On the process aspect in mathematics through genuine problem-solving

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The process aspect of mathematics is important to school teachers on the basis of subject didactic research. Through it, we can achieve better commitment to studies at school, more applied skills, and deeper learning, for example. However, in the conventional subject studies at the university level this approach may remain vague. Also, the concept as such needs some clarification. The author of this chapter sets himself a genuine problem through which he wants to achieve a clearer picture of the process aspect. GeoGebra software is quite a central tool here. Many experimental features then come along; many conjectures and hypotheses arise and very often they turn out wrong, too. So, in a sense, the process aspect seems to be a chain or a network of conjectures and refutations; it may also include some qualitative reasoning needed for a better understanding. Such features are common in other sciences, and also in everyday life. At the same time, however, something promising may appear, demanding

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validation: here, at the latest, the system aspect of mathematics, together with the toolbox view, comes naturally into play. One can also apply Polya's terminology and say that demonstrative reasoning and plausible reasoning meet in the process aspect. In the end, the author also creates some more general connections of this writing process to certain themes in the subject didactics, i.e. in the contents of his teaching as a university instructor.

Introduction

The didactic triangle is one of the first things to be studied in the subject teacher's pedagogical education. The two vertices of the didactic triangle are always labeled as teacher and learner. The third vertex is labeled according to the subject in the focus of teaching. In this chapter, this third vertex is mathematics. It is also a common practice to draw two-headed arrows between the vertices to express the mutual interactions there. The "classic" didactic triangle is quite simple as a model of the dynamic teaching and learning process at school. Even so, it creates a frame of many essential questions. For example, Schoenfeld (2012) has listed a few of them. The next three especially interesting ones in this context are gathered from that list:

- What is mathematics, and which version of it is the focus of classroom activities?
- What is the teacher's understanding (in a broad sense) of mathematics?
- How does the teacher mediate between the learner and mathematics, shaping the learner's developing understanding of mathematics?

A great deal of research has been published on the perspectives to mathematics by the professors of mathematics, teacher educators, and teachers of mathematics at school (e.g. Pehkonen 1999; Mura

1993; Törner & Pehkonen 1999; Viholainen et al. 2014; Tossavainen et al. 2017). The following classification of the dominant views seems to be the most typical one:

- Mathematics is a formal system, in which you have to write rigorous proofs with a precise and clear language (in short: "mathematics as a system", or plain S).
- Mathematics is a collection of calculation rules and routines, which will be applied along the circumstances ("mathematics as a toolbox", T).
- Mathematics is a dynamical process where everyone creates his or her own mathematics according to needs and abilities ("mathematics as a process", P).

The classification above has been reached by applying qualitative data analysis to the answers to the question "what is mathematics?" (Pehkonen 1999; Mura 1993). In an alternative approach, this classification has been used as a starting point for the analysis of data (Törner & Pehkonen 1999). It also has to be noticed that the S, T and P categories above do not mean that everyone is a member of just one class - quite on the contrary. Typically, everyone belongs to all classes, but in different magnitudes. Surprisingly, the P perspective was the least dominant among the professors; one could have expected the opposite because professors, in particular, produce something new in mathematics (Pehkonen 1999).

The S category above is well-known to everyone who has studied mathematics to some extent at the university level. For example, every student majoring in mathematics studies at least elementary mathematical analysis. Behind it is the axiomatic system of real numbers defined as the complete ordered field. Every proposition, say the fundamental theorem of analysis, has to be proved in this field (see, e.g., Poranen 2000, 20). The elementary mathematical analysis is already a huge and complicated system; studying it demands a lot of work and probably nearly everyone finds it a "finished system", which one just has to learn.

Respectively, "mathematics as a toolbox" (T) is familiar to every student of mathematics. The above-mentioned mathematical analysis has a never-ending number of applications in science, engineering, economics, geometry, etc. The same applies to other domains of mathematics which the prospective teacher usually studies. The applications may be familiar or not so well-known (see, e.g., Poranen & Haukkanen 2012; Abramovich & Brouwer 2003).

However, mathematics as a process (P), from the viewpoint of the prospective teachers, is surely much more difficult to describe. Especially the part "everyone creates his or her own mathematics…" can raise many questions, simply because the things are not usually studied in that way. We could also say that the ordinary tradition in teaching and learning mathematics at the university level is more educational than developmental – to use the terminology from Haapasalo & Kadijevich (2000).

Here, the author will not just content himself with some general level argument for the P perspective (however, never at the expense of the S and T perspectives in the subject teacher education); instead, he sets himself a genuine problem through which he wants to make the P perspective more accessible and understandable in teacher education, especially with consideration to the didactic triangle.

The following is a summary of quite a multidimensional problem-solving process with a few remarks and more general reflection.

