Eero Immonen

State Space Output Regulation Theory for Infinite-Dimensional Linear Systems and Bounded Uniformly Continuous Exogenous Signals

Eero Immonen

State Space Output Regulation Theory for Infinite-Dimensional Linear Systems and Bounded Uniformly Continuous Exogenous Signals

Thesis for the degree of Doctor of Technology to be presented with due permission for public examination and criticism in Tietotalo Building, Auditorium TB103, at Tampere University of Technology, on the 7th of July 2006, at 12 noon.

To Oili♡

Contents

Glossary

Notation	Description	
A, B, C, D	Parameters of the plant	1
$AP(\mathbb{R}, E)$	Almost periodic functions $\mathbb{R} \to E$	22
$A\vert_E$	Restriction of A to E	14
$BUC(\mathbb{R}, E)$	Bounded uniformly continuous functions $\mathbb{R} \to E$	18
$C_0^+(\mathbb{R}, E)$	BUC functions $\mathbb{R} \to E$ with $\lim_{t \to \infty} f(t) = 0$	40
E	General Banach space	22
F, G, J	Parameters of the error feedback controller	76
F, G, J, Γ	Parameters of the feedforward-feedback controller	110
H	Input and output space of the plant	1
$H(f_n, \omega_n)$	Scale of generalized Sobolev spaces of scalar $p-$periodic functions	25
$H(s)$	Transfer function $CR(s, A)B + D$ of the plant	199
$H_K(s)$	Transfer function $(C + DK)R(s, A + BK)B + D$ of the plant subject to a state feedback K	199
$H_{AP}(E, f_n, \omega_n)$	Scale of generalized Sobolev spaces of almost periodic functions $\mathbb{R} \to E$	23
$H_{per}^\gamma(0, p)$	Sobolev space of scalar $p-$periodic $\gamma-$differentiable functions	25

Notation	Description	
K, L	Parameters of the state feedback (feedforward) controller	34
P, Q, S	Parameters of the exosystem	17
$P_p(\mathbb{R}, E)$	$p-$periodic functions $\mathbb{R} \to E$	23
$R(\lambda, A)$	The resolvent operator $(\lambda I - A)^{-1}$	14
W	State space of the exosystem	17
X	State space of the error feedback controller	76
Z	State space of the plant	1
$[\&\mathcal{T}_{F,S}]$	The operator $\left(\begin{smallmatrix}\Pi \\ \Lambda\end{smallmatrix}\right) \to \mathcal{T}_{F,S}\Lambda$	126
$[\mathbf{C}\&\mathbf{D}]$	The operator $\left(\begin{smallmatrix}\Pi \\ \Lambda\end{smallmatrix}\right) \to \mathbf{C}\Pi + \mathbf{D}\Lambda$	126
$[\mathcal{T}_{A,S}\&\mathbf{B}]$	The operator $\left(\begin{smallmatrix}\Pi \\ \Lambda\end{smallmatrix}\right) \to \mathcal{T}_{A,S}\Pi + \mathbf{B}\Lambda$	125
$\Lambda(\mathbb{R}, E)$	BUC functions $\mathbb{R} \to E$ with $sp_C(f) \subset \Lambda$	28
$\mathcal{L}(E, H)$	Bounded linear operators $E \to H$	14
δ_0	Point evaluation operator centered at the origin	18
$\det(\cdot)$	Determinant of a square matrix	55
$\dim(\cdot)$	Dimension of a Banach space (possibly ∞)	16
$\mathbf{B}$	The operator $\Lambda \to BJ\Lambda$	125
$\mathbf{C}$	The operator $\Pi \to C\Pi$	125
$\mathbf{D}$	The operator $\Lambda \to DJ\Lambda$	125
$\mathcal{E}_f, \mathcal{H}_f$	The smallest closed translation invariant function space containing f	26
$\mathcal{H}$	A function space which is continuously embedded in $BUC(\mathbb{R}, H)$ and on which the left translation operators are strongly continuous isometries	18
$\mathcal{K}(W)$	Compact linear operators $W \to W$	89
$\mathcal{T}_{A,S}$	The operator $\Pi \to A\Pi - \Pi S$	125
$\mathcal{T}_{A_1, A_2}$	The operator $\Lambda \to A_1\Lambda - \Lambda A_2$	182
$\mathcal{T}_{F,S}$	The operator $\Lambda \to F\Lambda - \Lambda S$	125
$\mathcal{U}_{dist}$	Disturbance signal	1

Notation	Description	
$\mathcal{Z}$	A function space which is continuously embedded in $BUC(\mathbb{R}, Z)$ and on which the left translation operators are strongly continuous isometries	19
$\rho(A)$	Resolvent set of A	14
$\rho_\infty(A)$	Maximal connected component of $\rho(A)$ containing $+\infty$	14
$\sigma(A)$	Spectrum of A	14
$\sigma_A(A)$	Approximate point spectrum of A	14
$\sigma_P(A)$	Point spectrum of A	14
$e(t)$	Tracking error $y(t) - y_{ref}(t)$ at t	34
$sp_B(f)$	Bohr spectrum of f	45
$sp_C(f)$	Carleman spectrum of f	27
$u(t)$	Input of the plant at t	1
$w(t)$	State of the exosystem at t	17
$x(t)$	State of the error feedback controller at t	76
$y(t)$	Output of the plant at t	1
y_{ref}	Reference signal	1
$z(t)$	State of the plant at t	1
$\overset{s}{\hookrightarrow}$	Continuous embedding with left translation operators remaining strongly continuous isometries	18
EFRP	Error feedback output regulation problem	76
Extended regulator equations	Constrained operator Sylvester equations	77
FFRP	Feedforward-feedback output regulation problem	110
FRP	Feedforward output regulation problem	34

Notation	Description	
MIMO	Multi-input multi-output	6
Regulator equations	Constrained operator Sylvester equations	36
SISO	Single-input single-output	6
Sylvester equation	Operator equation of the form $XA_2 = A_1X + A_3$	221

Abstract

In this thesis we develop a state space output regulation theory for linear infinite-dimensional systems and bounded uniformly continuous exogenous reference/disturbance signals. The output regulation problems that we study involve the construction of such controllers which (i) stabilize the closed loop system consisting of the plant and the controller appropriately, (ii) achieve asymptotic tracking of the reference signals and rejection of the disturbance signals, and (iii) preferrably do this robustly with respect to small parameter variations in the control system.

We show how bounded uniformly continuous reference/disturbance signals are best generated using a (possibly infinite-dimensional) exogenous system. This exosystem utilizes a strongly continuous group of isometries on some Banach space and two bounded observation operators. The regulation of all signals in certain Banach subspaces of bounded uniformly continuous functions is shown to be equivalent to the regulation of all signals generated by such exosystems, with a suitable choice of the free parameters.

We conduct an extensive study of three controller configurations — feedforward controllers, error feedback controllers and hybrid feedforward-feedback controllers — for output regulation purposes. In particular, complete characterizations for the solvability of the three output regulation problems are obtained in terms of solutions of certain constrained operator Sylvester equations (regulator equations). We illustrate the abstract results with various examples and case studies, particularly from repetitive control applications.

We study robustness of the devised error feedback controllers using perturbation techniques. We also prove such a state space generalization of the Internal Model Principle which does not utilize any purely finite-dimensional concepts. This result describes the necessary and sufficient structure of all robustly regulating error feedback controllers, under appropriate closed loop stability assumptions.

We introduce the practical output regulation problem in which asymptotic tracking and disturbance rejection with a given accuracy only is required. Using perturbation techniques we present upper bounds for the norms of perturbations to the closed loop control systems' parameters such that practical output regulation with a desired accuracy occurs. Our results treat the above three controller configurations in a unified way.

Finally, we present a general methodology for the solution of the regulator equations in two (separate) cases. In the first case we assume that the plant is a single-input single-output (SISO) system, whereas in the second case we assume that the spectrum of the exosystem's generator is a discrete set. Both of these cases are important in practice, and they cover most of the applications that we have in mind — in particular the repetitive control problems for infinite-dimensional linear systems.

Foreword

This doctoral dissertation collects, unifies and completes the results of a study which I have conducted under the supervision of Professor Seppo Pohjolainen at the Institute of Mathematics, Tampere University of Technology, Finland, during the years 2002-2005. The results and ideas of this thesis are based on my research articles [40, 41, 42, 43, 44, 45, 46, 48, 49, 50, 51, 52, 53, 54], many of which have been written in collaboration with Professor Pohjolainen. The articles [40, 41, 42, 48, 49, 50, 51, 52, 53, 54] have been published (or have been accepted for publication) in peer-reviewed journals or conference proceedings, whereas the manuscripts [43, 44, 45, 46] are currently under review.

It is nowadays very common — and perfectly acceptable — for Ph.D. candidates in Finland to compose the doctoral thesis directly from a brief introduction to the research topic, followed by an appendix collecting the candidate's scientific publications. This approach did not seem adequate in my case, however. In the process of carrying out the research work for this thesis it became apparent that the results should appear in a monograph which is not a mere compilation of the papers cited above. Instead, a book-like treatise written in the distinct Anglo-Saxon way seemed to be necessary. There are two principal reasons for this. On the one hand, the sheer diversity of the results and the evolution of the notation from [51, 53] to e.g. [40, 43, 45, 48] rendered the composition of a concise introduction to the entire topic virtually impossible. On the other hand, I was shocked by the remark, made by one of my seniors at the 2004 annual meeting of the TISE graduate school, that nobody really cares about the doctoral thesis after the candidate's disputation day. If this indeed is the case, then why bother writing one at all? This to-the-point comment made me realize that unless the thesis is to sink into oblivion, there is only one way to write it: Every idea should be presented in such a way which is as easy as possible for the potential readers to grasp, without compromising mathematical rigour of course. The means to an end

here is the Anglo-Saxon approach, which hopefully facilitates the understanding of the results and their application in real-world control problems, although it inevitably results in a rather lengthy exposition of the material. However, in my view this is relatively small a price to pay for an additional insight into certain abstract methods of operator theory and harmonic analysis, which are quite alien in the control engineering community, but which in this thesis are shown to yield new and rather powerful results in output regulation theory.

Raisio, 9.6.2006

Eero Immonen

Chapter 1

Introduction

This thesis presents a state space output regulation theory for linear time-invariant control systems (*plants*) described by the following set of equations in the mild sense:

$$\dot{z}(t) = Az(t) + Bu(t) + \mathcal{U}_{dist}(t), \quad z(0) \in Z, \quad t \geq 0 \tag{1.1a}$$

$$y(t) = Cz(t) + Du(t), \quad t \geq 0 \tag{1.1b}$$

Here A generates a strongly continuous (i.e. C_0-) semigroup $T_A(t)$, $t \geq 0$, on a complex (possibly infinite-dimensional) Banach space Z. The continuous input $u : \mathbb{R}_+ = [0, \infty) \to H$ and the continuous output $y : \mathbb{R}_+ \to H$ take values in a complex (possibly infinite-dimensional) Banach space H, i.e. the output space of the plant (1.1) is the same as its input space. The state of the plant (1.1) is denoted by $z(t)$. The control operator $B \in \mathcal{L}(H, Z)$, the observation operator $C \in \mathcal{L}(Z, H)$ and the feedthrough operator $D \in \mathcal{L}(H)$, where $\mathcal{L}$ denotes bounded linear operators. The bounded uniformly continuous function $\mathcal{U}_{dist} : \mathbb{R} \to Z$ is an external disturbance signal affecting the plant's dynamical behaviour.

In very general terms, the output regulation problems that we study in this thesis involve the construction of such a *controller* for the plant (1.1) which

(*i*) stabilizes the closed loop system consisting of the plant and the controller appropriately;

(*ii*) drives the plant so that the output $y(t)$ asymptotically (as $t \to \infty$) tracks certain bounded uniformly continuous reference signals $y_{ref} : \mathbb{R} \to H$ in spite of the disturbances $\mathcal{U}_{dist}$;

1

(*iii*) preferrably does the above *robustly*, i.e. regardless of certain (small) perturbations to the parameters of the plant and the controller.

Since many physical phenomena — e.g. vibration and heat conduction [17] — and also many industrial processes [67] can be modelled by (possibly infinite-dimensional) linear systems of the form (1.1), it is obvious that output regulation problems of the above type play a prominent role in control theory. In Section 1.1 below we shall provide motivating examples of, as well as the background for, the particular output regulation problems solved in this thesis. On the other hand, Section 1.2 summarizes the contents and the main contributions of the present work, and in Section 1.3 we shall collect some notation and definitions used throughout this thesis.

1.1 Background and motivation

A distinguishing feature in the existing approaches towards the solution of the above output regulation problems is the assumption that the class of reference and/or disturbance signals consists of outputs of some autonomous linear dynamical system. This system is often called an *exogenous system* or *exosystem* in the literature, and many classical control problems can be formulated as an output regulation problem for some *particular* exosystem. For example, in the set-point control problems that occur frequently in applications the constant reference signals to be asymptotically tracked can be considered as outputs of an exosystem described by the differential equation $\dot{w}(t) = 0$, such that $y_{ref}(t) = w(t)$ for all $t \in \mathbb{R}$. A particular constant reference signal is in this case uniquely described by the initial state $w(0)$ of the exosystem. The case of constant disturbance signals can be treated in a completely analogous manner, and hence so can that case in which there are *both* constant reference signals *and* constant disturbance signals to be regulated. It is also easy to enlarge the class of exogenous signals under consideration from constants to linear combinations of sinusoids. However, the actual solution of the corresponding output regulation problem for such a class of exogenous signals is by no means a trivial procedure. The following example combines various parts of the seminal article [12] by C. I. Byrnes, I. Laukó, D. Gilliam and V. Shubov in order to illustrate some of the fundamental issues encountered in the solution of such output regulation problems for infinite-dimensional systems (1.1).

Example 1.1. *Consider a disturbance-free controlled one-dimensional heat equation on the unit*

interval $[0, 1]$ with Neumann boundary conditions, as described by the partial differential equation

$$\frac{\partial z(x,t)}{\partial t} = \frac{\partial^2 z(x,t)}{\partial^2 x} + 2\chi_{[\frac{1}{2},1]}(x)u(t), \tag{1.2a}$$

$$\frac{\partial z(0,t)}{\partial x} = \frac{\partial z(1,t)}{\partial x} = 0, \quad z(x,0) = \psi(x), \tag{1.2b}$$

$$y(t) = \int_0^1 z(x,t)2\chi_{[0,\frac{1}{2}]}(x)dx \tag{1.2c}$$

Here $\chi_{[\epsilon,\delta]}(x)$ denotes the characteristic function of the interval $[\epsilon, \delta]$, i.e. $\chi_{[\epsilon,\delta]}(x) = 1$ if $\epsilon \leq x \leq \delta$ and 0 otherwise. The measured output $y(t)$ represents the average temperature of the heated 1-D rod on the interval $[0, \frac{1}{2}]$, and the problem is to design a control law $u(t)$ for the system (1.2) such that $\lim_{t \to \infty} |y(t) - \sin(2t)| = 0$ for all initial temperature profiles $z(\cdot, 0) = \psi$ of the system (1.2). According to (1.2a) this control law $u(t)$ then specifies how the rod should be heated on the interval $[\frac{1}{2}, 1]$ in order to achieve the desired output behaviour.

The first step in the solution of the above output regulation problem is to formulate the system (1.2) as a plant (1.1). It is well-known (see [12, 17]) that this can be done by choosing $Z = L^2(0,1)$, $H = \mathbb{C}$ and by defining

$$A\psi = \frac{d^2\psi}{dx^2}, \quad \forall \psi \in \mathcal{D}(A) = \{ \psi \in H^2(0,1) \mid \frac{d\psi}{dx}(0) = \frac{d\psi}{dx}(1) = 0 \} \subset Z \tag{1.3}$$

$$Bu = 2\chi_{[\frac{1}{2},1]}(x)u, \quad \forall u \in \mathbb{C} \tag{1.4}$$

$$C\phi = \int_0^1 \phi(x)2\chi_{[0,\frac{1}{2}]}(x)dx, \quad \forall \phi \in Z \tag{1.5}$$

with $D = 0$ and $\mathcal{U}_{dist} = 0$. The operator A is an unbounded, self-adjoint, densely defined linear operator which generates a C_0-semigroup on Z. On the other hand, the operators $B : \mathbb{C} \to Z$ and $C : Z \to \mathbb{C}$ are bounded and linear [12].

The remaining steps in the solution of the output regulation problem are the construction of an exogenous system that can generate the reference signal $y_{ref}(t) = \sin(2t)$ and the construction of a controller which can regulate all signals generated by this exosystem. It is easy to see that the finite-dimensional linear exosystem

$$\dot{w}(t) = Sw(t), \quad S = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}, \quad w(0) \in W = \mathbb{C}^2 \tag{1.6a}$$

$$y_{ref}(t) = Qw(t), \quad Q = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{1.6b}$$

utilized in [12] can generate the desired reference signal. Indeed, for $w(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in W$ we have that $y_{ref}(t) = Qe^{St}w(0) = Qw(t)$ for all $t \in \mathbb{R}$.

As regards the controller design, the following observations were made in [12]:

(i) *The natural first attempt, namely driving the system with the desired output, i.e.* $u(t) = \sin(2t) = Qw(t)$ *for all t, results in a bounded output* $y(t)$ *which does not oscillate about zero. In other words, the plant output* $y(t)$ *contains an undesirable dc-bias which must be eliminated in order to achieve output regulation.*

(ii) *The aforementioned dc-bias in the plant output can be removed by incorporating the (exponentially) stabilizing state feedback term* $Kz(t) = -0.5\langle z(t), \mathbf{1}\rangle_Z$ *in the control law. Here* $\langle \cdot, \cdot \rangle_Z$ *is the inner product on Z and* $\mathbf{1} \in Z$ *is the constant function* $\mathbf{1}(x) = 1$ *for all* $x \in [0,1]$*. Moreover,* $A + BK$ *generates an exponentially stable* C_0-*semigroup* $T_{A+BK}(t)$ *on Z, i.e.* $\|T_{A+BK}(t)\| \le Me^{-\omega t}$ *for some* $M \ge 1, \omega > 0$ *and all* $t \ge 0$*. The control law* $u(t) = Kz(t) + \sin(2t) = Kz(t) + Qw(t)$*, utilizing a stabilizing state feedback and the desired output, results in a plant output* $y(t)$ *which appears to converge to a periodic trajectory oscillating about zero, as desired, but the resulting amplitude and phase are not those of the desired output* $y_{ref}(t) = \sin(2t)$*.*

(iii) *The feedforward part* $Lw(t)$ *of a control law* $u(t) = Kz(t) + Lw(t)$ *which does achieve asymptotic tracking of* $y_{ref}(t) = Qw(t)$ *for any* $w(0) \in W$ *(in particular for* $w(0) = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$*) can be found by solving the so called regulator equations*

$$\Pi S = A\Pi + B\Gamma \tag{1.7a}$$

$$C\Pi = Q \tag{1.7b}$$

for bounded linear operators $\Pi \in \mathcal{L}(W, Z)$ *and* $\Gamma \in \mathcal{L}(W, \mathbb{C})$*, such that* $\operatorname{ran}(\Pi) \subset \mathcal{D}(A)$*. In this particular example we can take* $L = \Gamma - K\Pi$ *where K is the above exponentially stabilizing state feedback operator, and the operators* $\Pi = \left(\Pi_1(\cdot)\ \Pi_2(\cdot) \right)$*,* $\Gamma = \left(\gamma_1\ \gamma_2 \right) = \left(\frac{\Re(H(i2))}{|H(i2)|^2}\ -\frac{\Im(H(i2))}{|H(i2)|^2} \right)$ *are defined using*

$$\Pi_1(x) = \gamma_1\Re([R(i2, A)B](x)) - \gamma_2\Im([R(i2, A)B](x)), \quad \forall x \in [0,1] \tag{1.8}$$

$$\Pi_2(x) = \gamma_1\Im([R(i2, A)B](x)) + \gamma_2\Re([R(i2, A)B](x)), \quad \forall x \in [0,1] \tag{1.9}$$

$$H(i2) = \frac{2\sinh(\frac{\sqrt{i2}}{2})}{i2\sqrt{i2}\cosh(\frac{\sqrt{i2}}{2})} \tag{1.10}$$

with $[R(i2, A)B](x) = \frac{\cosh(\sqrt{i2}x)}{i2\cosh(\frac{\sqrt{i2}}{2})}$ *for* $0 \le x \le \frac{1}{2}$ *and* $[R(i2, A)B](x) = \frac{2}{i2} - \frac{\cosh(\sqrt{i2}x)}{i2\cosh(\frac{\sqrt{i2}}{2})}$ *otherwise.*

(iv) If the state of the plant is not available for measurement but the tracking error $e(t) = y(t) - y_{ref}(t) = Cz(t) - Qw(t)$ is available for measurement, then a dynamic *error feedback controller*

$$\dot{x}(t) = Fx(t) + Ge(t), \quad x(0) \in X, \quad t \geq 0 \tag{1.11a}$$

$$u(t) = Jx(t) \tag{1.11b}$$

on the state space $X = L^2(0, 1) \times \mathbb{C}^2$ (in the mild sense) can be utilized. If we take

$$F = \begin{pmatrix} A + BK - G_1 C & B(\Gamma - K\Pi) + G_1 Q \\ -G_2 C & S + G_2 Q \end{pmatrix}, \quad G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}, \quad J = \begin{pmatrix} K & \Gamma - K\Pi \end{pmatrix}$$
$$\tag{1.12}$$

where $(G_1 u)(x) = 1.9u\mathbf{1}(x)$ for all $u \in \mathbb{C}$ and $0 \leq x \leq 1$, $G_2 = \begin{pmatrix} -3 \\ -3 \end{pmatrix}$, and the other operators as in the above, then the closed loop system operator $\mathcal{A} = \begin{pmatrix} A & BJ \\ GC & F \end{pmatrix}$ consisting of the plant and the controller (1.11) (with the exosystem (1.6) detached) generates an exponentially stable C_0-semigroup on $Z \times X$. Moreover, all reference signals $Qw(t)$, for $w(0) \in W$, and in particular $y_{ref}(t) = \sin(2t)$ if $w(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in W$, can be asymptotically tracked regardless of the initial states $z(0) \in Z$ and $x(0) \in X$ of the plant and the controller (1.11).

Alas, in many output regulation problems encountered in practice it is not sufficient to be able to regulate sinusoidal signals — or their linear combinations — only. Instead, a more realistic goal is to be able to asymptotically track and/or reject *general* periodic signals of some fixed period length $p > 0$. For example, these general periodic signals can be repetitive commands for mechanical systems such as industrial robots, or they can be periodic disturbances arising from rotational motion. Unfortunately, vast majority of the otherwise very useful recent output regulation results for infinite-dimensional systems, e.g. those in [12, 33], are not applicable in this situation. This is because they utilize *finite-dimensional* exosystems described by linear matrix differential equations, as in Example 1.1. Clearly such exosystems can only generate those periodic signals which are linear combinations of sinusoids, i.e. trigonometric polynomials.

