sum_{d_1\in\{1,p\}}\mu(d_1)\cdots\sum_{d_j\in\{1,p\}}\mu(d_j) - p^{\nu}\mu(p)^j + p^{\nu-1}\mu(p)^j\Big)$$
$$\times \sum_{d_{j+1}\in\{1,p\}}\mu(d_{j+1})\cdots\sum_{d_m\in\{1,p\}}\mu(d_m) = \begin{cases} (-1)^{j-1}(p^{\nu}-p^{\nu-1}), & \text{if } j=m,\\ 0, & \text{otherwise.}\end{cases}$$

Therefore, since g is symmetric in the variables, we deduce

$$g(p^{\nu_1},\dots,p^{\nu_k}) = \begin{cases} 1, & \text{if } \nu_1=\cdots=\nu_k=0,\\ (-1)^{j-1}\varphi(p^{\nu}), & \text{if a number } j\ge 1 \text{ of } \nu_1,\dots,\nu_k \text{ is equal to } \nu\ge 1,\\ & \text{while all others are zero,}\\ 0, & \text{otherwise.}\end{cases} \tag{4.1}$$

Furthermore, let

$$h(n_1,\dots,n_k) = \sum_{d_1\mid n_1,\dots,d_k\mid n_k} \frac{g(d_1,\dots,d_k)}{\varphi(d_1)\cdots\varphi(d_k)},$$

which is also multiplicative and symmetric in the variables. Let $p^{\nu_1},\dots,p_k^{\nu_k}$ be any prime powers and assume, without loss of generality, that for some $t\ge 0$, one has $\nu_1\ge\cdots\ge\nu_t\ge 1$, $\nu_{t+1}=\cdots=\nu_k=0$.

If $t=0$, then $h(1,\dots,1)=1$. If $t\ge 1$, then

$$h(p^{\nu_1},\dots,p^{\nu_k}) = \sum_{d_1\mid p^{\nu_1},\dots,d_t\mid p^{\nu_t}} \frac{g(d_1,\dots,d_t,1,\dots,1)}{\varphi(d_1)\cdots\varphi(d_t)}.$$

Let $d_1=p^{\beta_1},\dots,d_t=p^{\beta_t}$, with $0\le\beta_1\le\nu_1,\dots,0\le\beta_t\le\nu_t$. For any subset M of $\{1,\dots,t\}$ such that $\#M=j$ $(1\le j\le t)$ let $\beta_m=\nu$ $(1\le\nu\le\nu_{\max M})$ for every $m\in M$ and $\beta_m=0$ for $m\notin M$. Then, according to (4.1),

$$\frac{g(d_1,\dots,d_t,1,\dots,1)}{\varphi(d_1)\cdots\varphi(d_t)} = \frac{(-1)^{j-1}\varphi(p^{\nu})}{\varphi(p^{\nu})^j} = \frac{(-1)^{j-1}}{\varphi(p^{\nu})^{j-1}}.$$

We deduce that

$$h(p^{\nu_1},\dots,p^{\nu_k}) = 1 + \sum_{j=1}^{t}\sum_{\substack{M\subseteq\{1,\dots,t\}\\ \#M=j}}\sum_{\nu=1}^{\nu_{\max M}} \frac{(-1)^{j-1}}{\varphi(p^{\nu})^{j-1}}$$

$$= 1 + \sum_{j=1}^{t} (-1)^{j-1} \sum_{\substack{M \subseteq \{1,\dots,t\} \\ \#M = j}} \sum_{\nu=1}^{\nu_{\max M}} \frac{1}{\varphi(p^\nu)^{j-1}}.$$

Here, with the notation $A := \nu_{\max M}$, we have for $j \geq 2$,

$$K_j := \sum_{\nu=1}^{\nu_{\max M}} \frac{1}{\varphi(p^\nu)^{j-1}} = \frac{1}{(p-1)^{j-1}} \sum_{\nu=1}^{A} \frac{1}{p^{(j-1)(\nu-1)}}$$

$$= \frac{p^{j-1}}{(p-1)^{j-1}(p^{j-1}-1)} \left(1 - \frac{1}{p^{A(j-1)}}\right),$$

and for $j = 1$, $K_1 = A$.

That is,

$$h(p^{\nu_1}, \dots, p^{\nu_k}) = 1 + (\nu_1 + \dots + \nu_t) + \sum_{j=2}^{t} \frac{(-1)^{j-1} p^{j-1}}{(p-1)^{j-1}(p^{j-1}-1)} \sum_{\substack{M \subseteq \{1,\dots,t\} \\ \#M = j}} \left(1 - \frac{1}{p^{A(j-1)}}\right)$$

$$= 1 + (\nu_1 + \dots + \nu_t) + \sum_{j=1}^{t-1} \frac{(-1)^j p^j}{(p-1)^j (p^j - 1)} \left(\binom{t}{j+1} - \sum_{\substack{M \subseteq \{1,\dots,t\} \\ \#M = j+1}} \frac{1}{p^{Aj}} \right).$$

□

Corollary 4.2. $(n_1 = \dots = n_k = n)$ *For any* $n, k \in \mathbb{N}$,

$$\sum_{\substack{1 \le a_1 \le n \\ (a_1, n) = 1}} \cdots \sum_{\substack{1 \le a_k \le n \\ (a_k, n) = 1}} [(a_1 - 1, n), \dots, (a_k - 1, n)]$$

$$= \varphi(n)^k \prod_{p^\nu || n} \left(1 + k\nu + \sum_{j=1}^{k-1} (-1)^j \binom{k}{j+1} \frac{p^j}{(p-1)^j (p^j - 1)} \left(1 - \frac{1}{p^{\nu j}}\right) \right).$$

In the case $k = 2$ this gives the formula (1.5), while for $k = 1$ we reobtain Menon's identity (1.1).

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