Problem Q and its background

The father figure of the rich tradition of mathematical problem solving G. Polya (1887–1985) presents in his book (1973, 122 - 123) the question (**q1**):

> Given two points and a straight line, all in the same plane, both points on the same side of the line. On the given straight line, find a point from which the segment joining the two given points is seen under the greatest possible angle.

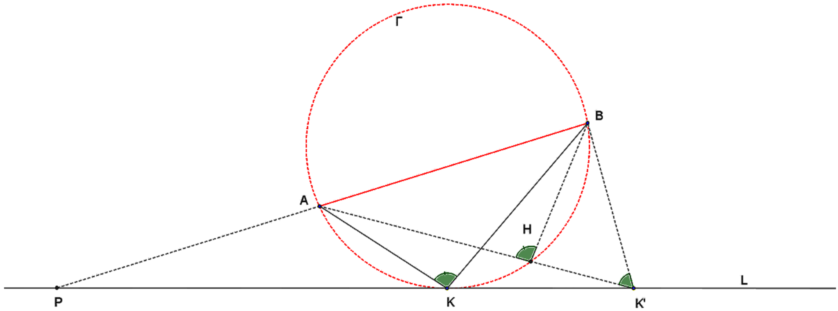

Figure 1. Question q1: How to find point K from line L so that the segment AB is seen from it under the greatest possible angle.

After a few quite qualitative considerations Polya says that we have to draw a circle passing through the given points A, B, and which touches line L: the vertex of the maximum angle is the touching point K (cf. Fig. 1). However, he does not reveal how all this can be done as a genuine classic geometric construction (i.e. with compass and ruler without a scale (see, e.g., Lehtinen, Merikoski & Tossavainen 2007, 79–84, 125–129). Nor does he give any proof for his proposition. He also seems to leave aside the fundamental question concerning the existence of the geometric solution. All these questions proved to be difficult to the author, although the entire essence of the problem (**Q**, below) is not yet here. One reason for the difficulties was that Polya's original figure contained just the segment AB, line L, the circle Γ, and the touching point K – without any further explanations. Our figure (Fig. 1) contains more essential elements, and their crucial role will be discussed further on in this chapter.

Later in his book, Polya (1973, 142–144) introduces another question (**q2**), which is likely to be much better known:

> Given two points and a straight line, all in the same plane, both points on the same side of the line. On a given straight line, find a point such that the sum of its distances from the two given points be minimum.

A widely known and elegant solution to this problem is visible in Figure 2:

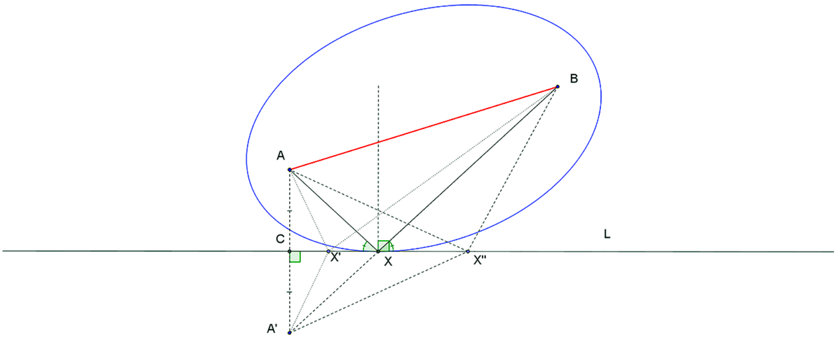

Figure 2. Question q2: How to find point X from line L so that the polygonal line AXB is as short as possible.

One geometric solution to question **q2** is quite simple: We first construct the mirror image A' of point A with respect to line L; then the solution is the point X intersection of segment A'B and line L. The solution (path AXB) is easy to prove to be true by using some elementary geometry (see Fig. 2). Respectively, it is easy to show that angles AXX' ("the angle of incidence") and BXX" ("the angle of reflection") are equal. The paths AX'B, and AX"B are two arbitrary routes from point A to point B through line L. By using the mirroring with respect to line L, we see that AX' = A'X' (AX" = A'X"), and that AX' + X'B (AX" + X"B) > AX + XB = A'X + XB because segment A'B gives the shortest distance between the two points A' and B.

It may also be interesting to see that we can draw an ellipse with the focal points A, and B, so that it touches line L at point X, i.e. we adjust the sum AX + XB to give the constant distance of the points of that ellipse from its focal points A and B (Fig. 2). We could generally deduce some central properties for all ellipses on the grounds of this observation. However, now we have to omit this line of inquiry.