In the linear case an *infinite-dimensional* exogenous system *must* be used if such periodic signals which have an infinite number of distinct frequency components are to be generated [47, 55]. The so-called *repetitive control* scheme addresses the problem of asymptotically tracking arbitrary (but sufficiently regular) $p-$periodic reference signals, as generated by a particular infinite-dimensional exosystem, for *finite-dimensional* plants. This scheme employs frequency domain methods and it

has proven to be quite effective in practice; see e.g. [36, 92, 96, 95] and the references therein. We next illustrate the principles of repetitive control and the related design challenges for single-input single-output (SISO) systems.

According to [36] any sufficiently regular scalar-valued $p-$periodic reference signal y_{ref} can be generated by an infinite-dimensional exosystem including a time-lag element corresponding to the period length $p > 0$, with an appropriate initial function y_{ref}^1 corresponding to a period of the desired signal; see Figure 1.1 and [36, 96]. A closed loop error feedback control system *incorporating*
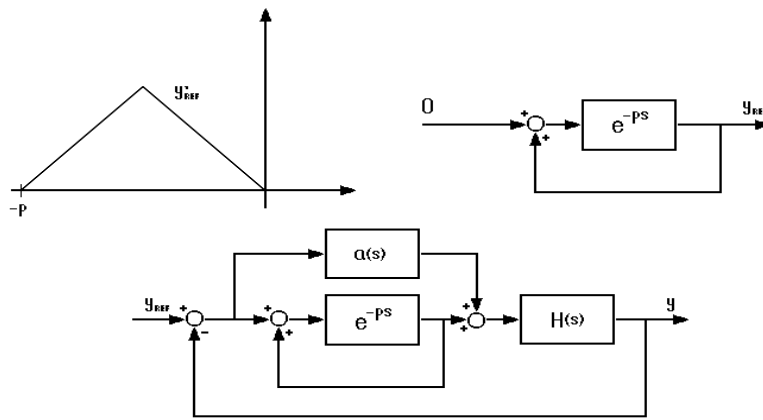
Figure 1.1: The initial state y_{ref}^1 of the exosystem generating a $p-$periodic reference signal y_{ref} (top left), the generator of y_{ref} (top right), and a repetitive control system (bottom).

this generator of $p-$periodic signals is called a repetitive control system [36]. Figure 1.1 depicts a repetitive control system where $H(s)$ is the transfer function of a SISO linear time-invariant finite-dimensional plant and $a(s)$ is an appropriate proper stable rational function ($a(s) = 0$ is also possible [96]). The following genuinely positive results about the repetitive control system of Figure 1.1 are well-known:

- Assume that $[1 + a(s)H(s)]^{-1}H(s)$ is a proper stable rational transfer function and that $\sup_{\omega \in \mathbb{R}} \left| [1 + a(i\omega)H(i\omega)]^{-1} [1 + (a(i\omega) - 1)H(i\omega)] \right| < 1$. Then the minimal state space realization of the closed loop repetitive control system in Figure 1.1 is exponentially stable and $\lim_{t \to \infty} e(t) = \lim_{t \to \infty} [y(t) - y_{ref}(t)] = 0$ for all continuous $p-$periodic reference signals y_{ref} (Theorem 1 in [36]).

- If the above assumptions hold for $a(s) = 0$, then $e(t) = y(t) - y_{ref}(t)$ tends to 0 exponentially for all continuous $p-$periodic reference signals y_{ref} (Theorem 5.9 in [96]).

However, it turns out that the assumptions of these positive results are notoriously difficult to satisfy in practice. In particular, the following negative result is well-known:

- If $H(s)$ is a strictly proper rational transfer function, then the closed loop repetitive control system in Figure 1.1 (i.e. its minimal state space realization) cannot be exponentially stable (Proposition 2 in [36]).

This general lack of exponential stabilizability of closed loop repetitive control systems also generalizes for multi-input multi-output (MIMO) systems [36]. It is known to be solely caused by the infinite number of poles on the imaginary axis, resulting from the incorporation of the above generator of $p-$periodic signals in the closed loop system [36, 96]. Unfortunately, in a sense it is also *necessary* to incorporate this generator of $p-$periodic signals in any stable closed loop error feedback control system achieving asymptotic tracking of $p-$periodic functions, as indicated by the following generalization of the Internal Model Principle[1] of Francis and Wonham [32]:

- Any pseudorational SISO unity feedback servo control system (see Figure 1.2 and [96]) which is internally stable (i.e. the minimal state space realization is exponentially stable), and for which $e(t) = y(t) - y_{ref}(t)$ tends to 0 exponentially for any $p-$periodic reference signal y_{ref}, must contain the *internal model* $(e^{ps} - 1)^{-1}$ of the dynamical behaviour of the reference signals in the forward path of the closed loop system (Theorem 5.12 in [96]).

In the repetitive control literature the above dilemma is principally resolved by considering the so-called *modified* repetitive control scheme [36, 92]. In this scheme the above exact internal model $(e^{ps} - 1)^{-1}$ is combined with a low-pass filter to facilitate exponential closed loop stabilization — and hence also output regulation. Unfortunately, perfect output regulation is lost in the process as the low-pass filter moves the high frequency poles of the internal model away from the imaginary axis to the closed left-half complex plane [92].

Fortunately, the above dilemma is only related to the lack of *exponential* stabilizability of closed loop *error feedback* repetitive control systems. Consequently, it is reasonable to inquire

---

[1]For finite-dimensional systems (1.1) and finite-dimensional exosystems this celebrated principle describes the necessary and sufficient structure of robust (i.e. structurally stable) regulating error feedback controllers.
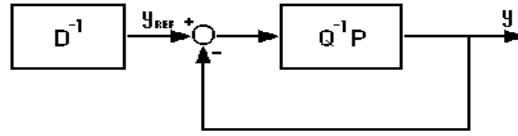
Figure 1.2: A pseudorational unity feedback servo system. Here D, P and Q are Laplace transforms of compactly supported distributions, with supports in $(-\infty, 0]$, satisfying the conditions of pseudorationality; they are entire functions satisfying certain growth estimates [95, 96].

whether the regulation of periodic signals is more generally possible if simpler open loop (i.e. feedforward) control and/or some weaker notion of closed loop stability are utilized. Since the existing repetitive control results only cover finite-dimensional plants [36, 92, 96, 95], it would be particularly important to establish realistic and general conditions under which output regulation of general periodic signals can also be achieved for *infinite-dimensional* plants. These open problems were the initial motivation for the research described in the present thesis; we conclude this section by elaborating some more on the possibilities and the difficulties related to solving them.

The state space output regulation theory developed by Byrnes et al. in [12] makes it possible to utilize both error feedback controllers and simple open loop (feedforward) controllers for the solution of certain output regulation problems for infinite-dimensional systems. As in item (*iii*) of Example 1.1, for their feedforward controllers only exponential stabilizability of the plant — and *not* the exogenous signal generator — is required; it is a separate feedforward control law ($Lw(t)$ in item (*iii*) of Example 1.1) that makes output regulation possible in this case. However, the results of [12] only apply to signals generated by certain finite-dimensional linear exosystems, and it is not at all trivial to generalize these results for infinite-dimensional linear exogenous systems. Moreover, Byrnes et al. [12] do not address the issue of robustness, even in the case of error feedback control (which is well-known to yield nice robustess properties in the case of finite-dimensional plants [29]). These problems are compounded by the fact that prior to attempting a generalization of the results of [12] for general periodic reference/disturbance signals, one should first identify the simplest possible infinite-dimensional exosystem generating $p-$periodic signals in the state space domain. To the author's knowledge little research on this topic has been reported in the literature [47, 55]; in particular, the repetitive control scheme is mostly based on frequency

domain techniques.

As regards closed loop stability, in the state space domain there is a notion of *strong stability* whose application in repetitive control problems has apparently not been studied before. A C_0-semigroup $T_A(t)$ on a Banach space Z is called strongly stable if $\lim_{t \to \infty} T_A(t)z = 0$ for all $z \in Z$. Since no uniform decay rate for $\|T_A(t)z\|$ needs to exist, strong stability is a considerably milder requirement for a C_0-semigroup than exponential stability. However, especially from the point of view of systems theory, the concept of strong stability of a C_0-semigroup is also considerably less well-understood than that of exponential stability. This is quite unfortunate because it seems to be a most natural stability notion for many systems described by partial differential equations [70]. The problem is that, unlike in the exponentially stable case, there seems to be no hope for a general duality between state space and frequency domain methods for such systems (1.1) where the operator A only generates a strongly stable C_0-semigroup [2, 17, 70, 96]. Consequently, it is not at all trivial to establish state space generalizations for the frequency domain repetitive control results of [36, 92, 95, 96], by utilizing the concept of a strongly stable C_0-semigroup. This problem is compounded by the fact that the theory of strongly stable Banach space C_0-semigroups is a branch of abstract harmonic analysis, whose methods are occasionally quite involved from the application point of view.

1.2 Contributions and organization

As was mentioned in Section 1.1, the initial motivation for the research described in this thesis was to generalize the results of [12] and [36, 92, 95, 96] in an effort to develop an output regulation theory for infinite-dimensional systems (1.1) and general periodic exogenous signals, by utilizing state space methods and the concept of a strongly stable C_0-semigroup. However, it turned out to be possible to achieve somewhat more, as indicated in the following list of the main contributions of this thesis:

- The construction and a detailed analysis of the *simplest possible* exogenous systems generating arbitrary *bounded uniformly continuous* reference/disturbance signals.

- The construction and a detailed analysis of three controllers (feedforward, error feedback and hybrid feedforward-error feedback) for the asymptotic tracking/rejection of the exogenous sig-

nals generated by the above-mentioned exosystems, including complete characterizations for the solvability of the related output regulation problems in terms of the corresponding regulator equations, and including repetitive control *even for strictly proper infinite-dimensional plants*.

- A detailed analysis of the stabilizability properties of the utilized exogenous systems, and a detailed analysis of the strong stabilization of the closed loop system for two error feedback controllers.

- A robustness (i.e. structural stability) analysis for the designed error feedback controllers, including such a state space generalization of the Internal Model Principle [32] which does not utilize any purely finite-dimensional concepts.

- The development of the mathematical foundations of practical output regulation, i.e. *approximate* asymptotic tracking/rejection of the exogenous signals generated by the above-mentioned exosystems.

- A general treatment of the solution of the regulator equations for infinite-dimensional systems (1.1) and the above-mentioned (possibly infinite-dimensional) exosystems.

- The presentation of several examples and case studies which illustrate, among other things, the new discovery made in this thesis that the *smoothness* of the exogenous signals crucially affects the solvability of the output regulation problem at hand, and also contributes to the robustness properties of the devised controllers.

We point out, however, that the theory of this thesis is *not applicable*

- for unbounded reference/disturbance signals (e.g. polynomials),

- in those point control and point observation problems which would require the use of *unbounded* operators B and/or C in the plant (1.1).

From the practical point of view, the requirement for boundedness of the exogenous signals is not at all restrictive. However, allowing for unbounded operators B and C in the plant (1.1) would notably enlarge the class of systems to which the output regulation theory of this thesis is applicable (see e.g. [84] and the references therein). Since the (possible) infinite-dimensionality

of the exosystem and the (possible) lack of exponential stabilizability of the closed loop system already introduce considerable mathematical complexity to the output regulation problems studied in the present work, allowing for unbounded B and C seemed unneccessary at this stage, because it would complicate the mathematics even more.

In the remainder of this section we shall provide a brief summary of the contents of this thesis and we shall point out the author's scientific publications contributing to the related parts of the thesis. More detailed comparison to the related earlier work will be provided at the beginning of each individual chapter.

Chapter 2: The exogenous system

We shall construct and analyse in detail the simplest possible exosystems generating various classes of bounded uniformly continuous exogenous reference/disturbance signals. These exosystems always utilize the generator of an isometric C_0-group on some Banach space W, plus one bounded observation operator for the reference signals and one for the disturbance signals. While conventionally the exogenous signals are assumed to be generated by some arbitrary (finite-dimensional) exosystem [12], here we devise a new and completely opposite approach: Given a class of exogenous signals we want to construct (i.e. realize) the exosystem such that precisely the desired class of signals is generated. The results of this chapter are based on those in [40, 41, 42, 43, 45, 46, 48, 49, 50, 51, 52, 53, 54].

Chapter 3: Feedforward output regulation theory

We shall pose and completely solve a feedforward output regulation problem (FRP) for infinite-dimensional plants (1.1) and the exosystems of Chapter 2. We shall generalize the results of e.g. [12, 29, 72] by completely characterizing the solvability of the FRP in terms of strong stabilizability of the plant and the solvability of the regulator equations (3.10). Various applications and case studies — including repetitive control, asymptotic tracking of individual signals and the effect of system zeros on output regulation — as well as illustrative examples are presented. The results of this chapter are based on those in [41, 42, 46, 49, 51, 52, 53, 54].

Chapter 4: Error feedback output regulation theory

We shall pose and completely solve an error feedback output regulation problem (EFRP) for infinite-dimensional plants (1.1) and the exosystems of Chapter 2. We shall generalize the results of e.g. [12, 24, 29, 32, 33, 36, 80, 87, 92, 95, 96] by completely characterizing the solvability of the EFRP in terms of strong stabilizability of the closed loop system consisting of the plant and the controller, and in terms of the solvability of the extended regulator equations (4.3). We shall also show that under certain assumptions it is necessary for output regulation that the exosystem's dynamics is embedded in the controller's dynamics; this is well-known in the finite-dimensional case [29]. Finally, we shall consider in detail the strong stabilization of the closed loop system for certain infinite-dimensional generalizations of two widely used finite-dimensional controllers. This turns out to be a delicate problem for infinite-dimensional exosystems. The results of this chapter are based on those in [41, 42, 48, 49].

Chapter 5: A feedforward-error feedback controller

In order to avoid the difficulties encountered in the strong stabilization of the closed loop error feedback control systems (Chapter 4), we shall introduce a hybrid controller utilizing both error feedback and feedforward control for output regulation purposes. This controller is designed in such a way that the two parts are completely independent of each other: The error feedback part is designed to stabilize the closed loop system appropriately, whereas the feedforward part is tuned, using the regulator equations (3.10), to achieve output regulation. We shall completely characterize the existence of a feedforward-error feedback controller, and we shall present an example of its construction. To the author's knowledge the results of this chapter are new even for finite-dimensional systems; they are based on those in [40].

Chapter 6: Robustness and the internal model structure

We shall study the robustness (structural stability) of the error feedback controllers designed in Chapter 4. Our results show that the unique solvability of the extended regulator equations (4.3) implies a degree of conditional robustness in output regulation. Here conditional robustness of output regulation means robustness *on condition that* the closed loop stability is also robust; this weak robustness notion is needed because strong stability of the closed loop system is not in

general robust with respect to the perturbations that we consider. We shall also generalize the Internal Model Principle [32] for infinite-dimensional systems (1.1) and the exosystems of Chapter 2 using state space techniques, without relying on any purely finite-dimensional concepts. Our results (and methods) are new even for finite-dimensional systems. In particular they improve and simplify those in [29], and they present sufficient conditions for robustness of the error feedback controllers utilized in [12]. The results of this chapter are based on those in [43, 45].

Chapter 7: Practical output regulation

We shall develop the mathematical foundations of practical output regulation, i.e. *approximate* asymptotic tracking/rejection of the signals generated by the exosystems of Chapter 2. Our idea is to directly employ a perturbation analysis to an exactly regulating closed loop control system and the corresponding (extended) regulator equations which in a certain sense describe its *steady state* behaviour. Thus, the main results of this chapter are general upper bounds for the norms of additive, bounded, linear perturbations to the the parameters of the plant, the exosystem and the (hypothetical) controller, which solves the corresponding exact output regulation problem, such that practical output regulation with a given accuracy occurs. Our results cover practical state space output regulation for the controllers devised in Chapters 3, 4 and 5 in a unified way; to our knowledge these results — and the methods utilized in their proofs — are new even for finite-dimensional systems. These results should be particularly useful in the construction of finite-dimensional approximations for the controllers designed in the previous chapters. The results of this chapter are based on those in [44, 50].

Chapter 8 : Solving the regulator equations

We shall study the solution of the regulator equations (3.10); the same methods apply with obvious changes for the extended regulator equations (4.3) too. We shall consider the following separate cases: That of a SISO plant and that in which the spectrum of the exosystem generator is a discrete set (this occurs e.g. in all repetitive control problems). The latter case can then be easily modified to cover another separate case in which the spectrum of the exosystem generator is not discrete but the signals are known to be in certain Banach spaces that we construct in Chapter 2. The results of this chapter are extensively used in various examples throughout this

thesis. They generalize the well-known finite-dimensional results (see e.g. [12]) which show that the nonexistence of transmission zeros on the spectrum of the exosystem generator implies the solvability of the regulator equations (3.10). The results of this chapter are based on those in [40, 41, 54].

Chapter 9 : Conclusions

We shall draw the conclusions and we shall present some interesting directions for future investigations.

Appendix A

We shall collect some well-known results from spectral theory of closed operators, the theory of Sylvester operator equations and the theory of bi-continuous semigroups, for the reader's convenience.

1.3   Notation and definitions

For Banach spaces E and F, $\mathcal{L}(E, F)$ denotes the space of bounded linear operators $E \to F$. If E is continuously embedded in F, then this is denoted by $E \hookrightarrow F$. The resolvent set of a closed linear operator $A : E \to F$ is denoted by $\rho(A)$, whereas $\sigma(A)$ denotes its spectrum. The maximal connected component of $\rho(A)$ containing $+\infty$ is denoted by $\rho_\infty(A)$. The point spectrum of A is denoted by $\sigma_P(A)$, whereas $\sigma_A(A)$ denotes the approximate point spectrum of A. $R(\lambda, A)$ denotes (whenever it exists) the resolvent operator $(\lambda I - A)^{-1}$. If $\widetilde{E}$ is a subspace of E, then $A|_{\widetilde{E}}$ denotes the restriction of A to $\widetilde{E}$. If $A : E \to E$ generates a strongly continuous (or C_0-) semigroup in E, then this semigroup is in general denoted by $T_A(t)$. We say that $T_A(t)$ is strongly stable if $\lim_{t \to \infty} \|T_A(t)x\| = 0$ for each $x \in E$. The semigroup $T_A(t)$ is exponentially stable if $\|T_A(t)\| \leq Me^{-\omega t}$ for some $M, \omega > 0$ and all $t \geq 0$. For the sake of brevity in these circumstances we sometimes say that A is strongly/exponentially stable. The norm on E is denoted by $\|\cdot\|_E$ and if E is a Hilbert space, then its inner product is denoted by $\langle \cdot, \cdot \rangle_E$. Here the subscript E is occasionally included to clarify the space on which the norm or the inner product is defined. If $T_A(t)$ is a C_0-semigroup on E and $F \hookrightarrow E$, then we say that F is invariant for $T_A(t)$ (or

$T_A(t)-$invariant) if $T_A(t)x \in F$ for all $x \in F$. The real part of a complex number z is denoted by

$\Re(z)$, and the imaginary part of z is denoted by $\Im(z)$.

Chapter 2

The exogenous system

As stated in Chapter 1, it is a common — and mathematically convenient — assumption in the output regulation literature that the reference signals y_{ref} and disturbance signals $\mathcal{U}_{dist}$ are generated by a so called exogenous system. Quite often this exosystem is assumed to be a linear, finite-dimensional, neutrally stable[1] autonomous system of the form

$$\dot{w}(t) = Sw(t), \quad w(0) \in W, \tag{2.1a}$$

$$y_{ref}(t) = Qw(t) \tag{2.1b}$$

$$\mathcal{U}_{dist}(t) = Pw(t) \tag{2.1c}$$

where P, Q and S are matrices of appropriate dimensions and $\dim(W) < \infty$ (see e.g. [12, 24, 29, 32, 35, 80] and the references therein). Neutral stability of the exosystem (2.1) implies that the eigenvalues of S are simple (S has no nontrivial Jordan blocks) and that they are located on the imaginary axis $i\mathbb{R}$ [12]. Consequently the reference/disturbance signals that can be generated by the exosystem (2.1) are constants or trigonometric polynomials (see Definition 2.8).

Unfortunately many interesting signals which occur in practice are not constants or trigono- metric polynomials; they are general bounded and uniformly continuous functions. In this thesis we shall develop a state space output regulation theory for such signals by utilizing the following (possibly infinite-dimensional) exosystem, which is a direct generalization of the finite-dimensional exosystem (2.1):

---

[1]According to [12] this is equivalent to the origin being Lyapunov stable forward and backward in time.

Definition 2.1 (The exogenous system)**.** Let W be any Banach space such that S generates an isometric[2] C_0-group $T_S(t)$ on W. Let $P \in \mathcal{L}(W, Z)$ and let $Q \in \mathcal{L}(W, H)$. Then the exogenous system is described by the following set of equations in the mild sense:

$$\dot{w}(t) = Sw(t), \quad w(0) \in W \tag{2.2a}$$

$$y_{ref}(t) = Qw(t), \quad t \in \mathbb{R} \tag{2.2b}$$

$$\mathcal{U}_{dist}(t) = Pw(t), \quad t \in \mathbb{R} \tag{2.2c}$$

Remark 2.2. The free parameters of the exosystem are the state space W, the operators P, Q and S, and the initial state $w(0) \in W$. However, in output regulation problems we often fix W, P, Q and S at the outset. Then different reference/disturbance signals are generated by varying the initial state $w(0) \in W$.

It is clear that the reference and disturbance signals generated by the exosystem (2.2) are indeed always bounded and uniformly continuous functions, because P and Q are bounded operators and because $T_S(t)$ is an isometric C_0-group. A remarkable property of the general exogenous system (2.2) is that also the converse holds: We shall show in this chapter that *all* bounded uniformly continuous reference/disturbance signals can be generated using the above exosystem by an appropriate choice of the free parameters. Moreover, it turns out to be possible to choose the exosystem's free parameters in such a way that the generated reference/disturbance signals are in prespecified subspaces of bounded uniformly continuous functions. This is a very useful feature of the general exosystem (2.2), because then solvability of an abstract output regulation problem implies that all exogenous signals in the prespecified function spaces can be regulated. In particular, we can pose — and precisely answer — questions such as: What classes of bounded uniformly continuous reference signals can the output of a given plant (1.1) asymptotically track?

We now briefly outline the contents of this chapter.

Section 2.1: We shall show how the exosystem (2.2) is best constructed in order to accomplish the generation of reference/disturbance signals in prespecified (possibly infinite-dimensional) function spaces. The novel feature of our approach is that while in most of the related literature, e.g. [10, 11, 12, 33, 87], the chosen exogenous system fixes the class of exogenous signals, here the chosen class of exogenous signals determines the simplest possible exosystem capable

---

[2] The C_0-group $T_S(t)$ is isometric if $\|T_S(t)w\| = \|w\|$ for all $t \in \mathbb{R}$.

of generating the desired signals. The methods of this section have been developed in the author's articles [40, 41, 42, 43, 45, 46, 48, 49, 50, 51, 52, 53, 54].

Section 2.2: We shall introduce some interesting function spaces whose elements can be generated by the exosystem (2.2) if its parameters are chosen as indicated in Section 2.1. These function spaces — and the related exosystems (2.2) — are used in various examples throughout this thesis. A special case is an exogenous system which can generate $p-$periodic signals, i.e. a state space analogue of the exosystems utilized in the repetitive control literature [36, 92, 95, 96]. However, we point out that, in contrast to [36, 92, 95, 96], the exosystems that we utilize also make it possible to specify the required degree of *smoothness* of the periodic exogenous signals. This additional feature will have an enormous impact on the solvability of certain repetitive control problems later in this thesis.

2.1   Generation of prespecified spaces of signals

In this section we shall show how the free parameters of the exosystem (2.2) should be chosen in order to accomplish the generation of prespecified subspaces of bounded uniformly continuous signals. The reader is advised that in the subsequent chapters output regulation theory is often formulated for the general exosystem (2.2) only, while interesting applications of this theory can be directly found by fixing the exosystem's free parameters as indicated in this section and Section 2.2. Consequently, the reader should bear in mind that the remainder of this chapter principally consists of useful *choices* for the parameters of the exosystem (2.2).

2.1.1   Generation of reference signals in a prespecified space $\mathcal{H}$

Suppose that we want to generate those reference signals y_{ref} which are in a Banach space $\mathcal{H} \hookrightarrow BUC(\mathbb{R}, H)$ using the exosystem (2.2). This notation means that $\mathcal{H}$ is continuously embedded in the space of bounded uniformly continuous $H-$valued functions on $\mathbb{R}$ endowed with the $\sup-$norm ($\mathcal{H}$ need not have the $\sup-$norm). Let us also assume that $\mathcal{H} \overset{s}{\hookrightarrow} BUC(\mathbb{R}, H)$, which means that, in addition to $\mathcal{H} \hookrightarrow BUC(\mathbb{R}, H)$, the left translation operators $T_S(t)|_{\mathcal{H}}$ defined by $T_S(t)|_{\mathcal{H}} f = f(\cdot + t)$ for every $f \in \mathcal{H}$ constitute an isometric C_0-group on $\mathcal{H}$. Denote by $S|_{\mathcal{H}}$ the infinitesimal generator of $T_S(t)|_{\mathcal{H}}$. Of course, $S|_{\mathcal{H}}$ is just the differential operator $\frac{d}{dx}$ with a suitable domain of definition $\mathcal{D}(S|_{\mathcal{H}}) \subset \mathcal{H}$. Now let us fix the exosystem's free parameters as follows: The state space $W = \mathcal{H}$,

the operator $S = S|_{\mathcal{H}}$, $Q = \delta_0$ is the point evaluation operator defined by $\delta_0 f = f(0)$ for each $f \in \mathcal{H}$, $P \in \mathcal{L}(\mathcal{H}, Z)$ is arbitrary and the initial state $w(0)$ varies in $\mathcal{H}$. Clearly $\delta_0 \in \mathcal{L}(\mathcal{H}, H)$ because $\|\delta_0 f\|_H = \|f(0)\|_H \leq \|f\|_\infty \leq c\|f\|_{\mathcal{H}}$ for some $c \geq 0$ since $\mathcal{H} \hookrightarrow BUC(\mathbb{R}, H)$. Moreover, the following result is obvious.

Proposition 2.3. *Let $\mathcal{H}$ be as in the above. The exogenous system (2.2), with $W = \mathcal{H}$, $S = S|_{\mathcal{H}}$, $Q = \delta_0 \in \mathcal{L}(\mathcal{H}, H)$, $P \in \mathcal{L}(\mathcal{H}, Z)$ and $w(0) \in \mathcal{H}$, can generate all reference functions in $\mathcal{H}$ and only those. Moreover, every reference function $y_{ref} \in \mathcal{H}$ is generated by the choice $w(0) = y_{ref} \in \mathcal{H}$.*

Proof. Clearly $\delta_0 T_S(t)|_{\mathcal{H}} f = f(x + t)|_{x=0} = f(t)$ for every $f \in \mathcal{H}$ and $t \in \mathbb{R}$. This shows that the system (2.2) can generate every reference function $y_{ref} \in \mathcal{H}$ (and only those) by the choice $w(0) = y_{ref} \in \mathcal{H}$ of the initial state. $\square$

Proposition 2.3 shows how to fix the exosystem's free parameters to obtain the *simplest* possible generator for the reference signals in $\mathcal{H}$, in the sense that no superfluous reference signals (i.e. those outside of $\mathcal{H}$) are generated.

2.1.2 Generation of disturbance signals in a prespecified space $\mathcal{Z}$

It is clear that Proposition 2.3 only reveals how the exosystem's free parameters should be chosen if the class of *reference* signals in question is known and if the *disturbance* signals are only required to have dynamical properties which are similar to those of the reference signals. In this case the disturbance signals are implicitly assumed to play a less prominent role than the reference signals; P is only required to be a bounded linear operator $\mathcal{H} \to Z$. On the other hand, based on the construction of Subsection 2.1.1 it is quite apparent how the exosystem's free parameters should be chosen if, instead, we are interested in asymptotically rejecting some prespecified space of disturbance signals $\mathcal{Z} \overset{s}{\hookrightarrow} BUC(\mathbb{R}, Z)$ and if the reference signals are only required to have similar dynamical properties as the disturbances:

Proposition 2.4. *The exogenous system (2.2), with $W = \mathcal{Z} \overset{s}{\hookrightarrow} BUC(\mathbb{R}, Z)$, $S = S|_{\mathcal{Z}}$, $Q \in \mathcal{L}(\mathcal{Z}, H)$, $P = \delta_0 \in \mathcal{L}(\mathcal{Z}, Z)$ and $w(0) \in \mathcal{Z}$, can generate all disturbance functions in $\mathcal{Z}$ and only those. Moreover, every disturbance function $\mathcal{U}_{dist} \in \mathcal{Z}$ is generated by the choice $w(0) = \mathcal{U}_{dist} \in \mathcal{Z}$.*

2.1.3 Generation of reference signals in $\mathcal{H}$ and disturbance signals in $\mathcal{Z}$

It is now obvious that the above construction employing the translation group and point evaluation at the origin can be further extended to the case in which *both* reference signals *and* disturbance signals are in some prespecified spaces $\mathcal{H} \stackrel{s}{\hookrightarrow} BUC(\mathbb{R}, H)$ and $\mathcal{Z} \stackrel{s}{\hookrightarrow} BUC(\mathbb{R}, Z)$ respectively. In this case we can let[3] $W = \mathcal{H} \times \mathcal{Z}$ and define $T_S(t)\left(\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} T_S(t)|_{\mathcal{H}} f_1 \\ T_S(t)|_{\mathcal{Z}} f_2 \end{smallmatrix}\right)$ for all $\left(\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right) \in \mathcal{H} \times \mathcal{Z}$ and for all $t \in \mathbb{R}$. Then $T_S(t)$ is a strongly continuous and isometric group of linear operators on W. Moreover, it is obvious that we should let $Q = \left(\begin{smallmatrix} \delta_0 & 0 \end{smallmatrix}\right) \in \mathcal{L}(W, H)$ and $P = \left(\begin{smallmatrix} 0 & \delta_0 \end{smallmatrix}\right) \in \mathcal{L}(W, Z)$ to generate any given combination $\left(\begin{smallmatrix} y_{ref} \\ u_{dist} \end{smallmatrix}\right) \in \mathcal{H} \times \mathcal{Z}$ of reference and disturbance signals. In particular, we have:

Proposition 2.5. *Let $\mathcal{H}$ and $\mathcal{Z}$ be as in the above. The exogenous system (2.2), with $W = \mathcal{Y} = \mathcal{H} \times \mathcal{Z}$, $S = \left(\begin{smallmatrix} S|_{\mathcal{H}} & 0 \\ 0 & S|_{\mathcal{Z}} \end{smallmatrix}\right)$ (with $\mathcal{D}(S) = \mathcal{D}(S|_{\mathcal{H}}) \times \mathcal{D}(S|_{\mathcal{Z}})$), $Q = \left(\begin{smallmatrix} \delta_0 & 0 \end{smallmatrix}\right) \in \mathcal{L}(W, H)$ and $P = \left(\begin{smallmatrix} 0 & \delta_0 \end{smallmatrix}\right) \in \mathcal{L}(W, Z)$ and $w(0) \in W$, can generate all combinations of reference signals in $\mathcal{H}$ and disturbance signals in $\mathcal{Z}$, and only those. Moreover, every combination $\left(\begin{smallmatrix} y_{ref} \\ u_{dist} \end{smallmatrix}\right) \in \mathcal{Y}$ of reference/disturbance signals is generated by the choice $w(0) = \left(\begin{smallmatrix} y_{ref} \\ u_{dist} \end{smallmatrix}\right) \in \mathcal{Y}$.*

In the remainder of this thesis script capital letters $\mathcal{H}$ and $\mathcal{Z}$ always refer to some Banach function spaces of the above type. In addition, $S|_{\mathcal{H}}$ and $S|_{\mathcal{Z}}$ refer to the infinitesimal generators of the (strongly continuous and isometric) left translation groups $T_S(t)|_{\mathcal{H}}$ and $T_S(t)|_{\mathcal{Z}}$ on $\mathcal{H}$ and $\mathcal{Z}$ respectively.

2.1.4 Some additional remarks about generality of the exosystems

We have seen in Proposition 2.3, Proposition 2.4 and Proposition 2.5 that exogenous systems employing the left translation operators and point evaluations at the origin on suitable function spaces are the simplest possible in terms of signals that can be generated. Now we shall conclude this section by showing that such exosystems are also the most general ones in the following sense:

- No additional generality is achieved by considering exosystems (2.2) where S need not generate a translation group and neither P nor Q need be the point evaluation operator δ_0. In fact, the exosystem (2.2) can always be embedded in some exosystem described in Proposition 2.5.

---

[3]We can endow W e.g. with the norm $\left\| \left(\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right) \right\| = \sqrt{\|f_1\|_{\mathcal{H}}^2 + \|f_2\|_{\mathcal{Z}}^2}$ to obtain a Banach space.

- The state space W in Proposition 2.3, Proposition 2.4 and Proposition 2.5 cannot be any larger if we want the translation group to be strongly continuous. Since our main tool in this thesis is the theory of C_0-semigroups, we shall not be able to consider more general bounded exogenous signals than those which are also uniformly continuous.

Proposition 2.6. *For every Banach space W, every generator S of a bounded C_0-group $T_S(t)$, every $Q \in \mathcal{L}(W, H)$ and every $P \in \mathcal{L}(W, Z)$ there exists a closed (left-) translation invariant subspace $\mathcal{Y} \subset BUC(\mathbb{R}, H) \times BUC(\mathbb{R}, Z)$ such that[4] for every $w \in W$ we have $\left(\begin{smallmatrix} y_{ref} \\ \mathcal{U}_{dist} \end{smallmatrix}\right) = \left(\begin{smallmatrix} QT_S(\cdot)w \\ PT_S(\cdot)w \end{smallmatrix}\right) \in \mathcal{Y}$ and*

$$y_{ref}(t) = QT_S(t)w = \left(\begin{smallmatrix} \delta_0 & 0 \end{smallmatrix}\right)T_S(t)|_{\mathcal{Y}}\left(\begin{smallmatrix} y_{ref} \\ \mathcal{U}_{dist} \end{smallmatrix}\right) = \left(\begin{smallmatrix} \delta_0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} y_{ref}(\cdot+t) \\ \mathcal{U}_{dist}(\cdot+t) \end{smallmatrix}\right) \tag{2.3}$$

$$\mathcal{U}_{dist}(t) = PT_S(t)w = \left(\begin{smallmatrix} 0 & \delta_0 \end{smallmatrix}\right)T_S(t)|_{\mathcal{Y}}\left(\begin{smallmatrix} y_{ref} \\ \mathcal{U}_{dist} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & \delta_0 \end{smallmatrix}\right)\left(\begin{smallmatrix} y_{ref}(\cdot+t) \\ \mathcal{U}_{dist}(\cdot+t) \end{smallmatrix}\right) \tag{2.4}$$

Proof. Let $\mathcal{Y} = \overline{\mathrm{span}}\{ QT_S(\cdot)w_0 \mid w_0 \in W \} \times \overline{\mathrm{span}}\{ PT_S(\cdot)w_0 \mid w_0 \in W \} = \mathcal{H} \times \mathcal{Z}$, where both closures are taken with respect to the $\sup-$norm. It is clear that $\mathcal{Y} \subset BUC(\mathbb{R}, H) \times BUC(\mathbb{R}, Z)$, because $T_S(t)$ is a bounded C_0-group, and because $P \in \mathcal{L}(W, Z)$ and $Q \in \mathcal{L}(W, H)$. Since

$$T_S(t)|_{\mathcal{Y}} \begin{pmatrix} QT_S(\cdot)w_0 \\ PT_S(\cdot)w_0 \end{pmatrix} = \begin{pmatrix} QT_S(\cdot + t)w_0 \\ PT_S(\cdot + t)w_0 \end{pmatrix} = \begin{pmatrix} QT_S(\cdot)T(t)w_0 \\ PT_S(\cdot)T(t)w_0 \end{pmatrix} \in \mathcal{Y} \quad \text{for all } w_0 \in W \tag{2.5}$$

by continuity and linearity the space $\mathcal{Y}$ is closed and translation invariant. As a consequence of this, $T_S(t)|_{\mathcal{Y}}$ is an isometric C_0-group on $\mathcal{Y}$ [28]. Let $y_{ref}(t) = QT_S(t)w$ and $\mathcal{U}_{dist}(t) = PT_S(t)w$ for all $t \in \mathbb{R}$ and some arbitrary $w \in W$. Then $\left(\begin{smallmatrix} y_{ref} \\ \mathcal{U}_{dist} \end{smallmatrix}\right) \in \mathcal{Y}$ and $\left(\begin{smallmatrix} y_{ref}(t) \\ \mathcal{U}_{dist}(t) \end{smallmatrix}\right) = \left(\begin{smallmatrix} \delta_0 T_S(t)|_{\mathcal{H}} y_{ref} \\ \delta_0 T_S(t)|_{\mathcal{Z}} \mathcal{U}_{dist} \end{smallmatrix}\right) = \left(\begin{smallmatrix} \delta_0 & \delta_0 \end{smallmatrix}\right)T_S(t)|_{\mathcal{Y}}\left(\begin{smallmatrix} QT_S(\cdot)w \\ PT_S(\cdot)w \end{smallmatrix}\right) = \left(\begin{smallmatrix} \delta_0 & \delta_0 \end{smallmatrix}\right)T_S(t)|_{\mathcal{Y}}\left(\begin{smallmatrix} y_{ref} \\ \mathcal{U}_{dist} \end{smallmatrix}\right)$, which proves the result. $\square$

Proposition 2.7. *Let E be a Banach space and let $\mathcal{E} \hookrightarrow L^\infty(\mathbb{R}, E)$. If the left translation operators $T_S(t)|_{\mathcal{E}}$ constitute a C_0-group on $\mathcal{E}$, then $\mathcal{E} \subset BUC(\mathbb{R}, E)$.*

Proof. We only have to verify that every $f \in \mathcal{E} \hookrightarrow L^\infty(\mathbb{R}, E)$ is uniformly continuous. Clearly for every such f we have that $\sup_{s \in \mathbb{R}} \|f(s + t) - f(s)\| = \|T_S(t)|_{\mathcal{E}} f - f\|_\infty \leq c\|T_S(t)|_{\mathcal{E}} f - f\|_{\mathcal{E}} \to 0$ as $t \to 0$, because $\mathcal{E} \hookrightarrow L^\infty(\mathbb{R}, E)$ and because $T_S(t)|_{\mathcal{E}}$ is strongly continuous. This shows that $f \in BUC(\mathbb{R}, E)$. $\square$

---

[4]Here translation invariance on this product of function spaces is of course understood componentwise.

2.2 Some practically relevant exogenous signal spaces

In Section 2.1 we showed how the free parameters of the exosystem (2.2) should be chosen in order to generate signals in function spaces which are suitably embedded in related spaces of bounded uniformly continuous functions. In the present section we shall provide some interesting examples of such function spaces of exogenous signals. We have encountered three interesting scenarios; (i) the class of reference signals $\mathcal{H}$ is prespecified, (ii) the class of disturbance signals $\mathcal{Z}$ is prespecified, and (iii) both $\mathcal{H}$ and $\mathcal{Z}$ are prespecified. To cover all of these three possibilities, below we shall consider general Banach spaces $\mathcal{E} \stackrel{s}{\hookrightarrow} BUC(\mathbb{R}, E)$ where E is any Banach space. We point out that although many of the function spaces $\mathcal{E}$ that we consider below are well known in the harmonic analysis literature (see e.g. [2, 38, 58]), they seem to be less known in the control theory literature.

2.2.1 Spaces of almost periodic and periodic signals

Many reference/disturbance signals which occur in practice are periodic functions or, more generally, almost periodic functions. There are several equivalent definitions for almost periodic functions (see [2, 38]), but we choose to give the following one from [2] which is quite intuitive and which illustrates the relation between trigonometric polynomials, periodic functions and almost periodic functions.

Definition 2.8. A trigonometric polynomial in $BUC(\mathbb{R}, E)$ is a function $p_N : \mathbb{R} \to E : t \to p_N(t) = \sum_{n=1}^{N} a_n e^{i\lambda_n t}$ where $N < \infty$, and $a_n \in E$ and $\lambda_n \in \mathbb{R}$ for each $1 \leq n \leq N$. A function $f : \mathbb{R} \to E$ is called almost periodic if it can be approximated uniformly (i.e. in the $\sup-$norm) on $\mathbb{R}$ by trigonometric polynomials. We let $AP(\mathbb{R}, E)$ denote the linear space of almost periodic functions $\mathbb{R} \to E$.

Proposition 2.9. *The space $\mathcal{E} = AP(\mathbb{R}, E)$ is a closed translation invariant subspace of $BUC(\mathbb{R}, E)$ and $T_S(t)|_{\mathcal{E}}$ constitutes an isometric C_0-group on $\mathcal{E}$.*

Proof. Since $AP(\mathbb{R}, E)$ is the $\sup-$norm closure of the linear space of trigonometric polynomials in $BUC(\mathbb{R}, E)$, it is a closed subspace of $BUC(\mathbb{R}, E)$. On the other hand, if a sequence $(p_n)_{n \in \mathbb{N}}$ of trigonometric polynomials uniformly approximates $f \in AP(\mathbb{R}, E)$, then for every $t \in \mathbb{R}$ the sequence $(T_S(t)|_{AP(\mathbb{R}, E)} p_n)_{n \in \mathbb{N}}$ of trigonometric polynomials uniformly approximates $T_S(t)|_{AP(\mathbb{R}, E)} f$. The

result follows by Subsection I.5.12 in [28]. □

Example 2.10. *The function $f : t \to \sin(t) + \sin(\sqrt{2}t)$ is in $AP(\mathbb{R}, \mathbb{R})$. However f is not periodic. The reader is referred to [15] for more details.*

Example 2.11. *It is clear that whenever $\dim(E) < \infty$ and whenever the index set I is finite, the Banach space $\mathcal{E} = \mathrm{span}\{ e^{i\omega_n \cdot} y \mid y \in E, n \in I \} \overset{s}{\hookrightarrow} AP(\mathbb{R}, E) \subset BUC(\mathbb{R}, E)$ if we endow $\mathcal{E}$ with the supremum norm. In this case $\mathcal{E}$ is a finite-dimensional space consisting of trigonometric polynomials (including possibly constant functions). Moreover, the differential operator $S|_{\mathcal{E}}$ can be represented by a diagonal matrix. Signals in such spaces $\mathcal{E}$ are precisely those which can be generated using finite-dimensional exosystems (2.1).*

Example 2.12. *The space $P_p(\mathbb{R}, E)$ of all $p-$periodic $E-$valued continuous functions (endowed with the sup-norm) is a closed translation invariant subspace of $AP(\mathbb{R}, E)$ [2]. Hence clearly $\mathcal{E} = P_p(\mathbb{R}, E) \overset{s}{\hookrightarrow} BUC(\mathbb{R}, E)$ if we endow $\mathcal{E}$ with the sup-norm. We emphasize that $AP(\mathbb{R}, E)$ contains all continuous periodic functions $\mathbb{R} \to E$, regardless of the period length.*

The following example shows that $AP(\mathbb{R}, E)$ does not necessarily consitute all of $BUC(\mathbb{R}, E)$.

Example 2.13. *Let $E = c$, the Banach space of all convergent complex sequences $x = (x_n)_{n \in \mathbb{N}}$ with the supremum norm $\|x\| = \sup_{n \in \mathbb{N}} |x_n|$. Consider the function $t \to f(t) = (e^{i\frac{t}{n}})_{n \in \mathbb{N}}$ for all $t \in \mathbb{R}$. Then $f \in BUC(\mathbb{R}, E)$, but $f \notin AP(\mathbb{R}, E)$. We refer the reader to Example 4.6.5 in [2] for more details.*

Following [41], we shall next introduce a useful scale $H_{AP}(E, f_n, \omega_n)$ of generalized Sobolev spaces of almost periodic signals.

Definition 2.14. Let $I \subset \mathbb{Z}$, let $(\omega_n)_{n \in I} \subset \mathbb{R}$ be a sequence of distinct frequencies and let $(f_n)_{n \in I} \subset \mathbb{R}$ such that $f_n \geq 1$ for each $n \in I$ and $(f_n^{-1})_{n \in I} \in \ell^2$. The generalized Sobolev space $H_{AP}(E, f_n, \omega_n)$ of $E-$valued functions is defined as

$$\Big\{ u : \mathbb{R} \to E \mid u(t) = \sum_{n \in I} a_n e^{i\omega_n t} \text{ for each } t \in \mathbb{R} \text{ and } \sum_{n \in I} |f_n|^2 \|a_n\|_E^2 < \infty \text{ and } (a_n)_{n \in I} \subset E \Big\}$$

(2.6)

The linear operations of addition and scalar multiplication on $H_{AP}(E, f_n, \omega_n)$ are defined in the obvious way.

Remark 2.15. In Definition 2.14 it is possible that the sequence $(\omega_n)_{n\in I}$ has also other points of accumulation besides $\pm i\infty$, so that it need not consist of isolated points only. On the other hand, the index set I may also be finite; in this case $H_{AP}(E, f_n, \omega_n)$ is a finite-dimensional space whenever $\dim(E) < \infty$.