Question **q2** is not at all as boring as it may seem at first sight. For example, there are many connections to geometric optics in it (see, e.g., Young & Freedman 2000, 1055–1063; also Polya 1973, 143–147). However, these and many other interesting features linked to it are not our focus now. Instead, we are presented with a chance to set a genuine problem (at least to the author): **Is there some relation between questions q1 and q2**? Let us call this question or problem **Q**. We can set problem **Q** more precisely also: **Is there some relation between points K and X** (cf. Fig. 1. and Fig. 2)? Of course, in this context, we are looking for some possible interesting mathematical connections between points K and X. As far as the author has been able to assure, Polya himself has not posed question **Q**. He merely writes (1973, 143):

> In fact, both problems have exactly the same data, and even the unknown is of the same nature: Here, as there, we seek the position of a point on a given line for which a certain extremum is attained. The two problems differ only in the nature of this extremum: Here we seek to minimize the sum of two lines; there we sought to maximize the angle included by those two lines.

In the following we will assume that segment AB lies entirely above line L; further we assume that the distance of point B from line L is greater than the distance of point A from it (see Fig. 1 and Fig. 2). The case AB || L will be investigated separately (see *The AB II L case* in this chapter) as will the case AB ⊥ L (see *The general case*). So from now on $0^\circ < \alpha < 90^\circ$ where α is the angle between the segment AB and line L, and the cases $\alpha = 0^\circ$, $\alpha = 90^\circ$ will be discussed separately. Here, we also prefer classic geometric reasoning to differential calculus, etc., to keep things as simple and observable as possible.

Some preliminary conjectures concerning question Q

Certainly, let us learn proving, but also let us learn guessing. (Polya 1973, Preface.)

In what follows, we will rely on *GeoGebra* (version 5.0); its correct functioning in measuring angles, for example, is unquestionable.

The first conjecture or hypothesis (**c1**) about **Q** could be quite straightforward. Point K, which defines the vertex of the maximum angle = point X which defines the shortest path (see Fig. 1 and Fig. 2). Some experiments by *GeoGebra*, however, show that **c1** cannot be true. However, there may be a special case where **c1** holds (see section *The AB II L case*), and perhaps, something else of interest may also emerge. One possibility in that direction could lie in the following consideration (Fig. 3):

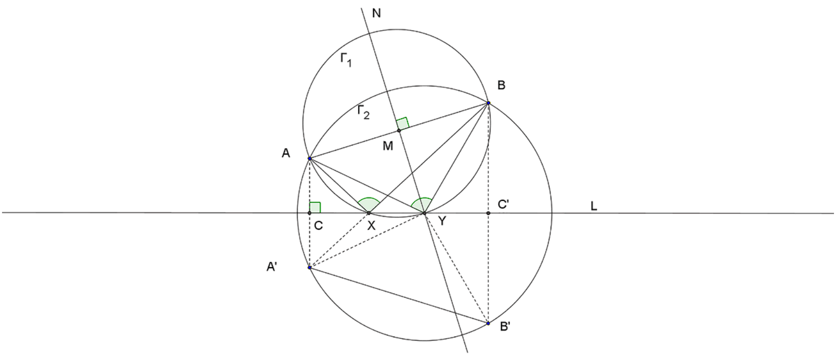

Figure 3. Angle AXB = angle AYB; angle BAY = angle AXC (angle BXC'). These observations provide two new methods to construct point X.

Let point Y be the intersection of the perpendicular bisector N of segment AB and line L (Fig. 3). Now we can construct circle Γ1 through points A, B and Y, and then prove that the circle also goes through point X (see Fig. 2). The details of the proof are omitted here, but the author thinks he succeeded in it by constructing another circle Γ2 passing through points A, B, B' and A', where A' and B' are the mirror images of points A and B with respect to line L (Fig. 3). Aided by this circle he could prove that angle AXB = angle AYB, so points X and Y must lie on the same circle Γ1. He also reached the conclusion that angle AXC = angle BAY (=angle BXC'), where point C is the intersection of segment AA' and line L.

By means of circle Γ1 we can construct point X (see Fig. 2) as the intersection of circle Γ1, and line L. We can also construct point X on the grounds of the observation that angle AXC = angle BAY, because triangle AXC is right-angled. Both of these methods are new to the author, and they developed as a sort of side effect by investigating hypothesis **c1**.

By observing circle Γ1 and its points X and Y (Fig. 3), we can state the second hypothesis (**c2**) concerning problem **Q**: the midpoint between X and Y is K (see Fig. 1 and Fig. 3). But *GeoGebra* is again ruthless… However, that midpoint may lie very near to point K. Furthermore, we can propose something a little bit weaker: point K always lies between points X and Y (**c3**). Using *GeoGebra*, the author has not succeeded to prove c3 wrong; yet, of course, it may not hold.