Proposition 2.16. *For all sequences $(f_n)_{n\in I}$ and $(\omega_n)_{n\in I}$ as in Definition 2.14 we have that $H_{AP}(E, f_n, \omega_n) \subset AP(\mathbb{R}, E)$. Moreover, $H_{AP}(E, f_n, \omega_n)$ is a Banach space with the norm $\|u\| = \sqrt{\sum_{n\in I} \|\hat{u}(n)\|_E^2 |f_n|^2}$. If E is a Hilbert space, then $H_{AP}(E, f_n, \omega_n)$ is a Hilbert space with the inner product $\langle u, v \rangle = \sum_{n\in I} \langle \hat{u}(n), \hat{v}(n) \rangle_E |f_n|^2$. Here $u(t) = \sum_{n\in I} \hat{u}(n) e^{i\omega_n t}$ and $v(t) = \sum_{n\in I} \hat{v}(n) e^{i\omega_n t}$ for every $t \in \mathbb{R}$.*

Proof. That $H_{AP}(E, f_n, \omega_n) \subset AP(\mathbb{R}, E)$ is evident because any function $\sum_{n\in I} a_n e^{i\omega_n \cdot}$ in the Sobolev space $H_{AP}(E, f_n, \omega_n)$ can be uniformly approximated by trigonometric polynomials $p_N = \sum_{|n|\leq N} a_n e^{i\omega_n \cdot}$. Moreover, it is easy to see that $H_{AP}(E, f_n, \omega_n)$ is a normed space (or inner product space if E is Hilbert space) with the norm $\|\cdot\|$. The completeness follows from the properties of the so-called Fourier-Bohr transformation. In fact, $H_{AP}(E, f_n, \omega_n)$ is the preimage of $\ell^2(E, f_n, \omega_n)$, the weighted (with weight $(f_n)_{n\in I}$) ℓ^2−space of all vector-valued sequences on $(\omega_n)_{n\in I}$, under the Fourier-Bohr transformation, with the norm inherited from that ℓ^2−space; see [41] and also [2, 63] for more details on the Fourier-Bohr transformation. $\square$

Clearly almost periodic functions in $H_{AP}(E, f_n, \omega_n)$ have the useful property that the coefficients in an approximation by trigonometric polynomials converge. Moreover, these generalized Sobolev spaces are suitable for the state space of the exosystem (2.2):

Theorem 2.17. $\mathcal{E} = H_{AP}(E, f_n, \omega_n) \overset{s}{\hookrightarrow} BUC(\mathbb{R}, E)$ *for all sequences $(f_n)_{n\in I}$ and $(\omega_n)_{n\in I}$ as in Definition 2.14.*

Proof. Let $u \in \mathcal{E}$ be arbitrary, such that $u(t) = \sum_{n\in I} \hat{u}(n) e^{i\omega_n t}$ for all $t \in \mathbb{R}$. Then by the Schwartz inequality we have

$$\sup_{t\in\mathbb{R}} \|u(t)\|_E = \sup_{t\in\mathbb{R}} \left\| \sum_{n\in I} \hat{u}(n) e^{i\omega_n t} \right\|_E \leq \sum_{n\in I} \|\hat{u}(n)\|_E \tag{2.7}$$

$$\leq \sqrt{\sum_{n\in I} \|\hat{u}(n)\|_E^2 |f_n^2|} \sqrt{\sum_{n\in I} |f_n^{-2}|} = c\|u\|_{\mathcal{E}} \tag{2.8}$$

for $c = \sqrt{\sum_{n\in I} |f_n^{-2}|} < \infty$. Hence $\mathcal{E} \hookrightarrow BUC(\mathbb{R}, \mathbb{C})$.

It remains to show that the left translation operators $T_S(t)|_{\mathcal{E}}$ constitute an isometric C_0-group on $\mathcal{E}$. The semigroup property is elementary to verify. Moreover, for every $t \in \mathbb{R}$ we have $u(\cdot + t) = \sum_{n \in I} \hat{u}(n) e^{i\omega_n(\cdot + t)} = \sum_{n \in I} \hat{u}(n) e^{i\omega_n t} e^{i\omega_n \cdot}$. Then $\|T_S(t)|_{\mathcal{E}} u\|_{\mathcal{E}}^2 = \sum_{n \in I} |f_n|^2 \|\hat{u}(n) e^{i\omega_n t}\|_E^2 = \sum_{n \in I} |f_n|^2 \|\hat{u}(n)\|_E^2 = \|u\|_{\mathcal{E}}^2$ for each $t \in \mathbb{R}$, so that the space $\mathcal{E}$ is invariant for $T_S(t)|_{BUC(\mathbb{R},E)}$. This also shows that $T_S(t)|_{\mathcal{E}}$ is an isometry for each $t \in \mathbb{R}$.

We shall prove strong continuity of $T_S(t)|_{\mathcal{E}}$ to establish the result. To this end let $\epsilon > 0$. Let $K = \sup_{0 \leq t \leq 1, n \in I} |e^{i\omega_n t} - 1|^2$. Then there exists $N \in \mathbb{N}$ such that $\sum_{|n| \geq N} \|\hat{u}(n)\|_E^2 |f_n|^2 < \frac{\epsilon^2}{2K}$ and there exists $t_0 > 0$ such that for $0 < t < t_0$ it is true that $\sup_{|n| < N} |e^{i\omega_n t} - 1|^2 < \frac{\epsilon^2}{2(1+\|u\|_{\mathcal{E}}^2)}$. We may then estimate for $0 < t < t_0$ as follows

$$\|T_S(t)|_{\mathcal{E}} u - u\|_{\mathcal{E}}^2 = \sum_{n \in I} |e^{i\omega_n t} - 1|^2 \|\hat{u}(n)\|_E^2 |f_n|^2 \tag{2.9}$$

$$= \sum_{|n| < N} |e^{i\omega_n t} - 1|^2 \|\hat{u}(n)\|_E^2 |f_n|^2 \tag{2.10}$$

$$+ \sum_{|n| \geq N} |e^{i\omega_n t} - 1|^2 \|\hat{u}(n)\|_E^2 |f_n|^2 \tag{2.11}$$

$$\leq \sup_{|n| < N} |e^{i\omega_n t} - 1|^2 \sum_{|n| < N} \|\hat{u}(n)\|_E^2 |f_n|^2 \tag{2.12}$$

$$+ K \sum_{|n| \geq N} \|\hat{u}(n)\|_E^2 |f_n|^2 \tag{2.13}$$

$$< \frac{\epsilon^2}{2} + \frac{\epsilon^2}{2} = \epsilon^2 \tag{2.14}$$

This shows that $T_S(t)|_{\mathcal{E}}$ is strongly continuous and the proof is complete. $\qquad\square$

The following subclass of the spaces $H_{AP}(\mathbb{C}, f_n, \omega_n)$, consisting of p-periodic functions only, is used as an illustrative example in many parts of this thesis.

Definition 2.18. Let $I \subset \mathbb{Z}$, let $p > 0$ and let $\omega_n = \frac{2\pi n}{p}$ for all $n \in I$. Let $(f_n)_{n \in I} \subset \mathbb{R}$ such that $f_n \geq 1$ for each $n \in I$ and $(f_n^{-1})_{n \in I} \in \ell^2$. The Sobolev space $H(f_n, \omega_n) = \left\{ u : \mathbb{R} \to \mathbb{C} \mid u(t) = \sum_{n \in I} a_n e^{i\omega_n t} \text{ for each } t \in \mathbb{R}, \sum_{n \in I} |f_n|^2 |a_n|^2 < \infty \text{ and } (a_n)_{n \in I} \subset \mathbb{C} \right\}$.

Remark 2.19. If $I = \mathbb{Z}$, $\gamma > \frac{1}{2}$, $p > 0$, $\omega_n = \frac{2\pi n}{p}$ and $f_n = \sqrt{1 + \omega_n^2}^{\gamma}$ for each $n \in \mathbb{Z}$, then $H(f_n, \omega_n)$ reduces to the standard Sobolev space $H_{per}^{\gamma}(0, p)$ of $\gamma-$differentiable $p-$periodic functions [56].

Remark 2.20. Whenever $\mathcal{E}$ is any one of the Sobolev spaces $H(f_n, \omega_n)$, the generator $S|_{\mathcal{E}}$ of the translation C_0-group on $\mathcal{E}$ is a Riesz spectral operator in the sense of Curtain and Zwart [17]. A Riesz spectral operator is a closed linear operator on a Hilbert space with the following properties: It has simple eigenvalues $\{\,\lambda_n \mid n \in I\,\}$, $I \subset \mathbb{Z}$, such that the closure $\overline{\{\,\lambda_n \mid n \in I\,\}}$ is totally disconnected, and the corresponding eigenvectors $(\psi_n)_{n \in I}$ constitute a Riesz basis. In the case of Sobolev spaces $H(f_n, \omega_n)$ the eigenvalues of $S|_{\mathcal{E}}$ are just the complex frequencies $i\omega_n$, $n \in I$, and the Riesz basis $(\psi_n)_{n \in I}$ is in fact an orthonormal basis of weighted exponentials. Moreover, $[S|_{\mathcal{E}} u](t) = \sum_{n \in I} i\omega_n \langle u, \psi_n \rangle \psi_n(t) = \sum_{n \in I} i\omega_n \hat{u}(n) e^{i\omega_n t}$ whenever $u \in \mathcal{D}(S|_{\mathcal{E}})$.

2.2.2 The smallest closed translation invariant space containing a given signal

In some practical situations it may be desirable to regulate some given *individual* signal $f \in \mathcal{E}$ only. Hence an important question is: What is the simplest exosystem employing Proposition 2.3, Proposition 2.4 or Proposition 2.5 which is capable of generating at least the signal f? The following result answers this question.

Proposition 2.21. *Let $f \in \mathcal{E} \stackrel{s}{\hookrightarrow} BUC(\mathbb{R}, E)$. Then the restriction of $T_S(t)|_{\mathcal{E}}$ to the space $\mathcal{E}_f = \overline{\operatorname{span}}\{\,T_S(t)|_{\mathcal{E}} f \mid t \in \mathbb{R}\,\}$ (closure in $\mathcal{E}$) constitutes an isometric C_0-group. Moreover, $\mathcal{E}_f$ is the smallest closed $T_S(t)|_{\mathcal{E}}-$invariant subspace of $\mathcal{E}$ containing f.*

Proof. It is clear that if we can establish that $\mathcal{E}_f$ is the smallest closed $T_S(t)|_{\mathcal{E}}-$invariant subspace of $\mathcal{E}$ containing f, then the restriction of $T_S(t)|_{\mathcal{E}}$ to the space $\mathcal{E}_f$ constitutes an isometric C_0-group (see Subsection I.5.12 in [28]). But it is evident by construction that $\mathcal{E}_f$ is a closed $T_S(t)|_{\mathcal{E}}-$invariant subspace contained in $\mathcal{E}$ such that $f \in \mathcal{E}_f$. On the other hand, the space $\operatorname{span}\{\,T_S(t)|_{\mathcal{E}} f \mid t \in \mathbb{R}\,\}$ must be contained in every closed translation invariant subspace of $\mathcal{E}$ which contains f. Hence also $\mathcal{E}_f = \overline{\operatorname{span}}\{\,T_S(t)f \mid t \in \mathbb{R}\,\}$ must be contained in every such space. $\qquad\square$

According to Proposition 2.21 the state space W of the exosystem (2.2) employing Proposition 2.3, Proposition 2.4 or Proposition 2.5 should be chosen to be $\mathcal{E}_f$, if only $f \in \mathcal{E}$ is to be generated. By the above result we can make the following useful convention.

Definition 2.22. We say that a given signal $f \in \mathcal{E}$ can be regulated if and only if all signals in $\mathcal{E}_f$ can be regulated.

For example, by this convention a given reference signal $y_{ref} \in \mathcal{H}$ can be asymptotically tracked if and only if every reference signal in the function space $\mathcal{H}_{y_{ref}}$ can be asymptotically tracked. This convention is natural, especially so long as a linear exogenous system (2.2) is used: The regulatability of a given signal is equivalent — by definition — to the regulatability of all signals generated by the simplest exosystem (2.2) capable of generating the given signal. We point out that if for some fixed $Q \in \mathcal{L}(W, H)$ and $w \in W$ *some* exosystem (2.2) generates $y_{ref} = QT_S(\cdot)w \in \mathcal{H}$, then by only varying the initial state $w \in W$, the *same* exosystem can always generate at least the reference functions in $\mathrm{span}\{T_S(t)|_{\mathcal{H}}y_{ref} \mid t \in \mathbb{R}\}$ which is always a dense subspace of $\mathcal{H}_{y_{ref}}$. On the other hand, if this exosystem is also observable in the sense that $QT_S(\cdot)w_n \to f \in \mathcal{H}$ implies $w_n \to w \in W$, then it can generate all reference functions in $\mathcal{H}_{y_{ref}}$. In fact, in this case every $y \in \mathcal{H}_{y_{ref}}$ is of the form $QT_S(\cdot)w$ for some $w \in W$.

Remark 2.23. Although the Banach space $\mathcal{E}$ need not in general be separable, for all $f \in \mathcal{E}$ the space $\mathcal{E}_f$ of Proposition 2.21 is always separable (cf. Proposition 4.3.11 in [2]). This feature may be important in some applications (see e.g. Theorem 4.22).

2.2.3 Signals having a prespecified spectral content

Some interesting and useful function spaces $\mathcal{E} \stackrel{s}{\hookrightarrow} BUC(\mathbb{R}, E)$ can also be constructed by restricting the spectral properties of bounded uniformly continuous functions. In order to do this, we shall need the concept of Carleman spectrum [2, 38, 90]:

Definition 2.24. The Carleman transform of an arbitrary $f \in BUC(\mathbb{R}, E)$ is defined by

$$\widetilde{f}(\lambda) = \begin{cases} \int_0^\infty e^{-\lambda t} f(t)dt, & \Re(\lambda) > 0 \\ -\int_{-\infty}^0 e^{-\lambda t} f(t)dt, & \Re(\lambda) < 0 \end{cases} \tag{2.15}$$

A point $\lambda_0 \in \mathbb{R}$ is called regular point of f if $\widetilde{f}$ can be continued analytically into a neighbourhood of $i\lambda_0$. The complement in $\mathbb{R}$ of the set of regular points is called the Carleman spectrum of f and it is denoted by $sp_C(f)$.

We remark that $sp_C(\cdot)$ is a subset of $\mathbb{R}$, but in this thesis we shall also employ its transformation to $i\mathbb{R}$ which is induced by the map $\lambda \to i\lambda$.

Remark 2.25. By Lemma 4.6.8 in [2] for every $f \in BUC(\mathbb{R}, E)$ we have that $isp_C(f) = \sigma(S|_{\mathcal{E}_f})$. Here $\mathcal{E}_f$ is as in Proposition 2.21 with closure taken in the sup-norm.

Proposition 2.26. *Let $\Lambda \subset \mathbb{R}$ be a closed set. Then $\mathcal{E} = \Lambda(\mathbb{R}, E) = \{ f \in BUC(\mathbb{R}, E) \mid sp_C(f) \subset \Lambda \}$, is a closed translation invariant subspace of $BUC(\mathbb{R}, E)$ and $T_S(t)|_{\mathcal{E}}$ constitutes an isometric C_0-group on $\mathcal{E}$.*

Proof. This result is just Corollary 1.3 in [38]. □

Example 2.27. *If $\Lambda = \{ \omega_n \mid n \in I \}$ is a finite set of distinct frequencies, then $\Lambda(\mathbb{R}, E)$ consists of trigonometric polynomials only (cf. Theorem 4.8.7 and Corollary 4.5.9 in [2]).*

Example 2.28. *If $\Lambda \subset \mathbb{R}$ is a closed countable set and if E does not contain an isomorphic copy of c_0 (the Banach space of numerical sequences converging to 0)[5], then $\Lambda(\mathbb{R}, E) \subset AP(\mathbb{R}, E)$ (see e.g. Example 4 in [90]).*

Remark 2.29. The spaces $\Lambda(\mathbb{R}, E)$ are useful in the construction of approximations for bounded uniformly continuous functions $\mathbb{R} \to E$. In fact, by Remark 2.25, Proposition A.1 and Lemma A.8, for every $f \in \mathcal{E} \stackrel{s}{\hookrightarrow} BUC(\mathbb{R}, E)$ and every $\epsilon > 0$ there exists $n \in \mathbb{N}$ and $f_n \in \mathcal{E}$ such that $sp_C(f_n) \subset [-n, n]$ and $\|f - f_n\| < \epsilon$. Furthermore, according to Lemma A.8, this approximation f_n of f can be generated by an exosystem (2.2) where the operator S is *bounded* (although in general W is still infinite-dimensional). As we shall see in the subsequent chapters, the output regulation problems that we study are considerably easier to solve if S is a bounded operator. In practice such approximations with bounded operators S can be constructed via convolutions by suitable summability kernels; see [58] and Proposition A.9. In the scalar case ($E = \mathbb{C}$), for example, $sp_C(\mu K(\mu x) * f) \subset [-\mu, \mu]$ for each $\mu > 0$. Here K denotes the Fejér kernel (see e.g. [58] p. 159).

---

[5]This is the case, for example, whenever E is reflexive [2].

Chapter 3

Feedforward output regulation

In this chapter we shall introduce and completely solve a relatively simple feedforward output regulation problem (FRP) using the exosystem (2.2). The simplicity of this problem is based on the open loop structure of the controller: Assuming that the plant has been appropriately stabilized by a state feedback, our goal is to find an input to the plant (i.e. a feedforward control) such that its steady state output is equal to a prespecified reference function, regardless of the disturbance.

Although it is well-known that such feedforward controllers do not in general lead to a robust (i.e. structurally stable) design, they have received noticeable attention in the literature during the past three decades. Besides the aforementioned simplicity, a key motivation for their study is the observation that more complex — and more realistic — error feedback regulation problems can often be formulated as feedforward regulation problems for certain extended systems (see Chapter 4 for details). Moreover, there is some practical benefit in the use of feedforward controllers: Control action is immediately taken with the onset of a disturbance, whereas with error feedback controllers control action is not taken until there is a change in the outputs of the system [20]. Finally, we are interested in feedforward controllers because, in contrast to the classical repetitive control scheme (see Chapter 1), it turns out that for such open loop controllers the stabilization of the closed loop system is not an overwhelming problem even if the exogenous signals were general $p-$periodic functions.

For finite-dimensional linear systems, feedforward output regulation problems were studied in the 1970s e.g. by Davison, Francis, Wonham and others (see e.g. [20, 22, 25, 29] and the references therein). In particular, the regulator equations characterizing the solvability of the feedforward

regulation problem for appropriately stabilized finite-dimensional plants were first[1] presented by Francis in [29]. Similar methods were used by Davison in [25] in the construction of feedforward controllers for output regulation purposes. In addition to this, the effect of transmission zeros on the existence of regulating feedforward controllers were studied in [25]. Roughly stated, in the case of stable finite-dimensional plants and (unstable) finite-dimensional exosystems (2.1), a regulating feedforward controller exists provided that there are no transmission zeros on the spectrum of the system operator of the exosystem. For plants (1.1) with $\dim(H) < \infty$ this, by definition, means that the transfer function $H(s)$ of the plant should be invertible for every $s \in \sigma(S)$, i.e. $\det(H(s)) \neq 0$ for all $s \in \sigma(S)$ [12].

Many authors have also constructed feedforward controllers for infinite-dimensional systems and finite-dimensional exosystems, e.g. Pohjolainen [72] and Byrnes et al. [12]. The finite-dimensional feedforward regulation theory of Davison [20, 22] was generalized in [72] for exponentially stable systems, under the additional assumption that the system operator A of the plant generates an analytic C_0-semigroup. Subsequently, in [12] Byrnes et al. generalized the finite-dimensional feedforward regulation theory of Francis [29] for infinite-dimensional plants having zero feedthrough, i.e. $D = 0$. For exponentially stabilizable systems they proved a complete characterization for the existence (and construction) of a regulating feedforward controller in terms of solvability of the regulator equations. Furthermore, for square plants Byrnes et al. [12] established that under the exponential stabilizability assumption (and under certain additional assumptions about the spectra $\sigma(A)$ and $\sigma(S)$) a feedforward output regulation problem is solvable regardless of the choice of the matrices P and Q in the exosystem (2.1) if and only if there are no transmission zeros on $\sigma(S)$.

In this chapter we shall generalize the feedforward regulation theory of infinite-dimensional systems and finite-dimensional exosystems in [12, 72] to allow for bounded uniformly continuous reference/disturbance signals generated by the possibly infinite-dimensional exogenous system (2.2). We shall also not require the pair (A, B) in the plant to be exponentially stabilizable; in certain cases it is actually sufficient that the pair (A, B) is only weakly stabilizable. It should be pointed out, however, that Pohjolainen [72] allows for a degree of unboundedness in the operators B and C; this is not the case here.

We next review the contents of this chapter in more detail, and we shall more precisely indicate

---

[1]It should be pointed out, though, that the key Lemma 1 in [29] which leads to the regulator equations is due to W. M. Wonham.

the respective contributions of this thesis.

Section 3.1: We shall define the feedforward regulation problem FRP. This is the same problem as in [12], except that we treat general bounded uniformly continuous reference/disturbance functions under weaker stabilizability assumptions, and we allow for $D \neq 0$ in the plant.

Section 3.2: We shall show that if the pair (A, B) in the plant is strongly stabilizable using $K \in \mathcal{L}(Z, H)$ and if the regulator equations (3.10) due to Francis [29] (see also [12]) have a solution (Π, Γ), then the FRP can be solved. Morever, the control law $u(t) = Kz(t) + (\Gamma - K\Pi)w(t)$ solves the FRP. The main result of this section, Theorem 3.6, was essentially proved for finite-dimensional exosystems (2.1) in Theorem IV.1 of [12] under the additional assumptions that (A, B) is exponentially stabilizable and $D = 0$. The results of this section are based on those in [46, 51, 54]. We point out that the actual solution of the regulator equations (3.10) is studied later in this thesis; see Chapter 8 and Section 3.5.

Section 3.3: We shall investigate whether the solvability of the regulator equations (3.10) is also necessary for the solvability of the FRP. In order to do this, we shall first define regularity of an operator $\Delta \in \mathcal{L}(W, Z)$ for a C_0−semigroup $T_A(t)$ and characterize this property of Δ by the solvability of the Sylvester operator equation $\Pi S = A\Pi + \Delta$ in $\mathcal{D}(S)$. Thereafter, we shall define, and present examples of, exosystems (2.2) which generate so called admissible reference signals. Roughly stated, if the exosystem (2.2) generates admissible reference signals, then we do not try to asymptotically track signals which vanish at $+\infty$. Our main result (Theorem 3.16) is: If a control law $u(t) = Kz(t) + Lw(t)$ solves the FRP, if $BL + P \in \mathcal{L}(W, Z)$ is regular for $T_{A+BK}(t)$ and if the exosystem (2.2) generates admissible reference signals, then the regulator equations (3.10) *necessarily* have a solution (Π, Γ) and $L = \Gamma - K\Pi$. If $A + BK$ also generates an exponentially stable C_0−semigroup, then the regularity of $BL + P$ for $T_{A+BK}(t)$ need not be explicitly required. If, in addition, output regulation is exponentially fast (i.e. $\|e(t)\|$ decays exponentially), then the exosystem need not even generate admissible reference signals. Together with the results of Section 3.2 these results provide various generalizations for Theorem IV.1 in [12] for bounded uniformly continuous reference/disturbance signals and plants which are not necessarily exponentially stabilizable. Moreover, D may be nonzero here. The results of this section are based on those in [41, 42, 49], and they completely describe the extent to which it is necessary and sufficient for the solvability of the FRP to

be able to solve the regulator equations (3.10).

Section 3.4: As a case study we shall consider regulation of a single given function $f \in \mathcal{E} \stackrel{s}{\hookrightarrow} BUC(\mathbb{R}, E)$ for certain Banach spaces E and $\mathcal{E}$ as in Chapter 2. According to the convention made in Definition 2.22 this amounts to solving the FRP for $W = \mathcal{E}_f = \overline{\text{span}}\{ T_S(t)|_{\mathcal{E}} f \mid t \in \mathbb{R} \}$. We are then regulating all exogenous signals — namely those in $\mathcal{E}_f$ — generated by the simplest exosystem (2.2) which can also generate the given signal f. While the elements of the function space $\mathcal{E}_f$ are easy to work out if f is a trigonometric polynomial, this is not necessarily the case if we only know that $f \in BUC(\mathbb{R}, E)$. In particular, $\mathcal{E}_f$ may in general be an infinite-dimensional space even though it is the closure of spans of translates of one single signal f only. The purpose of this case study is to characterize elements of the spaces $\mathcal{E}_f$ using knowledge of f. This information can be used to find out what other signals — besides f — can also necessarily be regulated. Although our methods employ fairly standard vector-valued harmonic analysis, they seem to be entirely new in control literature. The results of this section are based on those in [46].