The AB || L case

It is quite natural to try to get a clear view of the special case where segment AB is parallel to line L (Fig. 4):

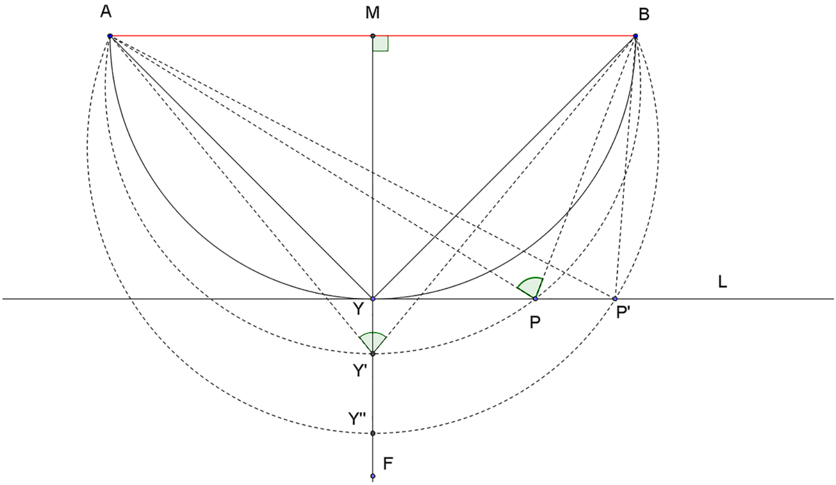

Figure 4. AB || L.

Let MF be the perpendicular bisector of AB and let point Y be the intersection of MF and L. Further, let points Y' and Y'' lie at MF so that MY < MY' < MY''. The supplementary adjacent angle of an angle in any triangle is always greater than the other two angles of that triangle. Thus angle BY'Y < angle BYM, and from this follows that angle BY'A < angle BYA. Respectively, we conclude that angle BY''A < angle BY'A (Fig. 4).

We can draw the circle through points A, Y' and B and name as P one of its two intersections with line L (Fig. 4). Now the vertices of angles BPA and BY'A are on the same circle opposing the same chord AB, so angle BPA = angle BY'A; further, especially, angle BPA < angle BYA. We also see that angle BP'A < angle BPA, and this is equivalent with YP < YP' (Fig. 4).

To sum up, angle BYA is the maximum angle in the case of AB || L, where Y is the intersection of the perpendicular bisector of AB and L; the greater the distance of point P from point Y along line L, the smaller angle BPA is. Equivalently, the greater the distance of point Y' along MF from point M (below AB), the smaller angle BY'A is. Also, clearly, BYA is the shortest path from B to A through L (Fig. 4). So now, point K = point X (see Fig. 1 and Fig. 2). We see that the new methods (see *Some preliminary conjectures concerning question* Q) to find point X also works in this special case; then, of course, point X = point Y.

The general case

First, we have to go back to Figure1 and question **q1**. Polya's basic idea there seems to be correct: let point K be the touching point of line L and the circle passing through points A and B. Further, let K' be an arbitrary point (≠ K) on line L and let point H be the intersection of segment AK' and circle Γ; in triangle HK'B, angle K' is smaller than the supplementary adjacent angle AHB (= angle K) of angle BHK'. Thus, angle K' < angle K, i.e. segment AB is seen from point K under the greatest possible angle. We may further define the intersection of the lines AB and L, and name it P; then PK' is a tangent line of circle Γ, and point K on it is the common point of this tangent and the circle. (Fig. 1)

Polya himself did not give an exact explanation (or proof) for the existence of point K, like that above. Now we know that point K exists, but is there a genuine geometric construction to it? To try to find it, we should first recall the principle of analysis and synthesis from the mathematical problem solving tradition (see, e.g., Haapasalo 2012). By analyzing the Figure 1, we may come to a construction idea (synthesis) with the help of the Secant Theorem (Fig. 5):

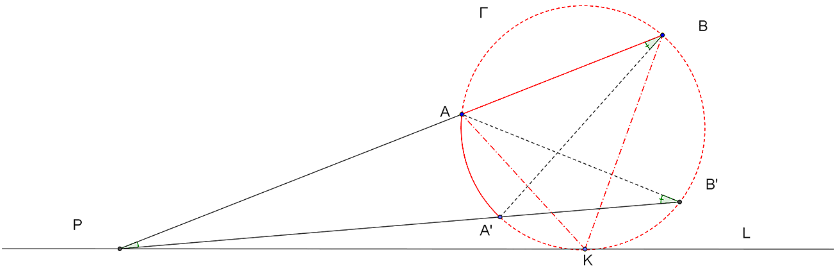

Figure 5. The Secant Theorem: PA · PB = PA' · PB'

Triangles PA'B, and PAB' have angle A'PA in common. Further, angle A'B'A = angle ABA' (corresponding to the same arc, A'A), so these triangles are similar. Thus PA'/PB = PA/PB', i.e. PA · PB = PA' · PB' (Fig. 5). Let, then, secant PB' rotate around point P so that it goes closer and closer to the tangent PK (cf. also Fig. 1). We see that then

$$PA \cdot PB \rightarrow PK \cdot PK = (PK)^2$$

Thus PK = the geometric mean of PA and PB, i.e. PK/PB = PA/PK. The geometric mean here (and more generally) is constructible, but we now omit the details of its proof (see, e.g., Väisälä 1965).

In other words, to solve the question **q1** geometrically, we have to construct the circle with that geometric mean as a radius and P as the centre. The intersection of this circle and line L gives the vertex point K of the maximum angle. It is interesting to note that the method works in the AB ⊥ L case, too (see Fig. 7). We may also observe that triangles PKB and PAK are similar. Then, for example, KA/KB = PA/PK. Obviously, we have found a solution to the next construction problem, too, i.e. given two points A, B above line L; construct a circle passing through the given points, and touching the given line L.

The author has to admit (with a little embarrassment) that he faced great troubles here. He had some private correspondence with J. Merikoski, the emeritus professor of mathematics (University of Tampere). Merikoski offered an idea on how to prove Polya's proposition about the existence of vertex K of the maximum angle.

He also showed the role of the Secant Theorem in the geometric construction of that point. So, now we know how to construct both points K, and X, separately. However, the main question, **Q**, still does not have a solution.

Even before the correspondence with professor Merikoski, the author had found by himself quite a promising procedure to define vertex K of the maximum angle, and there and then a conjecture (**c4**) concerning the main question **Q**, also (Fig. 6).

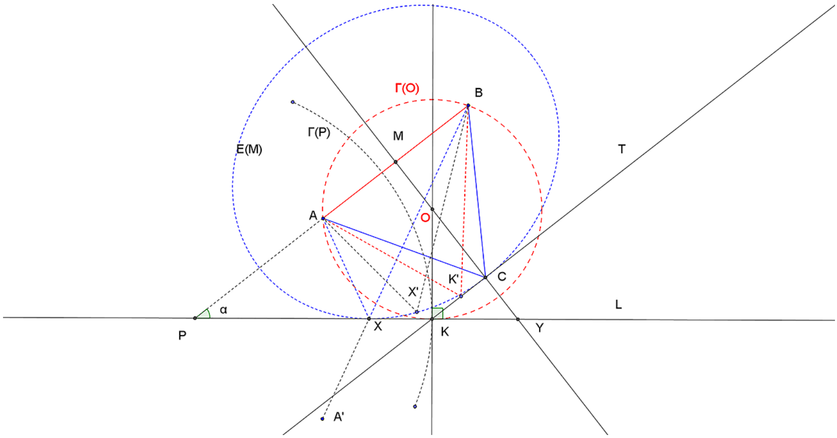

Figure 6. Conjecture c4: Points X and K are connected through a multiphase minimization-maximization-decreasing-maximization process starting from point X.

Again we have to first find point X, minimizing the length of the path from A to B, through line L (see Fig. 2). Of course, then we also see segment AB in an angle AXB. Next, we construct through point X an ellipse E(M) with the focal points A and B. Let the intersection go closer to L of E(M) and the perpendicular bisector MY be C. Then, we draw line T through point C, parallel to AB, and mark the intersection of it, and line L with K. (Fig. 6)

On the grounds of the observations (by *GeoGebra*), point K constructed in this way is the vertex of the maximum angle. Naturally, it also seems to be the same point that we get by using the equation $PA \cdot PB = (PK)^2$ (of course, in the contrary case we should reject conjecture **c4**). Through minor calculation, we can easily show our conjecture **c4** to be true in the special AB ⊥ L case (Fig. 7).