Section 3.5: In this case study section we shall focus on a concrete application, namely the construction of a feedforward control law which achieves the asymptotic tracking of $p-$periodic reference signals in the (generalized) Sobolev spaces $H(f_n, \omega_n)$ introduced in Chapter 2. We shall apply Proposition 2.3 for $W = \mathcal{H} = H(f_n, \omega_n)$, and solve the corresponding FRP for exponentially stabilizable SISO plants which do not have transmission zeros on $\sigma(S|_{\mathcal{H}}) = \{ i\omega_n \mid n \in I \}$. By making the formal method presented in [10, 11] mathematically rigorous we shall first explicitly solve the regulator equations (3.10). This allows us to explicitly resolve the regulating control law (in particular the operator L) in terms of the solution operators Π and Γ of the regulator equations (3.10). We shall then derive a verifiable condition (3.55) which completely characterizes both the solvability of the FRP and those periodic functions which a given exponentially stabilizable plant can asymptotically track. Our results demonstrate an interesting new phenomenon which is not present in the case of finite-dimensional exosystems [12, 23, 29, 35, 72]: The nonexistence of transmission zeros on $\sigma(S)$ is not enough to guarantee the solvability of the output regulation problem. In fact, the reference functions must also be *smooth enough* and the *high frequency damping* of the plant must also be modest enough in order for output regulation to be possible. The necessary and sufficient smoothness-damping

combination is completely characterized by the condition (3.55). The results of this section are contained in [54].

Section 3.6: As shown in Section 3.5, the nonexistence of transmission zeros on $\sigma(S)$ is not always sufficient to guarantee the solvability of the FRP, unless W is finite-dimensional [12]. As regards the necessity of nonexistence of zeros of the stabilized feedforward control system on $\sigma(S)$ for asymptotic tracking (using Proposition 2.3) of reference signals in some given Banach space $\mathcal{H} \hookrightarrow BUC(\mathbb{R}, H)$, in this case study section we shall prove the following results under Assumption 3.42:

- If $y_{ref} \in AP(\mathbb{R}, H)$ can be regulated (i.e. if $\mathcal{H}_{y_{ref}}$ can be regulated), then the stabilized feedforward control system does not have zeros on the Bohr spectrum of y_{ref} (cf. Theorem 3.46).

- If $y_{ref} \in BUC(\mathbb{R}, H)$ is ergodic at $\lambda \in \mathbb{R}$, with $M_\lambda y_{ref} \neq 0$, and if y_{ref} can be regulated (i.e. if $\mathcal{H}_{y_{ref}}$ can be regulated), then the stabilized feedforward control system does not have a zero at $i\lambda$ (cf. Theorem 3.48).

- If $y_{ref} \in BUC(\mathbb{R}, H)$ can be regulated (i.e. if $\mathcal{H}_{y_{ref}}$ can be regulated), then the stabilized feedforward control system does not have a zero at $i\lambda$ whenever $\lambda \in sp_C(y_{ref})$ is an isolated point (cf. Theorem 3.49).

- If for some space $\mathcal{H}$ the FRP is solvable, then the stabilized feedforward control system does not have zeros on the point spectrum of $S|_{\mathcal{H}}$ (cf. Theorem 3.47).

The above results have been developed in [46], and they are partial[2] generalizations of those in Section V of [12], because the neutrally stable finite-dimensional linear exosystems employed in [12] always generate almost periodic (and hence also totally ergodic; see [2]) reference signals. For SISO systems the results of this section fully generalize those in Section V of [12]. We conclude Section 3.6 with a discussion of the limitations of the above results. This discussion illustrates the inherent complexity of the relation between nonexistence of system zeros and asymptotic tracking of reference signals, in the case of an infinite-dimensional exosystem (2.2).

---

[2]The essential differences are the assumption that there are no disturbances and the use of a different, perhaps more natural notion of a system zero.

Section 3.7: We shall present some examples of feedforward output regulation in the setup of Proposition 2.3. The examples are mostly as in the papers [49, 54].

3.1 The feedforward regulation problem FRP

In this section we shall define the feedforward regulation problem FRP. Since output regulation means closed loop stability and asymptotic tracking of reference signals under disturbances, in the case of feedforward controllers we are led to study the following problem.

Definition 3.1 (FRP). The task in the feedforward regulation problem is to find operators $K \in \mathcal{L}(Z, H)$ and $L \in \mathcal{L}(W, H)$ having the following properties.

1. The pair (A, B) is strongly stabilizable using K, i.e. $A + BK$ generates a strongly stable C_0-semigroup $T_{A+BK}(t)$ on Z.

2. As the control law $u(t) = Kz(t) + Lw(t)$ is applied to the plant, in the extended system on $Z \times W$ described (in the mild sense) by the equations

$$\dot{z}(t) = (A + BK)z(t) + (BL + P)w(t), \quad t \geq 0 \tag{3.1a}$$

$$\dot{w}(t) = Sw(t), \quad t \in \mathbb{R} \tag{3.1b}$$

   the tracking error $e(t) = y(t) - y_{ref}(t) = (C + DK)z(t) + (DL - Q)w(t) \to 0$ as $t \to \infty$ regardless of the initial conditions $z(0) \in Z$ and $w(0) \in W$.

Remark 3.2. It is implicitly assumed in Definition 3.1 that the free parameters W, P, Q and S of the exogenous system (2.2) are fixed. However, the initial state $w(0) \in W$ of the exosystem need not be fixed.

Remark 3.3. If some bounded operators K and L solve the FRP in the case that the exosystem's parameters are fixed as in Proposition 2.3, then all reference signals in the function space $\mathcal{H} \stackrel{s}{\hookrightarrow} BUC(\mathbb{R}, H)$ can be asymptotically tracked in the presence of certain disturbances. Moreover, by construction the control law $u(t) = Kz(t) + Lw(t) = Kz(t) + LT_S(t)|_{\mathcal{H}} y_{ref}$ achieves asymptotic tracking of $y_{ref} \in \mathcal{H}$. Similarly, the solvability of the FRP in the case that the exosystem's parameters are chosen as in Proposition 2.4 implies that all disturbance signals in $\mathcal{Z} \stackrel{s}{\hookrightarrow} BUC(\mathbb{R}, Z)$ can be asymptotically rejected and certain reference signals can be tracked. Again by construction

the control law $u(t) = Kz(t) + Lw(t) = Kz(t) + LT_S(t)|_{\mathcal{Z}}\mathcal{U}_{dist}$ achieves asymptotic rejection of $\mathcal{U}_{dist} \in \mathcal{Z}$. Finally, the solvability of the FRP in the case that the exosystem's parameters are chosen as in Proposition 2.5 implies that all reference signals in $\mathcal{H}$ can be asymptotically tracked and all disturbance signals in $\mathcal{Z}$ can be asymptotically rejected. Moreover, by construction the control law $u(t) = Kz(t) + Lw(t) = Kz(t) + L\left(\begin{smallmatrix} T_S(t)|_{\mathcal{H}}y_{ref} \\ T_S(t)|_{\mathcal{Z}}\mathcal{U}_{dist} \end{smallmatrix}\right)$ regulates any given exogenous signals $y_{ref} \in \mathcal{H}$ and $\mathcal{U}_{dist} \in \mathcal{Z}$

Remark 3.4. In Definition 3.1 we do not require exponential stabilizability of the pair (A, B) in the FRP, as was done in [12, 72].

3.2 Sufficient conditions for the solvability of the FRP

In this section we shall show that if the regulator equations (3.10) can be solved, and if the pair (A, B) can be strongly stabilized, then the FRP can be solved. Moreover, we shall explicitly construct a control law $u(t) = Kz(t) + Lw(t)$ which solves the FRP in this case. We shall need the following lemma, which turns out to be of fundamental importance also later on in this thesis.

Lemma 3.5. Let X_1 and X_2 be Banach spaces, let A_1 generate a C_0-semigroup $T_{A_1}(t)$ on X_1, let A_2 generate a C_0-semigroup $T_{A_2}(t)$ on X_2 and let $A_3 \in \mathcal{L}(X_2, X_1)$. If there exists $\Pi \in \mathcal{L}(X_2, X_1)$ such that $\Pi(\mathcal{D}(A_2)) \subset \mathcal{D}(A_1)$ and Π satisfies the Sylvester type operator equation $\Pi A_2 = A_1\Pi + A_3$ in $\mathcal{D}(A_2)$, then $T_{A_1}(t)x_1(0) + \int_0^t T_{A_1}(t-s)A_3T_{A_2}(s)x_2(0)ds = \Pi T_{A_2}(t)x_2(0) + T_{A_1}(t)[x_1(0) - \Pi x_2(0)]$ for all $t \geq 0$, $x_1(0) \in X_1$ and $x_2(0) \in X_2$.

Proof. Since by our assumption $A_3 = \Pi A_2 - A_1\Pi$ in $\mathcal{D}(A_2)$, for each $x_2(0) \in \mathcal{D}(A_2)$ we have

$$\int_0^t T_{A_1}(t-s)A_3T_{A_2}(s)x_2(0)ds = \int_0^t T_{A_1}(t-s)\big[\Pi A_2 - A_1\Pi\big]T_{A_2}(s)x_2(0)ds \tag{3.2}$$

$$= \int_0^t \frac{d}{ds}T_{A_1}(t-s)\Pi T_{A_2}(s)x_2(0)ds \tag{3.3}$$

$$= \Pi T_{A_2}(t)x_2(0) - T_{A_1}(t)\Pi x_2(0) \quad \forall t \geq 0 \tag{3.4}$$

because $\mathcal{D}(A_2)$ is invariant for $T_{A_2}(t)$. Since $\mathcal{D}(A_2)$ is also dense in X_2, we can extend the equality

$$\int_0^t T_{A_1}(t-s)A_3T_{A_2}(s)x_2(0)ds = \Pi T_{A_2}(t)x_2(0) - T_{A_1}(t)\Pi x_2(0) \quad \forall t \geq 0 \tag{3.5}$$

to hold for every $x_2(0) \in X_2$. This is because for every $x_2(0) \in X_2$ there exists a sequence $(x_k)_{k\geq 0} \subset \mathcal{D}(A_2)$ such that $\|x_k - x_2(0)\|_{X_2} \to 0$ as $k \to \infty$. Now for every $t \geq 0$ and every $\epsilon > 0$

there exists $n \in \mathbb{N}$ such that

$$\left\| \int_0^t T_{A_1}(t-s)A_3 T_{A_2}(s)x_2(0)ds - \left[\Pi T_{A_2}(t)x_2(0) - T_{A_1}(t)\Pi x_2(0) \right] \right\| \leq \tag{3.6}$$

$$\left\| \int_0^t T_{A_1}(t-s)A_3 T_{A_2}(s)x_2(0)ds - \int_0^t T_{A_1}(t-s)A_3 T_{A_2}(s)x_n ds \right\| + \tag{3.7}$$

$$\left\| \int_0^t T_{A_1}(t-s)A_3 T_{A_2}(s)x_n ds - \left[\Pi T_{A_2}(t)x_n - T_{A_1}(t)\Pi x_n \right] \right\| + \tag{3.8}$$

$$\left\| \left[\Pi T_{A_2}(t)x_n - T_{A_1}(t)\Pi x_n \right] - \left[\Pi T_{A_2}(t)x_2(0) - T_{A_1}(t)\Pi x_2(0) \right] \right\| < \epsilon \tag{3.9}$$

The result now follows immediately. $\square$

Our main result in this section is the following.

Theorem 3.6. *Assume that the pair (A, B) is strongly stabilizable using $K \in \mathcal{L}(H, Z)$. If there exist $\Pi \in \mathcal{L}(W, Z)$ and $\Gamma \in \mathcal{L}(W, H)$ such that $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$ and the following regulator equations are satisfied:*

$$A\Pi + B\Gamma + P = \Pi S \quad in \; \mathcal{D}(S) \tag{3.10a}$$

$$C\Pi + D\Gamma = Q \quad in \; W \tag{3.10b}$$

then the control law $u(t) = Kz(t) + (\Gamma - K\Pi)w(t)$ solves the FRP.

Before proving this result, we hasten to advice the reader that the actual solution of the regulator equations (3.10) is deferred to Chapter 8, because those results are relatively independent of any particular output regulation problem that we study in this thesis. As we shall see, similar regulator equations also arise in other output regulation problems. Nonetheless, the reader is invited to take a look at the results of Chapter 8 at any time.

Proof of Theorem 3.6. Since by assumption $A + BK$ generates the strongly stable C_0-semigroup $T_{A+BK}(t)$, we only need to verify the condition 2 in Definition 3.1. Let $L = \Gamma - K\Pi \in \mathcal{L}(W, H)$. Since $\Pi S = (A + BK)\Pi + BL + P$ in $\mathcal{D}(S)$, by Lemma 3.5 we have

$$\int_0^t T_{A+BK}(t-\tau)(BL + P)T_S(\tau)w(0)d\tau = \Pi T_S(t)w(0) - T_{A+BK}(t)\Pi w(0) \tag{3.11}$$

for every $w(0) \in W$ and every $t \geq 0$. Consider then the composite operator $\mathcal{A}$ on the extended state space $Z \times W$ (see (3.1)) defined by

$$\mathcal{A} = \begin{pmatrix} A + BK & BL + P \\ 0 & S \end{pmatrix} \tag{3.12}$$

with $\mathcal{D}(\mathcal{A}) = \mathcal{D}(A + BK) \times \mathcal{D}(S)$. Since $A + BK$ generates the C_0-semigroup $T_{A+BK}(t)$ on Z and S generates the C_0-group $T_S(t)$ on W, it is clear that $\mathcal{A}$ generates a C_0-semigroup $T_{\mathcal{A}}(t)$ on $Z \times W$, because $BL + P \in \mathcal{L}(W, Z)$ (see also [17] Lemma 3.2.2). An easy calculation reveals that this semigroup is given by

$$T_{\mathcal{A}}(t) = \begin{pmatrix} T_{A+BK}(t) & \int_0^t T_{A+BK}(t - \tau)(BL + P)T_S(\tau)d\tau \\ 0 & T_S(t) \end{pmatrix} \tag{3.13}$$

$$= \begin{pmatrix} T_{A+BK}(t) & \Pi T_S(t) - T_{A+BK}(t)\Pi \\ 0 & T_S(t) \end{pmatrix} \tag{3.14}$$

Let $z(0) = z_0 \in Z$ and $w(0) = w_0 \in W$ be arbitrary. Then

$$T_{\mathcal{A}}(t)\begin{pmatrix} z_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} T_{A+BK}(t)(z_0 - \Pi w_0) + \Pi T_S(t)w_0 \\ T_S(t)w_0 \end{pmatrix} \tag{3.15}$$

Since by (3.10b) we have $(C + DK)\Pi + DL - Q = C\Pi + D\Gamma - Q = 0$, the explicit expression for the tracking error $e(t)$ is as follows:

$$e(t) = (C + DK)T_{A+BK}(t)(z_0 - \Pi w_0) + (C\Pi + D\Gamma - Q)T_S(t)w_0 \tag{3.16}$$

$$= (C + DK)T_{A+BK}(t)(z_0 - \Pi w_0) \quad \forall t \geq 0 \tag{3.17}$$

Since $T_{A+BK}(t)$ is strongly stable and $C + DK \in \mathcal{L}(Z, H)$, we have that $e(t) \to 0$ as $t \to \infty$. This shows that the asymptotic tracking/rejection condition of Definition 3.1 is also satisfied. The proof is then complete. $\square$

Remark 3.7. In the case that $H = \mathbb{C}^M$ for some $M \in \mathbb{N}$ it is actually sufficient in Theorem 3.6 for the asymptotic tracking/rejection to occur that the pair (A, B) is merely weakly stabilizable by K (i.e. that $\lim_{t\to\infty} f(T_{A+BK}(t)z) = 0$ for all $f \in \mathcal{L}(Z, \mathbb{C})$ and all $z \in Z$) and that Π and Γ solve the the regulator equations (3.10). In this case $C + DK \in \mathcal{L}(Z, \mathbb{C}^M)$ and so for every $z \in Z$ $\lim_{t\to\infty}(C + DK)T_{A+BK}(t)z = 0$, as is easily seen by utilizing weak stability componentwise. The conclusion follows via equations (3.16)-(3.17).

Remark 3.8. In order to be able to asymptotically track *one* given reference function $y_{ref} \in \mathcal{H} \hookrightarrow BUC(\mathbb{R}, H)$ in the case that the pair (A, B) is strongly stabilizable, by Proposition 2.3 and the proof of Theorem 3.6 it is clearly sufficient to find operators $\Pi \in \mathcal{L}(\mathcal{H}, Z)$ and $\Gamma \in \mathcal{L}(\mathcal{H}, H)$ for

which

$$A\Pi + B\Gamma + P = \Pi S|_{\mathcal{H}} \quad \text{in } \mathcal{D}(S|_{\mathcal{H}}) \tag{3.18a}$$

$$C\Pi + D\Gamma = \delta_0 \quad \text{in } \{\, T_S(t)|_{\mathcal{H}} y_{ref} \mid t \in \mathbb{R} \,\} \tag{3.18b}$$

However, by continuity and linearity the regulator equation $C\Pi + D\Gamma = \delta_0$ is then actually satisfied in $\mathcal{H}_{y_{ref}} = \overline{\text{span}}\{\, T_S(t)|_{\mathcal{H}} y_{ref} \mid t \in \mathbb{R} \,\}$. Since in addition $\mathcal{D}(S|_{\mathcal{H}_{y_{ref}}}) \subset \mathcal{D}(S|_{\mathcal{H}})$, all reference signals in $\mathcal{H}_{y_{ref}}$ can be asymptotically tracked by Theorem 3.6. This observation provides additional justification for the convention made Definition 2.22.

We again emphasize that the reader who wishes to learn about the actual solution of the regulator equations (3.10) is invited to take a look at Chapter 8 at any time. That part of this thesis is relatively independent of the other parts and it addresses the solvability of the equations (3.10) in general. On the other hand, the reader is adviced that the regulator equations (3.10) will be solved shortly in Section 3.5 for a simple special case of periodic signals and SISO systems in an example-like fashion. Before doing that, however, we shall discuss necessity of solvability of the regulator equations (3.10) for output regulation.

3.3 Necessary conditions for the solvability of the FRP

In Section 3.2 we proved that if strong stabilizability of (A, B) can be established then the output regulation problem FRP is solvable provided that the regulator equations (3.10) are solvable. In this section we discuss a converse question: Is it necessary to be able to solve the regulator equations (3.10) in order to be able to solve the corresponding output regulation problem FRP? Although it is well-known that Theorem 3.6 has a converse provided that the exosystem is finite-dimensional and the plant is also exponentially stabilizable [12], the question as to whether or not such a converse holds in the more general setting of Definition 3.1 turns out to be more difficult to answer decisively. There are certain cases in which a converse of Theorem 3.6 does hold: If the FRP is solvable and if certain additional assumptions hold, then the regulator equations (3.10) *must* have a solution. In the present section we shall give such results.

In order to study the necessity of solvability of the regulator equations (3.10) for the solvability of the FRP we have to identify the effect of both regulator equations on the dynamical behaviour of the closed loop system (3.1). Fortunately, this has already been done in the proof of Theorem 3.6

under the assumption that $A + BK$ generates a strongly stable C_0-semigroup and $L = \Gamma - K\Pi$. In particular, the first regulator equation $\Pi S = A\Pi + B\Gamma + P = (A + BK)\Pi + BL + P$ in $\mathcal{D}(S)$ guarantees that the state $z(t)$ of the controlled and disturbed plant can be decomposed into a sum of two parts, one of which decays to 0 as $t \to \infty$ and the other can be generated by the exogenous system (2.2) by an appropriate choice of its free parameters (see equation (3.15)). On the other hand, the second regulator equation $Q = C\Pi + D\Gamma = (C + DK)\Pi + DL$ in W guarantees that the unstable part of this decomposition of $z(t)$ cancels out the effect of the desired output in the tracking error $e(t)$ (see equations (3.16)-(3.17)).

In what follows we shall employ the concept of a *regular* operator $\Delta \in \mathcal{L}(W, Z)$ to formalize some of the above ideas (see Definition 3.9). This allows for a complete characterization of the solvability of the Sylvester operator equation $\Pi S = A\Pi + \Delta$ in $\mathcal{D}(S)$. The solvability of the second regulator equation is subsequently obtained by imposing certain auxiliary conditions for the reference signals or the speed of output regulation.

Definition 3.9. An operator $\Delta \in \mathcal{L}(W, Z)$ is said to be regular for the semigroup $T_A(t)$ if there exists $\Pi \in \mathcal{L}(W, Z)$ such that $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$ and such that for every $w \in W$ the function $t \to z(t) = \Pi T_S(t)w$ is a mild solution of the differential equation $\dot{z}(t) = Az(t) + \Delta T_S(t)w$ on the whole real line, i.e. z satisfies

$$z(t) = T_A(t - s)z(s) + \int_s^t T_A(t - \tau)\Delta T_S(\tau)wd\tau \quad \forall t \geq s \tag{3.19}$$

Remark 3.10. In (3.19) t and s need not be nonnegative. Moreover, $z(t) = \Pi T_S(t)w$, with $w \in W$, need not be the *only* mild solution of the differential equation $\dot{z}(t) = Az(t) + \Delta T_S(t)w$ on the real line.

Remark 3.11. If an operator $\Delta \in \mathcal{L}(W, Z)$ is regular for $T_A(t)$, then some mild solution of the differential equation $\dot{z}(t) = Az(t) + \Delta T_S(t)w$ on the whole line can always be generated using the exosystem (2.2) by an appropriate choice of the free parameters. For example, if $W = \mathcal{Z}$ is a closed translation invariant and operator invariant[3] subspace of $BUC(\mathbb{R}, Z)$ and if $\Delta = \delta_0 \in \mathcal{L}(\mathcal{Z}, Z)$ is regular for $T_A(t)$, then for all $f = \Delta T_S(\cdot)|_{\mathcal{Z}}f \in \mathcal{Z}$ there exists a mild solution $\Pi T_S(\cdot)|_{\mathcal{Z}}f \in \mathcal{Z}$ of the differential equation $\dot{z}(t) = Az(t) + f(t)$ on the real line. If this mild solution in $\mathcal{Z}$ is also unique, then the space $\mathcal{Z}$ is sometimes called regularly admissible in the literature, see e.g. [90].

---

[3]$\mathcal{Z}$ is operator invariant if for all $f \in \mathcal{Z}$ and all $P \in \mathcal{L}(\mathcal{Z}, Z)$ the function $t \to PT_S(t)|_{\mathcal{Z}}f \in \mathcal{Z}$ [90].

Lemma 3.12 below is crucial for the development of the main results in this section. Observe that we do have to assume any stability properties for $T_A(t)$ in it.

Lemma 3.12. *The operator equation $\Pi S = A\Pi + \Delta$ in $\mathcal{D}(S)$ has a solution $\Pi \in \mathcal{L}(W, Z)$ if and only if $\Delta \in \mathcal{L}(W, Z)$ is regular for $T_A(t)$.*

Proof. Let $\Pi \in \mathcal{L}(W, Z)$ such that $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$ and $\Pi S = A\Pi + \Delta$ in $\mathcal{D}(S)$. Let $w \in W$ be arbitrary and set $z(t) = \Pi T_S(t)w$. Then it is easy to see using Lemma 3.5 and an elementary change of variables that

$$\int_s^t T_A(t - \tau)\Delta T_S(\tau)wd\tau = \Pi T_S(t)w - T_A(t - s)\Pi T_S(s)w \tag{3.20}$$

for every $w \in W$ and every $t \geq s$. Hence for every $t \geq s$, we have

$$z(t) = T_A(t - s)\Pi T_S(s)w + \Pi T_S(t)w - T_A(t - s)\Pi T_S(s)w \tag{3.21}$$

$$= T_A(t - s)z(s) + \int_s^t T_A(t - \tau)\Delta T_S(\tau)wd\tau \tag{3.22}$$

so that $z(t)$ is a mild solution of the differential equation $\dot{z}(t) = Az(t) + \Delta T_S(t)w$ on the whole real line. In other words, Δ is regular for $T_A(t)$.

Conversely, suppose that there exists $\Pi \in \mathcal{L}(W, Z)$ such that $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$ and such that for every $w \in W$ the function $t \to z(t) = \Pi T_S(t)w$ is a mild solution of the differential equation $\dot{z}(t) = Az(t) + \Delta T_S(t)w$ on the whole real line. Let $w \in \mathcal{D}(S)$. Then since $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$ and since $\mathcal{D}(S)$ is invariant for $T_S(t)$, the function $t \to \Delta T_S(t)w$ is continuously differentiable, and we can differentiate both sides of the identity

$$\Pi T_S(t)w = T_A(t)\Pi w + \int_0^t T_A(t - \tau)\Delta T_S(\tau)wd\tau, \quad \forall t \geq 0 \tag{3.23}$$

(cf. Proposition 1.3.6 in [2]) and set $t = 0$ to obtain $\Pi S = A\Pi + \Delta$ in $\mathcal{D}(S)$. The proof is complete. $\square$

Remark 3.13. If $T_A(t)$ is exponentially stable, then by Corollary 8 in [88] (see also Section A.2), the operator equation $\Pi S = A\Pi + \Delta$ in $\mathcal{D}(S)$ has a (unique) solution $\Pi \in \mathcal{L}(W, Z)$ for every $\Delta \in \mathcal{L}(W, Z)$. Consequently, every operator $\Delta \in \mathcal{L}(W, Z)$ is regular for an exponentially stable C_0-semigroup.

One way to obtain the necessity of solvability of the regulator equations (3.10) for the solvability of the FRP is to restrict the class of reference signals as follows.

Definition 3.14. The exogenous system (2.2) generates admissible reference signals if for every $w \in W$ and each $Q \in \mathcal{L}(W, H)$ it is true that $QT_S(\cdot)w \in C_0^+(\mathbb{R}, H) = \{ f \in BUC(\mathbb{R}, H) \mid f(t) \rightarrow 0 \text{ as } t \rightarrow \infty \}$ only if $Qw = 0$.

For example, if the operator S in (2.2) generates a periodic C_0-group $T_S(t)$ on W, i.e. $T_S(t + p) = T_S(t)$ for each $t \in \mathbb{R}$ and some $p > 0$, then the exogenous system (2.2) generates admissible reference signals. In fact, the function $t \rightarrow QT_S(t)w$ is $p-$periodic for each $Q \in \mathcal{L}(W, H)$ and every $w \in W$. Consequently, $Qw = \lim_{n \to \infty} QT(np)w = 0$ for all $Q \in \mathcal{L}(W, H)$ and all $w \in W$ such that $QT_S(\cdot)w \in C_0^+(\mathbb{R}, H)$. More generally, we have the following:

Proposition 3.15. *The exogenous system* (2.2) *generates admissible reference signals provided that at least one of the four conditions below holds.*

1. *The reference signals $QT_S(\cdot)w$ are in $AP(\mathbb{R}, H)$ for all $Q \in \mathcal{L}(W, H)$ and all $w \in W$.*

2. *The spectrum $\sigma(S)$ is countable and H does not contain a closed subspace which is isomorphic to c_0 (the space of sequences converging to 0 with the sup-norm).*

3. *The spectrum $\sigma(S)$ is discrete.*

4. $\dim(W) < \infty$.

Proof. It can be shown (see e.g. [2] p. 290) that for every $f \in AP(\mathbb{R}, H)$ there exists a sequence $(t_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$, and $\|f(t_n + s) - f(s)\| \leq \frac{1}{n}$ for all $n \in \mathbb{N}$ and all $s \in \mathbb{R}$. Suppose that $QT_S(\cdot)w \in C_0^+(\mathbb{R}, H)$ for some $Q \in \mathcal{L}(W, H)$ and some $w \in W$. That $Qw = 0$ follows from the estimates $\|Qw\| \leq \|QT_S(0)w - QT_S(t_n)w\| + \|QT_S(t_n)w\| \rightarrow 0$ as $n \rightarrow \infty$.

Next suppose that $\sigma(S)$ is countable and that H does not contain a closed subspace which is isomorphic to c_0. Then for all $Q \in \mathcal{L}(W, H)$ and all $w \in W$ the function $QT_S(\cdot)w \in BUC(\mathbb{R}, H)$ is such that its Carleman spectrum $sp_C(QT_S(\cdot)w)$ satisfies $isp_C(QT_S(\cdot)w) \subset \sigma(S)$ (cf. Remark 4.6.2 in [2]). Consequently also $sp_C(QT_S(\cdot)w)$ is countable. Hence by Theorem 4.6.3 in [2] the function $QT_S(\cdot)w \in AP(\mathbb{R}, H)$. The conclusion now follows as in the above.

If $\sigma(S)$ is a discrete set, then by the above reasoning so is always $sp_C(QT_S(\cdot)w)$. Consequently, by Theorem 4.8.7 in [2], in this case $QT_S(\cdot)w \in AP(\mathbb{R}, H)$ and so the conclusion follows.

Finally, if $\dim(W) < \infty$, then since S generates an isometric C_0-group on W, $\sigma(S)$ must be discrete. The proof is complete. $\square$

Restricting our attention to such exosystems (2.2) that only generate admissible reference signals essentially means that we do not want to asymptotically track reference functions vanishing at $+\infty$. This assumption is natural and quite common in output regulation literature (see e.g. [12, 29]). It is obvious that the control $u(t) = Kz(t)$ would suffice for the asymptotic tracking of vanishing signals, provided that $A+BK$ generates a strongly stable C_0-semigroup on Z. We point out, however, that in contrast to the case $\dim(W) < \infty$, if $\dim(W) = \infty$, then the assumption $\sigma(S) \subset i\mathbb{R}$ alone does not guarantee that exosystem (2.2) generates admissible reference signals. For example, let $W = BUC(\mathbb{R}, \mathbb{C})$ and $S = S|_{BUC(\mathbb{R},\mathbb{C})}$ as in Proposition 2.3. Then every compactly supported infinitely smooth function f is in $W \cap C_0^+(\mathbb{R}, \mathbb{C})$ while $\sigma(S|_{BUC(\mathbb{R},\mathbb{C})}) = i\mathbb{R}$. In this case $\delta_0 T_S(t)f = f(t) \to 0$ as $t \to \infty$ but $\delta_0 f = f(0) \neq 0$ is possible.

We are now ready to present some conditions under which the solvability of the regulator equations (3.10) is necessary for the solvability of the FRP.

Theorem 3.16. *Assume that the exosystem (2.2) generates admissible reference signals. If the FRP is solvable for some control law $u(t) = Kz(t)+Lw(t)$ such that the operator $BL+P \in \mathcal{L}(W, Z)$ is regular for the semigroup $T_{A+BK}(t)$, then there exist $\Pi \in \mathcal{L}(W, Z)$ and $\Gamma \in \mathcal{L}(W, H)$ such that $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$, $L = \Gamma - K\Pi$ and the regulator equations (3.10) are satisfied.*

Proof. Since $BL + P$ is regular for $T_{A+BK}(t)$, by Lemma 3.12 there exists $\Pi \in \mathcal{L}(W, Z)$ such that $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A + BK) = \mathcal{D}(A)$ and $\Pi S = (A + BK)\Pi + BL + P$ in $\mathcal{D}(S)$. Let $\Gamma = L + K\Pi \in \mathcal{L}(W, H)$. Then Π and Γ solve the first regulator equation (3.10a).

We next show that also the second regulator equation is satisfied. Lemma 3.5 shows that since $\Pi S = (A + BK)\Pi + BL + P$ in $\mathcal{D}(S)$, we have for every $w \in W$ and every $t \geq 0$ that

$$\int_0^t T_{A+BK}(t - \tau)(BL + P)T_S(\tau)wd\tau = \Pi T_S(t)w - T_{A+BK}(t)\Pi w \qquad (3.24)$$

Let $w(0) = w \in W$ be arbitrary and take $z(0) = \Pi w \in Z$. Then the corresponding tracking error $e(t)$ is given by

$$e(t) = (C + DK)T_{A+BK}(t)[z(0) - \Pi w] + [(C + DK)\Pi + DL - Q]T_S(t)w \qquad (3.25)$$

$$= [(C + DK)\Pi + DL - Q]T_S(t)w \qquad (3.26)$$

$$= [C\Pi + D\Gamma - Q]T_S(t)w, \quad \forall t \geq 0 \qquad (3.27)$$

Now $[C\Pi + D\Gamma - Q]T_S(\cdot)w \in C_0^+(\mathbb{R}, H)$ because the FRP is solvable. Since the exosystem (2.2) generates admissible reference signals and since $C\Pi + D\Gamma - Q \in \mathcal{L}(W, H)$, we must have that

$C\Pi w + D\Gamma w - Qw = 0$. As $w \in W$ is arbitrary, also the second regulator equation (3.10b) is satisfied. $\qquad\square$

Remark 3.17. Since Theorem 3.16 generalizes the "necessity" part of Theorem IV.1 in [12] and since the latter result relies on exponential stability of $T_{A+BK}(t)$, it is somewhat surprising that the proof of Theorem 3.16 above does not explicitly employ even strong stability of $T_{A+BK}(t)$.

By combining the above result with those in Section 3.2 we obtain the following complete characterization for the solvability of the regulator equations and the FRP.

Corollary 3.18. *Let the pair (A, B) be strongly stabilizable using $K \in \mathcal{L}(Z, H)$ and assume that the exogenous system (2.2) generates admissible reference signals. Then the FRP is solvable using the control law $u(t) = Kz(t) + Lw(t)$, where $L \in \mathcal{L}(W, H)$ and $BL + P$ is regular for $T_{A+BK}(t)$, if and only if there exists $\Pi \in \mathcal{L}(W, Z)$ and $\Gamma \in \mathcal{L}(W, H)$ such that $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$, $L = \Gamma - K\Pi$ and the regulator equations (3.10) are satisfied.*

If the pair $A + BK$ also generates an exponentially stable C_0−semigroup, then the above regularity condition for $BL + P \in \mathcal{L}(W, Z)$ can be dropped:

Theorem 3.19. *Assume that $K \in \mathcal{L}(Z, H)$ stabilizes the pair (A, B) exponentially and assume also that the exosystem (2.2) generates admissible reference signals. Then the FRP is solvable using the control law $u(t) = Kz(t) + Lw(t)$, where $L \in \mathcal{L}(W, H)$, if and only if there exists $\Pi \in \mathcal{L}(W, Z)$ and $\Gamma \in \mathcal{L}(W, H)$ such that $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$, $L = \Gamma - K\Pi$ and the regulator equations (3.10) are satisfied.*

Proof. Theorem 3.6 covers the sufficiency part of this assertion. On the other hand, the necessity part is proved as in the proof of Theorem 3.16; however, instead of Lemma 3.12 we rely on Corollary 8 in [88] (see also Section A.2) to ensure the unique solvability of the first regulator equation. $\qquad\square$

Finally, if $A + BK$ generates an exponentially stable C_0−semigroup and if we can solve the FRP in such way that output regulation is exponentially fast, then we can also dispense with the assumption that the exogenous system (2.2) only generates admissible reference signals. We arrive at another complete characterization for the solvability of the FRP:

Theorem 3.20. *Assume that $K \in \mathcal{L}(Z, H)$ stabilizes the pair (A, B) exponentially. Then there exists $L \in \mathcal{L}(W, H)$ such that the control law $u(t) = Kz(t) + Lw(t)$ solves the FRP and $\|e(t)\| \leq$*

$Me^{-\omega t}[\|z(0)\| + \|w(0)\|]$ *for all $t \geq 0$ and some $M, \omega > 0$ which do not depend on the initial conditions $z(0) \in Z$ and $w(0) \in W$, if and only if $L = \Gamma - K\Pi$ where $\Pi \in \mathcal{L}(W, Z)$ and $\Gamma \in \mathcal{L}(W, H)$ satisfy the regulator equations* (3.10).

Proof. In the sufficiency part we refer the reader to the proof of Theorem 3.6. In particular, equation (3.17) yields $\|e(t)\| \leq \|(C+DK)T_{A+BK}(t)(z(0)-\Pi w(0))\| \leq \|C+DK\|\|T_{A+BK}(t)\|[\|z(0)\| + \|\Pi\|\|w(0)\|] \leq Me^{-\omega t}[\|z(0)\| + \|w(0)\|]$ for all $t \geq 0$ and some $M, \omega > 0$ which do not depend on the initial states $z(0) \in Z$ and $w(0) \in W$.

On the other hand, in the necessity part we first deduce (by exponential stability and Corollary 8 in [88]; see Section A.2) that there exists a unique $\Pi \in \mathcal{L}(W, Z)$ such that $\Pi S = (A+BK)\Pi + BL + P$ in $\mathcal{D}(S)$. Moreover, $e(t) = (C + DK)T_{A+BK}(t)(z(0) - \Pi w(0)) + (C + DK)\Pi + DL - Q)T_S(t)w(0)$ for every $z(0) \in Z$ and $w(0) \in W$.

Assume that there exists $w_0 \in W$ such that $\|[(C+DK)\Pi+DL-Q]w_0\| > 0$. Let $t_0 > 0$ be such that $Me^{-\omega t_0}[\|\Pi\|\|w_0\| + \|w_0\|] < \|[(C + DK)\Pi + DL - Q]w_0\|$, and set $w(0) = T_S(-t_0)w_0 \in W$ and $z(0) = \Pi w(0) \in Z$. Then the corresponding tracking error $e(t)$ satisfies

$$\|e(t_0)\| = \|[(C + DK)\Pi + DL - Q]T_S(t_0)w(0)\| \tag{3.28}$$

$$= \|[(C + DK)\Pi + DL - Q]T_S(t_0)T_S(-t_0)w_0\| \tag{3.29}$$

$$= \|[(C + DK)\Pi + DL - Q]w_0\| \tag{3.30}$$

$$\leq Me^{-\omega t_0}[\|\Pi\|\|w_0\| + \|w_0\|] \tag{3.31}$$

because $\|z(0)\| = \|\Pi w(0)\| \leq \|\Pi\|\|T_S(-t_0)w_0\| = \|\Pi\|\|w_0\|$ (recall that $T_S(t)$ is an isometric group). But this is clearly a contradiction. Hence $[(C + DK)\Pi + DL - Q]w_0 = 0$. The proof is complete. $\qquad\square$

The above results show that the regulator equations (3.10) must often be solvable for Π and Γ, *and* that it is often also necessary to choose $L = \Gamma - K\Pi$ in order to achieve output regulation in the sense of the FRP. Consequently, although the feedforward controller is very simple — and hence quite appealing — it not necessarily robust even with respect to perturbations in the stabilizing feedback operator K.

3.4 A case study: Regulation of individual signals

Suppose that we are interested in regulating one given signal f in the sense of the FRP only. Here $f \in \mathcal{E} \stackrel{s}{\hookrightarrow} BUC(\mathbb{R}, E)$ where the Banach spaces E and $\mathcal{E}$ are determined by the goal that we have. For example, if f is some particular reference function that we want to asymptotically track, then $E = H$ and $\mathcal{E} \hookrightarrow BUC(\mathbb{R}, H)$. By the convention made in Definition 2.22, regulating f is equivalent to regulating all signals generated by the simplest exosystem (2.2) capable of generating f, i.e. regulating all signals in the smallest closed translation invariant space $\mathcal{E}_f \subset \mathcal{E}$ containing f. Using the constructions of Proposition 2.3, Proposition 2.4 or Proposition 2.5 the latter is then equivalent to solving the FRP for $W = \mathcal{E}_f$ and appropriate choices of P, Q which also depend on the regulation problem at hand. However, the space $\mathcal{E}_f$ may in general be infinite-dimensional in spite of the fact that it is the closure of spans of translates of only one function f. Consequently it is both important and interesting to investigate what *other* signals, besides f, can also *necessarily* be regulated under these circumstances. This can be done by applying some standard methods of harmonic analysis to study the content of $\mathcal{E}_f$.

Recall in the following that almost periodic functions are those which can be uniformly approximated by trigonometric polynomials (see Definition 2.8).

Theorem 3.21. *Let $f \in AP(\mathbb{R}, E)$ and define*

$$a(\lambda, f) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda t} f(t) dt \quad \forall \lambda \in \mathbb{R} \tag{3.32}$$

Then for sup-norm closures we have $\mathcal{E}_f = \overline{\mathrm{span}}\{\, a(\lambda, f)e^{i\lambda \cdot} \mid \lambda \in sp_B(f)\,\}$, where the Bohr spectrum $sp_B(f) = \{\, \lambda \in \mathbb{R} \mid a(\lambda, f) \neq 0\,\}$ [38]. If in addition f is $p-$periodic and has the Fourier series $\sum_{n \in \mathbb{Z}} a_n e^{i\omega_n t}$, then $\mathcal{E}_f = \overline{\mathrm{span}}\{\, a_n e^{i\omega_n \cdot} \mid a_n \neq 0\,\}$.

Proof. The first part of the result is just Corollary 1.4 in [38]. If f is a $p-$periodic function with the Fourier series $\sum_{n \in \mathbb{Z}} a_n e^{i\omega_n t}$, then its Carleman spectrum $sp_C(f) = \{\, \omega_n \mid a_n \neq 0\,\}$ by Example 1.3 and Theorem 1.14 in [38]. But we also have that $sp_C(f) = \overline{sp_B(f)}$ by Proposition 1.2 in [38]. Hence trivially $sp_B(f) = \{\, \omega_n \mid a_n \neq 0\,\}$. Moreover, it is not difficult to see that $a(\omega_n, f) = a_n$ for every $n \in \mathbb{Z}$. By the first part of the theorem, we then have that $\mathcal{E}_f = \overline{\mathrm{span}}\{\, a_n e^{i\omega_n \cdot} \mid a_n \neq 0\,\}$. $\square$

Corollary 3.22. *Let $f \in \mathcal{E} \hookrightarrow AP(\mathbb{R}, E)$. Consider the space $\mathcal{E}_f$ with closure in $\mathcal{E}$. Then $\mathcal{E}_f \subset \overline{\mathrm{span}}\{\, a(\lambda, f)e^{i\lambda \cdot} \mid \lambda \in sp_B(f)\,\}$ (closure in the sup-norm).*

Proof. This result follows immediately, because $\mathcal{E} \hookrightarrow AP(\mathbb{R}, E)$. □

As an example of the above we consider p−periodic scalar reference functions in a generalized Sobolev space $H(f_n, \omega_n)$ (cf. Chapter 2).

Example 3.23. *Consider a Sobolev space $\mathcal{H} = H(f_n, \omega_n)$ for some sequences $(\omega_n)_{n \in I}$ and $(f_n)_{n \in I}$. Let $y_{ref}(t) = \sum_{n \in I} \hat{y}(n) e^{i\omega_n t} \in \mathcal{H}$. If $\hat{y}(n) \neq 0$ for each $n \in I$, then $\mathcal{H}_{y_{ref}} = \mathcal{H}$. This fact can be proved as follows. Since $\mathcal{H}_{y_{ref}} \subset \mathcal{H} = \overline{\operatorname{span}}\{ e^{i\omega_n \cdot} \mid n \in I \}$, with closures in $\mathcal{H}$, it is sufficient to show that the functions $t \to e^{i\omega_n t} \in \mathcal{H}_{y_{ref}}$ for every $n \in I$. Since $\mathcal{H}$ is a Hilbert space, its closed subspace $\mathcal{H}_{y_{ref}}$ is also a Hilbert space. Moreover, since $T_S(t)|_{\mathcal{H}_{y_{ref}}}$ is a uniformly bounded C_0−group, according to Corollary 4.3.5 in [2], for each $n \in I$ the C_0−group $e^{-i\omega_n t} T_S(t)|_{\mathcal{H}_{y_{ref}}}$ is Cesáro ergodic. This, on the other hand, by Proposition 4.3.1 in [2] means that for each $f \in \mathcal{H}_{y_{ref}}$ and $n \in I$ the limit $x_f^n = \lim_{t \to \infty} \frac{1}{t} \int_0^t e^{-i\omega_n s} T_S(s)|_{\mathcal{H}_{y_{ref}}} f ds \in \ker(-i\omega_n I + S|_{\mathcal{H}_{y_{ref}}}) \subset \mathcal{H}_{y_{ref}}$. Now for the particular choice $f = y_{ref} \in \mathcal{H}_{y_{ref}}$ we have $x_f^n \in \mathcal{H}_{y_{ref}}$ and $x_f^n(0) \neq 0$ for every $n \in I$ because in this case $x_f^n(0) = \delta_0 \lim_{t \to \infty} \frac{1}{t} \int_0^t e^{-i\omega_n s} T_S(s)|_{\mathcal{H}_{y_{ref}}} y_{ref} ds = \lim_{t \to \infty} \frac{1}{t} \int_0^t e^{-i\omega_n s} y_{ref}(s) ds = \hat{y}(n)$. By Lemma II.1.9 in [28] we have $e^{-i\omega_n t} T_S(t)|_{\mathcal{H}_{y_{ref}}} x_f^n = x_f^n$ for each $n \in I$ and $t \in \mathbb{R}$. This shows that the function $t \to \delta_0 T_S(t)|_{\mathcal{H}_{y_{ref}}} x_f^n = x_f^n(t) = e^{i\omega_n t} x_f^n(0) \in \mathcal{H}_{y_{ref}}$ for every $n \in I$.*

The above example shows that in order to be able to track *all* reference signals in $H(f_n, \omega_n)$ in the sense of the FRP, it is necessary and sufficient to be able to track *one* function $y_{ref} \in H(f_n, \omega_n)$ with nonzero Fourier coefficients in the sense of the FRP.

We now turn to the general case, in which the function f to be regulated is not necessarily almost periodic. Unfortunately, because of this extra generality we can only obtain partial information on the space $\mathcal{E}_f$.