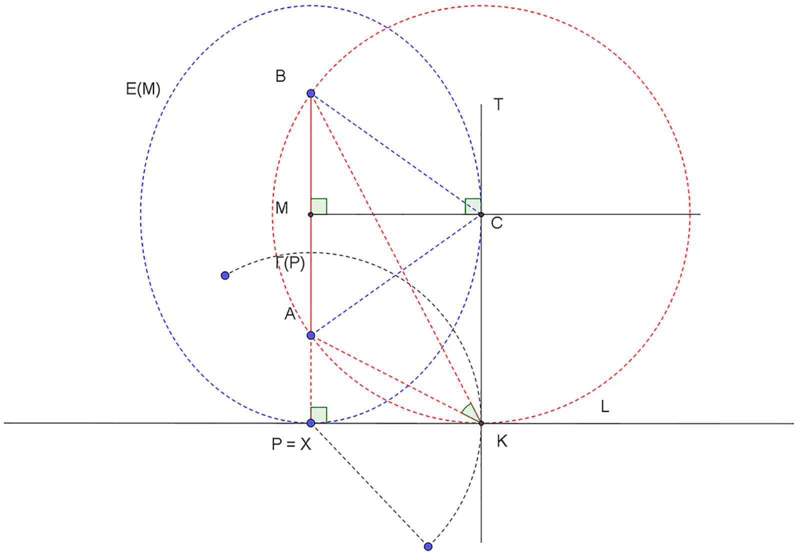

Figure 7. The AB ⊥ L case

In this case (Fig. 7), the length of the shortest path AXB (see Fig. 2) is clearly = AP + PA + AB = 2AP + AB, and P = X. Thus, AC (= BC) = AP + AB/2, and $(MC)^2 = (AC)^2 - (AB/2)^2$, i.e. $(MC)^2 = AP(AP + AB)$, i.e. MC = the geometric mean of the segments PA, and PB, i.e. MC = PK. Further, clearly MC || PK, therefore, line T through point C, parallel to AB, intersects line L at point K (Fig. 7). It may be interesting to note that now angle AXB = angle AYB = 0˚ also, if we consider that point Y is the "intersection" of lines L and MC at infinity. Respectively, then the second result that we found in section 3 also holds true (cf. Fig. 3).

In the general case (Fig. 6), we content ourselves with *plausible reasoning* – instead of *demonstrative reasoning*, i.e. we leave the (possible) true proof for another time; both these terms of reasoning come from Polya (1973).

So, let us choose a "moving point" X' from the ellipse E(M) (Fig. 6). Then always AX' + X'B = AX + XB, and angle AX'B increases as we move closer to MY along the ellipse E(M). We reach the maximum size of this angle at point C.

Now we have an isosceles triangle ACB, the base of which is AB and sides AC = BC = (AX + XB)/2. Next, we construct a line T parallel to AB through the vertex C of this triangle. At this point, at the latest, it could also be a good idea to take a glance at Figure 4 and recall the considerations there.

We will move again, but now along line T to the left using point K' (see Fig. 6). Then angle AK'B < angle ACB, and angle AK'B decreases all the time as point K' moves to the left. But there is a point where line T meets line L. We stop the moving there and conjecture that this point is the vertex of the maximum angle AKB.

In addition, the (plausible or "qualitative") reasoning above not only includes the fact that the area of triangle AKB = the area of triangle ACB, it also gives us some evidence for conjecture **c3**. If we are right here, we have found a genuine constructive connection between points X, and K. We may have to exclude the ellipses from the set of allowable geometric constructions, but the role of the ellipse E(M) is not crucial in the reasoning: clearly, we can also get point C (see Fig. 6) without it (starting from point X).

In our figures (Fig. 2, Fig. 6, Fig. 7), we have generated the ellipses by *GeoGebra*. If we allow such use and drawing of the ellipses, we may, perhaps, also start from point K. Angle CKY = α (see Fig. 6), and therefore we can find point C starting from point K. We should then prove that an ellipse through this point and with the focal points A, B, touches line L and that this touching point = X.

Some remarks on the Q process

The author had not encountered question **q1** earlier. Therefore, it (as part of the main question **Q**) served well as a starting point for examining the process aspect (P). In the beginning, the author thought that it could not be very hard to find a geometric (constructive) solution to it – if it only happened to exist. He also thought that in case of possible difficulties, analytic geometry or some differential calculus would quickly help. However, question **q1** remained difficult (for the author), and it was sometimes quite frustrating, too. The author learned, little by little, a lot of things about the possible constructive solution of **q1**, but it did not help. He found out, for example, that the centre O of that circle Γ(O), which passes through point K, must lie on the perpendicular bisector MY, so KO = OA (= OB); angle KOY must be equal to angle KPA (Fig. 6), etc. In practice, the author executed many experiments using *GeoGebra*; it took, however, a lot of time before reaching something promising (cf. Fig. 6).

The author abstained from online help. He had to be on his own so to speak; i.e. he had to use just his existing knowledge structures (hoping that there is something) to achieve a true understanding and experience of the process aspect. Still, some standard textbooks used in his teaching at school and university were "allowed" (see References).

The author did not, however, find or invent anything directly useable from his knowledge structure, or from the "allowable textbooks". Afterwards, it is easy to say that he should have been able to find something. For example, the legendary Finnish textbook of Geometry (Väisälä 1965, 120) contains the Secant Theorem, and some exercises that could have been helpful (there was, however, not a word about some possible connections to question **q1**); the author has used the Secant Theorem in his own teaching and has even applied it to a geometric interpretation of division (cf. the S and T perspectives), etc. So, The Secant Theorem must have been somewhere in the author's mind. However, he was unable to connect it to question **q1**. Therefore, we have here an example of the classic transfer problem also (see, e.g., Ropo 1999, 158; Greeno, Collins, & Resnick 1996, 21).