Definition 3.24. A function $f \in \mathcal{E} \hookrightarrow BUC(\mathbb{R}, E)$ is called ergodic at $\eta \in \mathbb{R}$ with respect to $T_S(t)|_{\mathcal{E}}$ if the mean $M_\eta f = \lim_{t \to \infty} \frac{1}{t} \int_0^t e^{-i\eta\tau} T_S(\tau)|_{\mathcal{E}} f d\tau$ converges in $\mathcal{E}$ (cf. [2] Definition 4.3.10).

Proposition 3.25. *Let $f \in \mathcal{E} \hookrightarrow BUC(\mathbb{R}, E)$ be ergodic at $\eta \in \mathbb{R}$, with $M_\eta f \neq 0$. Then there exists $x \in E$, $x \neq 0$, such that $M_\eta f = e^{i\eta \cdot} x \in \mathcal{E}_f$ (closure in $\mathcal{E}$).*

Proof. By the assumptions

$$M_\eta f = \lim_{t \to \infty} \frac{1}{t} \int_0^t e^{-i\eta\tau} T_S(\tau)|_{\mathcal{E}} f d\tau \neq 0 \tag{3.33}$$

Since $e^{-i\eta t}T_S(t)|_{\mathcal{E}}$ is a bounded C_0-group generated by $-i\eta I + S|_{\mathcal{H}}$, by Proposition 4.3.1 in [2] we have $M_\eta f \in \ker(-i\eta I + S|_{\mathcal{E}})$. Hence for every $t \in \mathbb{R}$, $T_S(t)|_{\mathcal{E}}M_\eta f = e^{i\eta t}M_\eta f$ by Lemma II.1.9 in [28]. But this implies that

$$(M_\eta f)(t) = \delta_0 T_S(t)|_{\mathcal{E}}M_\eta f = e^{i\eta t}\delta_0 M_\eta f = e^{i\eta t}x \tag{3.34}$$

for $x = \delta_0 M_\eta f = [M_\eta f](0) \in E$. Now $M_\eta f \in \mathcal{E}_f$, since the integral in (3.33) can in fact be calculated in $\mathcal{E}_f$ (by definition this space is closed in $\mathcal{E}$). The claim now follows, because $x \neq 0$ (for otherwise $M_\eta f = 0$). $\qquad\square$

Corollary 3.26. *Let $f \in \mathcal{E} \hookrightarrow BUC(\mathbb{R}, E)$ and define the set*

$$\Omega_f = \{\, \eta \in \mathbb{R} \mid 0 \neq M_\eta f = \lim_{t\to\infty} \frac{1}{t}\int_0^t e^{-i\eta\tau}T_S(\tau)|_{\mathcal{E}}f d\tau \in \mathcal{E} \,\} \tag{3.35}$$

Then for closures in $\mathcal{E}$ we have $\overline{\mathrm{span}}\{\, [M_\eta f](0)e^{i\eta\cdot} \mid \eta \in \Omega_f \,\} \subset \mathcal{E}_f$.

The following result shows that isolated points λ in the Carleman spectrum $sp_C(f)$ of $f \in BUC(\mathbb{R}, E)$ always induce a nontrivial element $xe^{i\lambda\cdot}$ in $\mathcal{E}_f$.

Proposition 3.27. *Let $f \in BUC(\mathbb{R}, E)$ and assume that $\lambda \in \mathbb{R}$ is an isolated point in $sp_C(f)$. Then there exists a nonzero $x \in E$ such that the function $e^{i\lambda\cdot}x \in \mathcal{E}_f$ (closure in the sup-norm).*

Proof. By the results of Section A.1 we have $isp_C(f) = \sigma(S|_{\mathcal{E}_f})$, whenever $\mathcal{E}_f$ is closed in the sup-norm. Consequently, under our assumptions $i\lambda$ is an isolated point in $\sigma(S|_{\mathcal{E}_f})$. Since $S|_{\mathcal{E}_f}$ generates a bounded C_0-group on $\mathcal{E}_f$, by Gelfand's $T = I$ Theorem (cf. Corollary 4.4.9 in [2] or Section A.1) $i\lambda$ is an eigenvalue of $S|_{\mathcal{E}_f}$. This implies that there exists a nonzero $y \in \mathcal{E}_f$ such that $T_S(t)|_{\mathcal{E}_f}y = e^{i\lambda t}y$ for every $t \in \mathbb{R}$. Consequently $y(t) = \delta_0 T_S(t)|_{\mathcal{E}_f}y = e^{i\lambda t}\delta_0 y = e^{i\lambda t}y(0)$ for each $t \in \mathbb{R}$. The claim now follows upon the choice $x = y(0)$, because $x \neq 0$ (otherwise $y = 0$). $\qquad\square$

We remark that a complete characterization of $\mathcal{E}_f$, such as in Theorem 3.21, in terms of the Carleman spectrum $sp_C(f)$ alone does not generally seem to exist for those functions $f \in BUC(\mathbb{R}, E)$ which are not almost periodic. In fact, according to [58] (p. 170), even in the scalar-valued case it is only possible to completely characterize the weak* closure of spans of translates of f (and not $\mathcal{E}_f$ which is the $\|\cdot\|_{\mathcal{E}}$-closure of spans of translates of f) in terms of $sp_C(f)$. In the framework of this thesis we have to restrict our attention to $\mathcal{E}_f$ in order to use the theory of C_0-semigroups in the solution of the FRP.

3.5 A case study: Periodic tracking for exponentially stabilizable SISO systems

In order to convince the reader of the value and applicability of our general results, in this section we shall we shall solve the FRP and the regulator equations (3.10) in the case that the reference signals are in generalized Sobolev spaces $H(f_n, \omega_n)$ of $p-$periodic functions (see Chapter 2) and the disturbance signals are also known to be $p-$periodic[4]. Throughout this section our standing assumption for the plant is the following:

Assumption 3.28. *The plant* (1.1) *is a SISO system (i.e. $H = \mathbb{C}$), and the pair (A, B) is exponentially stabilizable by a given (fixed) operator $K \in \mathcal{L}(Z, \mathbb{C})$.*

According to Proposition 2.3 a feasible way to model the signals is to choose the free parameters of the exosystem (2.2) as in the following assumption which we shall also pose throughout this section.

Assumption 3.29. $W = \mathcal{H} = H(f_n, \omega_n)$, $S = S|_{\mathcal{H}}$, $Q = \delta_0 \in \mathcal{L}(\mathcal{H}, \mathbb{C})$, $P \in \mathcal{L}(\mathcal{H}, Z)$ *and* $w(0) = y_{ref} \in \mathcal{H}$.

Under the above assumptions Theorem 3.19 ensures that output regulation in the sense of the FRP is equivalent to the solvability of the regulator equations (3.10) for Π and Γ such that $L = \Gamma - K\Pi$. Below, we can sometimes explicitly solve these regulator equations provided that the stabilized plant does not have transmission zeros at the complex Fourier frequencies $i\omega_n$ of the reference signals. This allows us to explicitly resolve the regulating control law (i.e. the operator L) in terms of the solution operators Π and Γ. Furthermore, we can derive the condition (3.55) which completely characterizes the solvability of the FRP and simultaneously provides a verifiable answer to the question: What periodic signals can an exponentially stabilizable SISO control system asymptotically track?

Let us define $\phi_n(x) = e^{i\omega_n x}$ for each $x \in \mathbb{R}$ and $n \in I$. Clearly $(\phi_n)_{n \in I}$ constitutes an orthogonal basis in $\mathcal{H}$, and $\phi_n \in \mathcal{D}(S|_{\mathcal{H}})$ with $S|_{\mathcal{H}}\phi_n = i\omega_n\phi_n$ for each $n \in I$. Before proving our main results we shall first introduce transmission zeros for SISO systems, and we shall prove a spectral condition under which they are feedback invariant. By convention, throughout this section the

---

[4]A treatment of the solution of the regulator equations (3.10) in a considerably more general setup is to be presented Chapter 8 and the reader is invited to take a look at those results at any time.

transfer function $H(s)$ of the plant is given by $H(s) = CR(s, A)B + D$ for every $s \in \rho(A)$ and the transfer function $H_K(s)$ of the stabilized plant is given by $H_K(s) = (C + DK)R(s, A + BK)B + D$ for each $s \in \rho(A + BK)$.

Definition 3.30. The sequence of disturbance coefficients for the stabilized plant is defined by $(H_d(n))_{n \in I} = ((C + DK)R(i\omega_n, A + BK)P\phi_n)_{n \in I} \subset \mathbb{C}$.

Definition 3.31. The plant (respectively stabilized plant) has a transmission zero at $s = s_0$ if $H(s_0) = 0$ (respectively $H_K(s_0) = 0$).

For the case $D = 0$ the next result was stated in Lemma V.2 of [12].

Lemma 3.32. Let $s_0 \in \rho(A) \cap \rho(A + BK)$. Then the plant has a transmission zero at $s = s_0$ if and only if the stabilized plant has a transmission zero at $s = s_0$.

Proof. Let $s = s_0$ be a transmission zero of the plant. Clearly $CR(s_0, A)B + D = 0$ if and only if

$$\ker \begin{pmatrix} C & D \\ s_0 I - A & -B \end{pmatrix} \neq \{0\} \tag{3.36}$$

where the domain of definition of the operator $\mathcal{R} = \begin{pmatrix} C & D \\ s_0 I - A & -B \end{pmatrix}$ is $\mathcal{D}(A) \times \mathbb{C}$. Let $0 \neq \begin{pmatrix} x \\ u \end{pmatrix} \in \ker \mathcal{R}$. Then since $x = R(s_0, A)Bu$, we must have that $u \neq 0$. Moreover,

$$\begin{pmatrix} C & D \\ s_0 I - A & -B \end{pmatrix} \begin{pmatrix} I & 0 \\ K & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -K & I \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = 0 \tag{3.37}$$

which implies

$$\begin{pmatrix} C + DK & D \\ s_0 I - A - BK & -B \end{pmatrix} \begin{pmatrix} x \\ u - Kx \end{pmatrix} = 0 \tag{3.38}$$

Let $\mathcal{R}_K = \begin{pmatrix} C+DK & D \\ s_0 I - A - BK & -B \end{pmatrix}$ with $\mathcal{D}(\mathcal{R}_K) = \mathcal{D}(\mathcal{R})$. If $x = 0$, then $0 \neq \begin{pmatrix} 0 \\ u \end{pmatrix} \in \ker \mathcal{R}_K$. On the other hand, if $x \neq 0$, then $0 \neq \begin{pmatrix} x \\ u - Kx \end{pmatrix} \in \ker \mathcal{R}_K$. In any case $\ker \mathcal{R}_K \neq \{0\}$. By the above this means that $(C + DK)R(s_0, A + BK)B + D = 0$, i.e. that the stabilized plant has a transmission zero at $s = s_0$.

Similar arguments show that also the converse holds. We omit the details. $\square$

The following result is the key to the solution of the regulator equations (3.10) in this special case of SISO systems. It provides a means for solving these equations in finite-dimensional spaces spanned by exponentials.

Proposition 3.33. *Let $n \in I$. If the stabilized plant does not have a transmission zero at $s = i\omega_n$, and if we define the operator $L : \mathrm{span}\{\phi_n\} \to \mathbb{C}$ by $L(a\phi_n) = aH_K(i\omega_n)^{-1}[1 - H_d(n)] \in \mathbb{C}$ for all $a \in \mathbb{C}$ then the operators $\Pi : \mathrm{span}\{\phi_n\} \to Z$ and $\Gamma : \mathrm{span}\{\phi_n\} \to \mathbb{C}$ defined by*

$$\Pi(a\phi_n) = R(i\omega_n, A + BK)\big[BL(a\phi_n) + P(a\phi_n)\big] \quad \forall a \in \mathbb{C} \tag{3.39}$$

$$\Gamma(a\phi_n) = L(a\phi_n) + K\Pi(a\phi_n) \quad \forall a \in \mathbb{C} \tag{3.40}$$

satisfy $\Pi S|_{\mathcal{H}}\phi_n = A\Pi\phi_n + B\Gamma\phi_n + P\phi_n$ and $C\Pi\phi_n + D\Gamma\phi_n = \delta_0\phi_n = 1$. Hence by linearity Π and Γ satisfy the regulator equations (3.10) on the one-dimensional space $\mathrm{span}\{\phi_n\}$.

Proof. Since $S|_{\mathcal{H}}\phi_n = i\omega_n\phi_n$, it is clear that if we solve the equations

$$(A + BK)\Pi\phi_n + BL\phi_n + P\phi_n = i\omega_n\Pi\phi_n \tag{3.41a}$$

$$(C + DK)\Pi\phi_n + DL\phi_n = 1 \tag{3.41b}$$

for $\Pi\phi_n$ and $L\phi_n$, and then set $\Gamma\phi_n = L\phi_n + K\Pi\phi_n$, we simultaneously have $\Pi S|_{\mathcal{H}}\phi_n = A\Pi\phi_n + B\Gamma\phi_n + P\phi_n$ and $C\Pi\phi_n + D\Gamma\phi_n = \delta_0\phi_n = 1$. From (3.41a) we obtain

$$(A + BK)\Pi\phi_n + BL\phi_n + P\phi_n = i\omega_n\Pi\phi_n \quad \Leftrightarrow \quad \Pi\phi_n = R(i\omega_n, A + BK)(BL\phi_n + P\phi_n) \tag{3.42}$$

because $A + BK$ generates an exponentially stable C_0-semigroup. Applying this expression for $\Pi\phi_n$ to equation (3.41b) yields

$$(C + DK)R(i\omega_n, A + BK)(BL\phi_n + P\phi_n) + DL\phi_n = 1 \Leftrightarrow \tag{3.43}$$

$$\big[(C + DK)R(i\omega_n, A + BK)B + D\big]L\phi_n + (C + DK)R(i\omega_n, A + BK)P\phi_n = 1 \Leftrightarrow \tag{3.44}$$

$$H_K(i\omega_n)L\phi_n + H_d(n) = 1 \Leftrightarrow \tag{3.45}$$

$$H_K(i\omega_n)^{-1}\big[1 - H_d(n)\big] = L\phi_n \tag{3.46}$$

by the assumption that $H_K(i\omega_n) \neq 0$. The proof is complete. $\qquad\square$

Remark 3.34. By the above proof $\Pi\phi_n = R(i\omega_n, A+BK)\big[BL\phi_n+P\phi_n\big]$ and $L\phi_n = H_K(i\omega_n)^{-1}[1-H_d(n)]$ are in fact the unique solutions of the equations (3.41).

Theorem 3.35. *Suppose that for every $n \in I$, $s = i\omega_n$ is not a transmission zero of the stabilized plant. Let $\langle\cdot, \cdot\rangle$ denote the L^2 inner product on $\mathcal{H}$ and define*

$$Ly = \sum_{n \in I} H_K(i\omega_n)^{-1}[1 - H_d(n)]\langle y, \phi_n\rangle \tag{3.47}$$

for all such $y \in \mathcal{H}$ for which the series converges. Then the FRP is solvable using the control law $u(t) = Kz(t) + Lw(t)$ if and only if $L \in \mathcal{L}(\mathcal{H}, \mathbb{C})$ (so that $\mathcal{D}(L) = \mathcal{H}$).

Proof. (Necessity.) If the control law $u(t) = Kz(t) + Lw(t)$ solves the FRP, then by definition L must be in $\mathcal{L}(\mathcal{H}, \mathbb{C})$ and defined everywhere.

(Sufficiency.) Suppose that $L \in \mathcal{L}(\mathcal{H}, \mathbb{C})$. Then the linear operator $\Pi : \mathcal{H} \to Z$ defined by $\Pi w = \int_0^\infty T_{A+BK}(\tau)(BL + P)T_S(-\tau)|_{\mathcal{H}} w d\tau$ for each $w \in \mathcal{H}$ is in $\mathcal{L}(\mathcal{H}, Z)$. Since $T_S(-t)|_{\mathcal{H}}\phi_n = e^{-i\omega_n t}\phi_n$ and $R(i\omega_n, A+BK)z = \int_0^\infty e^{-i\omega_n t} T_{A+BK}(t)z dt$ for every $z \in Z$ (Proposition 5.1.5 in [2]), we have that $\Pi\phi_n = R(i\omega_n, A + BK)(BL + P)\phi_n$ for each n. In addition, since $\langle \phi_n, \phi_m \rangle = \delta_{m,n}$ (Kronecker delta), we have that $L\phi_n = H_K(i\omega_n)^{-1}\big[1 - H_d(n)\big]$ for each $n \in I$. Consequently the restrictions of Π and L satisfy the equations (3.41) for each $n \in I$. Using the closedness of A, the continuity of Π and $\Gamma = L + K\Pi$ and Remark 2.20, we obtain

$$\Pi S|_{\mathcal{H}} y = \sum_{n \in I} i\omega_n \langle y, \phi_n \rangle \Pi\phi_n = \sum_{n \in I} \langle y, \phi_n \rangle \big[A\Pi\phi_n + B\Gamma\phi_n + P\phi_n \big] \tag{3.48}$$

$$= \big[A\Pi + B\Gamma + P \big] y \quad \forall y \in \mathcal{D}(S|_{\mathcal{H}}) \tag{3.49}$$

$$\big[C\Pi + D\Gamma \big] y = \sum_{n \in I} \langle y, \phi_n \rangle \big[C\Pi + D\Gamma \big] \phi_n = \sum_{n \in I} \langle y, \phi_n \rangle = y(0) = \delta_0 y \quad \forall y \in \mathcal{H} \tag{3.50}$$

by the fact that $\langle y, \phi_n \rangle = \hat{y}(n)$ is the nth L^2 Fourier coefficient of $y \in \mathcal{H}$. Since now $L = \Gamma - K\Pi$ where $\Pi \in \mathcal{L}(\mathcal{H}, Z)$ and $\Gamma \in \mathcal{L}(\mathcal{H}, \mathbb{C})$ solve the regulator equations (3.10), the result follows from Theorem 3.19. $\qquad\square$

Remark 3.36. Under the assumptions of Theorem 3.35, for a given stabilizing state feedback $K \in \mathcal{L}(Z, \mathbb{C})$ it is actually *necessary* to use the particular operator L defined in (3.47) in a regulating feedforward control $u(t) = Kz(t) + Lw(t)$. This is because $L = \Gamma - K\Pi$ where Π and Γ solve the regulator equations (3.10), and in particular $L\phi_n = \Gamma\phi_n - K\Pi\phi_n$ for every $n \in I$. But then L is unique, because $L\phi_n$ is unique for every $n \in I$ (by the above) and $(\phi_n)_{n \in I}$ is an orthogonal basis.

Corollary 3.37. *Suppose that the assumptions of Theorem 3.35 are satisfied and that L defined in (3.47) is in $\mathcal{L}(\mathcal{H}, \mathbb{C})$, so that the FRP is solvable using $u(t) = Kz(t) + Lw(t)$. Then for every $y_{ref} \in H(f_n, \omega_n)$ the corresponding control law $u_{y_{ref}}(t)$ which achieves the asymptotic tracking of $y_{ref}(t)$ in the presence of the disturbance $\mathcal{U}_{dist}(t)$ is given by*

$$u_{y_{ref}}(t) = Kz(t) + \sum_{n \in I} H_K(i\omega_n)^{-1}[1 - H_d(n)]\hat{y}(n)e^{i\omega_n t}, \quad \forall t \geq 0 \tag{3.51}$$

where $y_{ref}(t) = \sum_{n \in I} \hat{y}(n)e^{i\omega_n t}$ for each $t \in \mathbb{R}$.

Proof. By construction, $u(t) = Kz(t) + Lw(t) = Kz(t) + LT_S(t)|_{\mathcal{H}}w(0)$ is the control law which achieves asymptotic tracking of $y_{ref} = w(0) \in \mathcal{H}$ and asymptotic rejection of $\mathcal{U}_{dist} = PT_S(\cdot)|_{\mathcal{H}}w(0)$. Hence it is sufficient to let $w(0) = y_{ref}$ and work out $Lw(t) = LT_S(t)|_{\mathcal{H}}w(0)$ as follows:

$$LT_S(t)|_{\mathcal{H}}w(0) = \sum_{n \in I} \hat{y}(n) LT_S(t)|_{\mathcal{H}}\phi_n = \sum_{n \in I} \hat{y}(n)e^{i\omega_n t}L\phi_n \tag{3.52}$$

$$= \sum_{n \in I} H_K(i\omega_n)^{-1}[1 - H_d(n)]\hat{y}(n)e^{i\omega_n t}, \quad \forall t \geq 0 \tag{3.53}$$

because $T_S(t)|_{\mathcal{H}}\phi_n = e^{i\omega_n t}\phi_n$ for each $n \in I$ and every $t \in \mathbb{R}$. $\square$

In particular, if there are no disturbances (i.e. $P = 0$), and if the plant is already exponentially stable then the control law (3.51) reduces to the remarkably simple one:

$$u_{y_{ref}}(t) = \sum_{n \in I} H(i\omega_n)^{-1}\hat{y}(n)e^{i\omega_n t} \tag{3.54}$$

The following corollary characterizes the solvability of the FRP by the asymptotic behaviour of $H_K(i\omega_n)^{-1}[1 - H_d(n)]$ as $n \to \pm\infty$.

Corollary 3.38. *Suppose that the assumptions of Theorem 3.35 are satisfied. Let L be defined as in (3.47). Then $u(t) = Kz(t) + Lw(t)$ solves the FRP if and only if*

$$\left(H_K(i\omega_n)^{-1}[1 - H_d(n)]f_n^{-1}\right)_{n \in I} \in \ell^2 \tag{3.55}$$

In the disturbance free case (i.e. $P = 0$) this condition reads $\left(H_K(i\omega_n)^{-1}f_n^{-1}\right)_{n \in I} \in \ell^2$.

Proof. Let $\psi_n = \frac{\phi_n}{\|\phi_n\|}$ for every $n \in I$. Then $(\psi_n)_{n \in I}$ constitutes an orthonormal basis in $\mathcal{H}$ with respect the natural inner product of $\mathcal{H}$. Here we denote this inner product by $\langle \cdot, \cdot \rangle_f$. Observe that we can write

$$L = \sum_{n \in I} f_n^{-1}H_K(i\omega_n)^{-1}[1 - H_d(n)]\langle \cdot, \psi_n \rangle_f \tag{3.56}$$

because $f_n\hat{y}(n) = f_n\langle y, \phi_n \rangle_{L^2} = \langle y, \psi_n \rangle_f$ for each $n \in I$ and $y \in \mathcal{H}$. Since $\mathcal{H}$ is a Hilbert space, by the Riesz Representation Theorem $L \in \mathcal{L}(\mathcal{H}, \mathbb{C})$ if and only if there exists a unique element $l \in \mathcal{H}$ such that $Lw = \langle w, l \rangle_f$ for every $w \in \mathcal{H}$. Then we must have that $L\psi_n = \langle \psi_n, l \rangle_f = f_n^{-1}H_K(i\omega_n)^{-1}[1 - H_d(n)]$, or, $\langle l, \psi_n \rangle_f = \overline{H_K(i\omega_n)^{-1}[1 - H_d(n)]}f_n^{-1}$ for every $n \in I$. But by the orthonormality of $(\psi_n)_{n \in I}$ the element l thus defined is in $\mathcal{H}$ if and only if

$$\sum_{n \in I} |\langle l, \psi_n \rangle_f|^2 = \sum_{n \in I} |H_K(i\omega_n)^{-1}[1 - H_d(n)]|^2|f_n|^{-2} < \infty \tag{3.57}$$

This and Theorem 3.35 give the desired result. □

Remark 3.39. In the case that $\mathcal{H}$ is finite-dimensional, the condition (3.55) is trivially satisfied under the assumptions of Theorem 3.35.