The author had set his mind too rigorously on looking for a solution to **q1** otherwise – not through point P (cf. Fig. 1, Fig. 5 and Fig. 6). Anyway, it was a bit difficult to deal with limited ability already with the problem **q1**. But the invention of the rather complicated hypothesis **c4** (see Fig. 6, Fig. 7) gives him some consolation. The author is quite convinced that the (partly qualitative) reasoning there is correct, or at least very fascinating. He may still be wrong.

Question **q2** (as the second part of the main question **Q**) was already quite familiar to the author. He has utilized it in his teaching of subject didactics, for example, by considering some functional working methods, and also in his teaching of geometry. Still, during this writing process he learned much more about it (see, for example Fig. 3). In many games, such as ice hockey, a player may pass the puck to another player via a wall; question **q2** gives the interesting geometry behind that. As a standard extreme value task, say, by handling root functions in differential calculus, it is not hard; however, in manual computing it demands quite a lot of work.

To many students in teacher education it seems to be important to realize that some extreme value tasks (such as **q2**) can be viewed in many different ways – also concretely without using, say, the typical arsenal of abstract differential calculus. We could present question **q1** in the context of differential calculus, too. By applying our pure geometric solution we could also deduce an analytic expression to point K, and this could be of some use there. But, even then, **q1** would probably be quite difficult in the context of differential calculus. Respectively, we could find (easily) an analytic expression for point X. Thus, it could be possible to investigate the main question **Q** through these analytic expressions for K, and X, too. However, in this chapter we have had to omit that possibility.

The main problem **Q** was difficult (for the author). This was not a surprise, at least not after those preliminary conjectures **c1, c2**, and many others which, one by one, proved incorrect. One reason for the difficulty was certainly question **q1** (as part of it), which bothered the author for a long time. However, perhaps because question **Q** was the author's "own", it always remained motivating and interesting; well, at times it was frustrating, too. All in all, though, **Q** was an excellent companion with whom the author could "talk" almost any time.

There still remains at least one crucial question we should consider. Does the narrative of the problem solving process above – let us call it the Q Process – shed some light on the general process aspect (P)? "Everyone creates his or her own mathematics…" Did something like that happen? Yes, a little. At the beginning, the author made many observations through GeoGebra. However, it was not at all clear how to utilize or interpret them, because they did not give any hints to straightforward inductive generalizations. So, in a way, the author had no concepts through which he could have created some order in the observations (yet, naturally, there were some starting points). Furthermore, we must remember, of course, that the whole question **Q** was genuinely new to the author.

For example, the new methods (to the author) to find point X on line L (see Fig. 3) were one of the attempts to create some order in the observations (they may also have a role when investigating conjecture **c4**). The same applies to the considerations in Fig. 4. The reasoning there was simple; however, it was also new to the author, and, as a matter of fact, it proved to be important in the (plausible) reasoning concerning conjecture **c4** (Fig. 6, Fig. 7). Conjecture **c4** remains unproven, but the whole complex thinking process there was certainly new to the author; we may also say that it was his own. So, to get some true understanding and experience of the P aspect, the genuine problem solving seems useful.

During the Q Process, conjectures turned up many times, and it was natural query about their proof (or falsifications). Therefore, in this way for example, the S aspect was present in the Q Process, too. If we think about some possible appliances, the maximum angle in games or lightning, for instance, we must be sure about the theory being applied – and so we have a natural motivation for proof in the S context.

The T aspect was also present in the Q Process. For example, the use of the Supplementary Adjacent Angle Theorem can also be seen from that perspective, and why not the Secant Theorem, as well. Thus, even these quite trivial remarks show that in the Q Process the S and T aspects were present, too.

As far as the author can tell, mathematics, more generally, affords endless possibilities to find such a good companion as the Q Process above. But it is not always necessary to try to solve a suitable or meaningful "problem of one's own". Any concept or structure from the "finished mathematics" may work as a starting point. Then, however, it could be a good idea to think about things, so to say, developmentally rather than educationally (cf. Haapasalo & Kadijevich 2000; Haapasalo 2004).

All in all, a good guideline could be found from Polya's treatise (1973) on the features of plausible reasoning, such as generalization, specialization, analogy and induction. Also, the article (Yrjänäinen & Ropo 2013) is a good presentation concerning the general characteristics of meaningful learning. One application of the ideas in that article is the paper by Poranen & Yrjänäinen (2015). There, the starting point was the consideration of the geometry of a car jack, as a (dynamical) rhombus.

The uncertainty and roughness which runs throughout this text is usually absent from the textbooks and basic studies of mathematics. This state of affairs has certainly many good and practical grounds. However, the system (S) and toolbox (T) views can then be given too much focus without giving the students enough possibilities to real or deeper understanding of the ideas. The most important reason to study mathematics is to learn to think; forgetting or hiding the process view (P) may be quite a poor solution in this regard. The textbooks could perhaps contain some of the author's narratives on their genuine problem solving, and not always just "success stories".

The prospective teachers of mathematics should write their own "meaning narratives" or stories about their own genuine problem solving. That could be part of their standard studies during their subject didactic (pedagogical) studies at the latest. Otherwise, they will not have enough means to get the learners to participate in and commit to their studies at school. This includes, naturally, that the learners are required to do something similar. Without any genuine experience of their own, the teachers cannot offer the learners the possibilities to do that. Also, without such experience, the prospective teachers cannot understand what Schoenfield (2012) means by his three questions presented at the beginning of this chapter.

Discussion

The author works as a university instructor in three basic domains in the Faculty of Education at the University of Tampere: in the subject teacher education teaching didactics of mathematical sciences, in primary school teacher education as well as a teacher of didactic mathematics (this is not to be confused with didactics of mathematics, see, e.g., Poranen & Silfverberg 2011). Before working at the university (since 2004), the author worked as a school teacher for about a quarter of a century. He taught there, among other things, mathematics and philosophy.

It takes one academic year to complete the didactical/pedagogical studies (60 ECTS) in subject teacher education. During that year, the students study general education, subject didactics, conduct subject didactic research, and do the teaching practice at school. Before starting the proper pedagogical studies, including the teaching practice at school, students must have completed a minimum of 50–60 ECTS of studies in mathematics if it is their major.

Classroom mathematics is something quite different compared to university mathematics. For example, Sorvali (2004), a Finnish mathematician, has written about some considerable differences between school and university mathematics where he questions the traditional thinking based on the idea that the best know-how in school mathematics is achieved only through studying mathematics as an axiomatic-deductive system (cf. S perspective). Instead, Sorvali proposes a kind of "observational mathematics" to be part of the prospective mathematics teachers' subject studies (cf. Q Process & P perspective). Klisinska (2009) deals with the complicated relationship between academic and classroom mathematics using the notion of didactic transposition of mathematical knowledge as a central theoretical instrument (cf. also Schoenfeld 2012). Much research pertaining to functional pedagogic knowledge of mathematics exists (e.g. Stylianides & Stylianides 2010).

As a consequence, one of the first natural steps in subject didactic studies is elaborating on each student's conceptions and perspectives of mathematics as a starting point in discussion on classroom mathematics. Then, usually, the S and T perspectives introduced at the beginning of this chapter emerge. On the other hand, school mathematics also comprises aims whereby the S and T perspectives may not be enough to teach successfully. For example, the students in grades 7–9 should be encouraged to find and utilize mathematics in their own lives; they should also have abilities to model and solve problems mathematically (POPS 2014, 374). There are similar emphases in the national [Finnish] curriculum for upper secondary school, too (cf. LOPS 2015, 129).

Respectively, since the 1980s, concepts related to the didactics of mathematics have considered mathematical proficiency quite broadly. At that time, concepts such as *strategic competence, adaptive reasoning*, and *productive disposition* were emphasized. By strategic competence we refer to the ability to formulate, represent and solve mathematical problems. Adaptive reasoning refers to the capacity for logical thought, reflection, explanation and justification, while productive disposition points to the habitual inclination to see mathematics as worthwhile, sensible and useful, a connection to a belief in hard work and one's own efficacy. In addition to those three features, *conceptual understanding* (i.e. comprehension of concepts, relations and operations) and *procedural fluency* (i.e. know-how in when and how to use different standard procedures) are also usually mentioned. The author thinks that the first three characteristics are not far from our P perspective here, and, respectively, that the last two features are close to the S and T perspectives.

In the literature of the didactics of mathematics, researchers have always emphasized that all of the five components are interwoven and interdependent on each individual's development of proficiency in mathematics (see, e.g., Kilpatrick et al. 2001; Joutsenlahti 2005). We have also seen above, in a concrete way, that the perspectives S, T, and P in our Q Process were interwoven, although the focus was on the P perspective.

Our Q Process and questions **q1** and **q2** as such are quite good examples, if we think, for example, about those emphases mentioned in our national curricula and the didactical emphases in mathematical proficiency. Or, to say it more clearly, the prospective teachers of mathematics should have some similar training and experiences, as this author had in his Q Process, to better meet the demands set forth in the curricula and to better understand the more general and significant P perspective (including, for example, the strategic competence in mathematical proficiency). Of course, they still have to possess a good know-how of many basic things in the traditional sense, i.e. in the senses of the S and T perspectives. And, of course, they are much cleverer in problem solving than this author, a senior instructor who, however, still wants eagerly to become a better teacher.

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