The above results formalize the intuitive idea that in order to be able to asymptotically track general $p-$periodic reference signals, the stabilized plant should not attenuate high frequency oscillations too drastically and at the same time the reference signals should be smooth enough. By the above, in the absence of transmission zeros it is true that all reference signals in $\mathcal{H} = H(f_n, \omega_n)$ can be regulated in the sense of the FRP if and only if the condition (3.55) holds, i.e. the operator L defined in (3.47) is bounded. We point out that this operator L is not in general bounded for *all* choices of the state space $W = \mathcal{H}$ of the exosystem; however, the condition (3.55) allows us to *choose* such a topology for W that L becomes bounded. The choice of this topology amounts to an appropriate choice of the sequence $(f_n)_{n \in I}$, which on the other hand completely determines the degree of smoothness of the reference signals.

But what if the condition (3.55) is not satisfied? In that case there must exist $y_{ref} \in \mathcal{H}$ which cannot be regulated in the sense of the FRP. Assume that there also exists some $y_1 \in \mathcal{H}$ which we can regulate in the sense that there exists $L \in \mathcal{L}(\mathcal{H}, \mathbb{C})$ such that $u(t) = Kz(t) + LT_S(t)|_{\mathcal{H}} y_1$ achieves asymptotic tracking of y_1 for each initial state $z(0) \in Z$ of the plant. In the absence of transmission zeros of the stabilized plant on $\sigma(S|_{\mathcal{H}})$, according to Theorem 3.19 and Proposition 3.33 all exponential functions $t \to e^{i\omega_n t} = \phi_n(t)$, $n \in I$ can be regulated. Hence we may in this case assume without loss of generality that y_1 has nonvanishing Fourier coefficients[5]. Exponential stability of $A + BK$ now guarantees the solvability of the first regulator equation $\Pi S|_{\mathcal{H}} = A\Pi + B\Gamma + P = A\Pi + B(L + K\Pi) + P$ in $\mathcal{D}(S|_{\mathcal{H}})$, whereas the second regulator equation $C\Pi + D\Gamma = C\Pi + D(L + K\Pi) = \delta_0$ is satisfied at least in the subset $\{ T_S(t)|_{\mathcal{H}} y_1 \mid t \in \mathbb{R} \} \subset \mathcal{H}$ by the periodicity of $T_S(t)|_{\mathcal{H}}$ and the tracking requirement[6]. Using extension by continuity and linearity it is straightforward to show that it is then actually possible to regulate all reference signals in $\mathcal{H}_{y_1} = \overline{\text{span}}\{ T_S(t)|_{\mathcal{H}} y_1 \mid t \in \mathbb{R} \}$. But by Example 3.23 then the space $\mathcal{H} = \mathcal{H}_{y_1}$, so that all reference signals in $\mathcal{H}$ can be regulated. This contradiction shows that the condition (3.55) also characterizes those Sobolev spaces $H(f_n, \omega_n)$ which *cannot* be regulated (in the sense of the FRP),

---

[5]Otherwise we could always add missing individual frequency components to y_1, because the individual frequencies can be regulated.

[6]Apply the argument used in the proof of Theorem 3.16 to see that $[C\Pi + D\Gamma - \delta_0]T_S(t)y_1 = 0$ for all $t \in \mathbb{R}$.

in the absence of transmission zeros of the stabilized plant on $\sigma(S|_{\mathcal{H}})$.

We conclude this section with some auxiliary results which in some cases simplify the verification of the condition (3.55) (see also Theorem 8.2 and the discussion in Section 8.3 for related results).

Theorem 3.40. *Let A generate an exponentially stable C_0-semigroup and let $A + BK$, for $K \in \mathcal{L}(Z, \mathbb{C})$, also generate an exponentially stable C_0-semigroup. Then there exist $m, M \geq 0$ (which do not depend on $n \in I$) such that $\|CR(i\omega_n, A)B\| \leq m\|CR(i\omega_n, A+BK)B\| \leq M\|CR(i\omega_n, A)B\|$ for each $n \in I$.*

Proof. By an elementary calculation, we have that $CR(i\omega_n, A + BK)B = CR(i\omega_n, A)B[I + KR(i\omega_n, A + BK)B]$ and that $CR(i\omega_n, A)B = CR(i\omega_n, A + BK)B[I - KR(i\omega_n, A)B]$ for every $n \in I$. Since A and $A + BK$ both generate exponentially stable C_0-semigroups, $\|R(i\omega_n, A)\|$ and $\|R(i\omega_n, A + BK)\|$ are uniformly bounded in n, according to the Riemann-Lebesgue Lemma [17]. The desired conclusion now follows by some obvious norm estimates. $\square$

According to Theorem 3.40, if $D = 0$, if there are no disturbances, and if both A and $A + BK$ generate exponentially stable C_0-semigroups, then $(H(i\omega_n)f_n^{-1})_{n \in I} \in \ell^2$ if and only if $(H_K(i\omega_n)f_n^{-1})_{n \in I} \in \ell^2$. In particular, in this case the capability of asymptotic tracking is an intrinsic property of the plant which is independent of the stabilizing feedback K. This can be seen by applying the above result for $A + BK_1$ and $A + BK_2 = (A + BK_1) + B(K_2 - K_1)$ where K_1 and K_2 are two different exponentially stabilizing state feedback operators for the pair (A, B).

Corollary 3.41. *Let A and $A + BK$, where $K \in \mathcal{L}(Z, \mathbb{C})$, generate exponentially stable analytic C_0-semigroups. Then the following hold.*

1. *For $D = 0$ the condition (3.55) holds if $\left(H(i\omega_n)^{-1}[1 - H_d(n)]f_n^{-1}\right)_{n \in I} \in \ell^2$.*

2. *For $D \neq 0$ the condition (3.55) holds if $H(i\omega_n) \neq 0$ for each $n \in I$ and if $\left([1 - H_d(n)]f_n^{-1}\right)_{n \in I} \in \ell^2$; in particular, whenever $H(i\omega_n) \neq 0$ for each $n \in I$ and $\sup_{n \in I} |H_d(n)| < \infty$.*

Proof. The case $D = 0$ is settled by Theorem 3.40 and Lemma 3.32 so we may let $D \neq 0$. By exponential stability and Lemma 3.32, for all $n \in I$ we have $H_K(i\omega_n) \neq 0$. Since A and $A + BK$ generate exponentially stable analytic semigroups, we have $\lim_{n \to \pm\infty} H_K(i\omega_n) = D \neq 0$. Consequently for some $\delta > 0$ we have $\delta < \inf_{n \in I} |H_K(i\omega_n)| < \sup_{n \in I} |H_K(i\omega_n)| < \infty$, and so

$$\left|H_K(i\omega_n)^{-1}[1 - H_d(n)]f_n^{-1}\right| \leq \frac{1}{\delta}\left|[1 - H_d(n)]f_n^{-1}\right| \quad \forall n \in I \tag{3.58}$$

This shows that condition (3.55) holds if $\left([1 - H_d(n)]f_n^{-1}\right)_{n \in I} \in \ell^2$. Finally, if $\sup_{n \in I} |H_d(n)| = M_0 < \infty$, then we may estimate $\sum_{n \in I} |f_n|^{-2}|1 - H_d(n)|^2 < (1 + M_0)^2 \sum_{n \in I} |f_n|^{-2} < \infty$. $\qquad\square$

3.6 A case study: Zeros and asymptotic tracking

We saw in Section 3.5 for periodic signals with $W = \mathcal{H} = H(f_n, \omega_n)$ that whenever a suitably stabilized SISO plant does not have transmission zeros on $\sigma(S)$, i.e. $H_K(i\omega_n) \neq 0$ for all $n \in I$, *and* also the condition (3.55) holds, all reference functions in $\mathcal{H}$ can be asymptotically tracked in the presence of certain disturbances. On the other hand, for general MIMO systems and *finite-dimensional* exosystems (2.1) it is well-known that a converse result also holds, at least if $D = 0$ and $\sigma(S) \subset \rho_\infty(A)$ (cf. Corollary V.1 and Lemma V.2 in [12]): If there *are* transmission zeros of the stabilized plant on $\sigma(S)$, i.e. $\det(CR(i\omega, A + BK)B) = 0$ for some $i\omega \in \sigma(S)$, then output regulation of all possible reference/disturbance signals cannot be achieved. But does this converse result remain true for infinite-dimensional exosystems (2.2)? In other words, is it still necessary for output regulation that there are no transmission zeros of the stabilized plant on $\sigma(S)$ if $\dim(W) = \infty$? This question turns out to be considerably more difficult to answer decisively than the corresponding one for finite-dimensional exosystems (2.1). In the present section we shall conduct a case study of this question under the following standing assumption, which by Proposition 2.3 also shows that we are only interested in the asymptotic tracking of a class $\mathcal{H} \hookrightarrow BUC(\mathbb{R}, H)$ of reference signals, without disturbance rejection.

Assumption 3.42. *The pair (A, B) is exponentially stabilizable using $K \in \mathcal{L}(Z, H)$, and the exosystem's free parameters are chosen as[7] $W = \mathcal{H} \overset{s}{\hookrightarrow} BUC(\mathbb{R}, H)$, $S = S|_{\mathcal{H}}$, $Q = \delta_0 \in \mathcal{L}(\mathcal{H}, H)$, $P = 0$, $w(0) = y_{ref} \in \mathcal{H}$. Moreover, the solvability of the FRP is equivalent to the solvability of the regulator equations (3.10) for Π and Γ such that $L = \Gamma - K\Pi$ in the regulating control law.*

Remark 3.43. Theorem 3.19 and Theorem 3.20 present conditions under which the solvability of the FRP is equivalent to the solvability of the regulator equations (3.10) for Π and Γ such that $L = \Gamma - K\Pi$ in the regulating control law. Moreover, by construction the solvability of the FRP in this case implies that all reference signals in the Banach function space $\mathcal{H}$ can be asymptotically tracked.

---

[7] The function space $\mathcal{H}$ will vary in the results of this section, and the exosystem is always assumed to be chosen in this way, according to the specific $\mathcal{H}$ in question.

In this section we shall be concerned of the following kind of system zeros:

Definition 3.44. Let $L \in \mathcal{L}(\mathcal{H}, H)$ be nonzero. We say that the stabilized feedforward control system (i.e. the plant (1.1) subject to the control $u(t) = Kz(t) + Lw(t)$) has a zero at $\lambda \in \rho(A + BK)$ if $H_K(\lambda)L = \left[(C + DK)R(\lambda, A + BK)B + D\right]L = 0$.

It is clear that if $H = \mathbb{C}^N$ (say), and if the stabilized plant does not have a transmission zero at $\lambda \in \rho(A + BK)$ (i.e. $\det(H_K(\lambda)) \neq 0$, as in [12]), then the stabilized feedforward control system cannot have a zero at λ. Conversely, in this case all zeros $\lambda \in \rho(A + BK)$ of the stabilized feedforward control system are transmission zeros of the stabilized plant in the above sense. In the SISO case the stabilized feedforward control system has a zero at $\lambda \in \rho(A + BK)$ if and only if it has a transmission zero at λ. In addition, in this case Lemma 3.32 guarantees that all transmission zeros of the stabilized plant in $\lambda \in \rho(A) \cap \rho(A + BK)$ coincide with those of the plant without stabilization. However, since a transmission zero of the stabilized (MIMO) plant need not be a zero of the stabilized (MIMO) feedforward control system, it is appropriate to provide some motivation for the above definition.

The reason why we do not, in general, employ the ordinary concept of a transmission zero in this section is as follows. Byrnes, Laukó, Gilliam and Shubov have shown (Corollary V.1 in [12]) for finite-dimensional exosystems (2.1) that under certain additional assumptions the FRP is solvable *for every operator* P and Q in the exosystem if and only if the transfer function of the stabilized plant is invertible on $\sigma(S)$, i.e. there are no transmission zeros on $\sigma(S)$ (this result also appears in a different form in the earlier work of Schumacher [80]). However, in this case study section we want to consider the *particular* problem where the asymptotic tracking of signals in a given function space $\mathcal{H}$, without disturbance rejection, is required; by Proposition 2.3 this can be done by *fixing* the operators $P = 0$ and $Q = \delta_0$ as done in Assumption 3.42. Consequently, according to the result of Byrnes et al. [12] cited above, even for a finite-dimensional $\mathcal{H}$ the existence of transmission zeros on $\sigma(S|_{\mathcal{H}})$ does not — in our particular problem — in general imply the impossibility of asymptotic tracking of signals in $\mathcal{H}$. However, it turns out that the existence of zeros of the stabilized feedforward control system, as in Definition 3.44, on $\sigma(S|_{\mathcal{H}})$ *does* sometimes imply this. We again emphasize that these zeros constitute a subset, usually also a proper one, of the transmission zeros of the stabilized plant.

The following lemma plays a key role in the proofs of the main results of this section.

Lemma 3.45. *Assume that for some $L \in \mathcal{L}(\mathcal{H}, H)$ the control law $u(t) = Kz(t) + Lw(t)$ solves the FRP. If for some $\lambda \in \mathbb{R}$ and for some nonzero $a \in H$ the function $t \to y(t) = ae^{i\lambda t} \in \mathcal{H}$, then $H_K(i\lambda)Ly = a$.*

Proof. By Assumption 3.42 there exist $\Pi \in \mathcal{L}(\mathcal{H}, Z)$ and $\Gamma \in \mathcal{L}(\mathcal{H}, H)$ which solve the regulator equations (3.10) and $L = \Gamma - K\Pi$. Consequently, we also have that $\Pi S|_{\mathcal{H}} = (A + BK)\Pi + BL$ in $\mathcal{D}(S|_{\mathcal{H}})$. But the solution $X \in \mathcal{L}(\mathcal{H}, Z)$ of equation $XS|_{\mathcal{H}} = (A + BK)X + BL$ in $\mathcal{D}(S|_{\mathcal{H}})$ is unique because $A + BK$ generates an exponentially stable C_0−semigroup (see e.g. [88], Corollary 8). This solution is given by

$$X = \int_0^\infty T_{A+BK}(t)BLT_S(-t)|_{\mathcal{H}}dt = \Pi \tag{3.59}$$

(the strong limit of a Riemann integral).

Let $y = ae^{i\lambda \cdot} \in \mathcal{H}$. Then since $C\Pi + D\Gamma = \delta_0$ in $\mathcal{H}$, and $L = \Gamma - K\Pi$, we also have that

$$(C + DK)\Pi y + DLy = \delta_0 y = y(0) = a \tag{3.60}$$

By the boundedness of $C + DK$ this yields

$$\int_0^\infty (C + DK)T_{A+BK}(t)BLT_S(-t)|_{\mathcal{H}}ydt + DLy = a \tag{3.61}$$

and so

$$\int_0^\infty (C + DK)T_{A+BK}(t)BLae^{i\lambda(\cdot-t)}dt + DLae^{i\lambda \cdot} = \tag{3.62}$$

$$\big[(C + DK)R(i\lambda, A + BK)B + D\big]Lae^{i\lambda \cdot} = \tag{3.63}$$

$$\big[(C + DK)R(i\lambda, A + BK)B + D\big]Ly = a \tag{3.64}$$

This shows that $H_K(i\lambda)Ly = a$, as was claimed. $\qquad\square$

We next present the main results of this section. Recall the convention made in Definition 2.22; under Assumption 3.42 it says that a signal $y_{ref} \in \mathcal{H}$ can be asymptotically tracked iff all signals in the smallest possible state space $\mathcal{H}_{y_{ref}} \subset \mathcal{H}$ of the exosystem generating y_{ref} can be asymptotically tracked.

Theorem 3.46. *Suppose that a given signal $y_{ref} \in AP(\mathbb{R}, H)$ can be asymptotically tracked in the sense of the FRP, using a control law $u(t) = Kz(t) + Lw(t)$. Then there cannot be zeros of the*

stabilized feedforward control system on the Bohr spectrum of y_{ref}, i.e the following implication holds for all $\lambda \in \mathbb{R}$:

$$\lambda \in sp_B(y_{ref}) \implies H_K(i\lambda)L \neq 0 \tag{3.65}$$

Proof. That $y_{ref} \in AP(\mathbb{R}, H)$ can be asymptotically tracked means, by the convention of Definition 2.22, that we can track all reference signals in $\mathcal{H}_{y_{ref}}$ (where closure is taken with respect to the norm of $AP(\mathbb{R}, H)$, i.e. the $\sup-$norm). This, on the other hand, means the solvability of the FRP for $W = \mathcal{H}_{y_{ref}}$ using the control law $u(t) = Kz(t) + Lw(t)$. Let $\lambda \in sp_B(y_{ref})$. By Theorem 3.21, there exists a nonzero $a \in H$ such that $y = ae^{i\lambda \cdot} \in \mathcal{H}_{y_{ref}}$. By Lemma 3.45, we have $H_K(i\lambda)Ly = a \neq 0$. Hence $H_K(i\lambda)L \neq 0$. $\qquad\square$

In the following result the reference signals need not be almost periodic.

Theorem 3.47. *Let $\mathcal{H} \overset{s}{\hookrightarrow} BUC(\mathbb{R}, H)$. If a control law $u(t) = Kz(t) + Lw(t)$ solves the FRP, then there cannot be zeros of the stabilized feedforward control system on the point spectrum of $S|_{\mathcal{H}}$, i.e. the following implication holds for all $\lambda \in \mathbb{R}$:*

$$i\lambda \in \sigma_P(S|_{\mathcal{H}}) \implies H_K(i\lambda)L \neq 0 \tag{3.66}$$

Proof. Let $i\lambda \in \sigma_P(S|_{\mathcal{H}})$. Then $S|_{\mathcal{H}}f = i\lambda f$ for some nonzero $f \in \mathcal{D}(S|_{\mathcal{H}})$. Hence also $T_S(t)|_{\mathcal{H}}f = e^{i\lambda t}f$ for every $t \in \mathbb{R}$. This shows that for every $t \in \mathbb{R}$ we have $\delta_0 T_S(t)|_{\mathcal{H}}f = f(t) = e^{i\lambda t}\delta_0 f = e^{i\lambda t}f(0)$ where $f(0) \neq 0$. Since $f \in \mathcal{H}$, the result follows by Lemma 3.45. $\qquad\square$

The above results show in particular that the standing assumption of Section 3.5 (i.e. no transmission zeros of the stabilized SISO plant on $\sigma(S)$) cannot in general be removed if output regulation for $\mathcal{H} = H(f_n, \omega_n)$ is to be achieved. Moreover, in the SISO case they generalize the well-known results of output regulation theory for finite-dimensional exogenous systems (see e.g. [12]).

Whereas in the above two theorems we could conclude that there are no zeros of the stabilized feedforward control system on certain subsets of $i\mathbb{R}$, the following two theorems are of individual nature; they apply to single points on the imaginary axis. Recall that a function $y_{ref} \in \mathcal{H} \overset{s}{\hookrightarrow} BUC(\mathbb{R}, H)$ is called ergodic at $\eta \in \mathbb{R}$ with respect to $T_S(t)|_{\mathcal{H}}$ if the mean $M_\eta y_{ref} = \lim_{t \to \infty} \frac{1}{t} \int_0^t e^{-i\eta\tau} T_S(\tau)|_{\mathcal{H}} y_{ref} d\tau$ converges in $\mathcal{H}$ (cf. Definition 3.24 and [2] Definition 4.3.10).

Theorem 3.48. *Suppose that a given signal $y_{ref} \in \mathcal{H} \stackrel{s}{\hookrightarrow} BUC(\mathbb{R}, H)$ can be regulated in the sense of the FRP using a control law $u(t) = Kz(t) + Lw(t)$. If y_{ref} is ergodic at $\lambda \in \mathbb{R}$, with $M_\lambda y_{ref} \neq 0$, then $H_K(i\lambda)L \neq 0$, i.e. the stabilized feedforward control system does not have a zero at $i\lambda$.*

Proof. By convention $\mathcal{H}_{y_{ref}}$ (closure in $\mathcal{H}$) can be regulated in the sense of the FRP using the control law $u(t) = Kz(t) + Lw(t)$. By Proposition 3.25, there exists $x \in H$, $x \neq 0$, such that $y = e^{i\lambda \cdot}x \in \mathcal{H}_{y_{ref}}$. By Lemma 3.45, $H_K(i\lambda)Ly \neq 0$ from which the claim follows. $\qquad\square$

Theorem 3.49. *Suppose that a given signal $y_{ref} \in BUC(\mathbb{R}, H)$ can be regulated in the sense of the FRP using a control law $u(t) = Kz(t) + Lw(t)$. Then for every isolated point $\lambda \in sp_C(y_{ref})$ we have $H_K(i\lambda)L \neq 0$, i.e. the stabilized feedforward control system does not have a zero at $i\lambda$.*

Proof. By convention $\mathcal{H}_{y_{ref}}$ (closure in the sup-norm) can be regulated in the sense of the FRP using the control law $u(t) = Kz(t) + Lw(t)$. By Proposition 3.27 there exists a nonzero $x \in H$ such that $xe^{i\lambda \cdot} \in \mathcal{H}_{y_{ref}}$. The claim immediately follows by Lemma 3.45. $\qquad\square$

Although the above results generalize some well-known ones from the output regulation theory of finite-dimensional exosystems [12], there are, unfortunately, certain subtle limitations in them. First of all, the Bohr spectrum $sp_B(y_{ref})$ of an almost periodic function y_{ref} is not necessarily all of the Carleman spectrum $sp_C(y_{ref})$ of y_{ref}: In general, we only have the equality for closures, i.e. $\overline{sp_B(y_{ref})} = sp_C(y_{ref})$ (cf. Proposition 1.2 in [38]). On the other hand $sp_C(y_{ref})$ does not necessarily consist entirely of isolated points [58]. Hence as regards Theorem 3.46 and Theorem 3.49, what one would *like* to have is the validity of an implication of the form

$$\forall \lambda \in \mathbb{R}: \quad \lambda \in sp_C(y_{ref}) \implies H_K(i\lambda)L \neq 0 \tag{3.67}$$

provided that a bounded uniformly continuous signal y_{ref} can be regulated using the control law $u(t) = Kz(t) + Lw(t)$ in the sense of the FRP (we call this the Setting 1).

On the other hand, since $T_S(t)|_{\mathcal{H}}$ is an isometric C_0-group on $\mathcal{H}$, we have $\sigma(S|_{\mathcal{H}}) = \sigma_A(S|_{\mathcal{H}})$ (the approximate point spectrum of $S|_{\mathcal{H}}$) by Proposition IV.1.10 in [28]. However, Theorem 3.47 only applies to the point spectrum of $S|_{\mathcal{H}}$. What one *wants* to have, in general, is the validity of an implication of the form

$$\forall \lambda \in \mathbb{R}: \quad i\lambda \in \sigma(S|_{\mathcal{H}}) \implies H_K(i\lambda)L \neq 0 \tag{3.68}$$

provided that a control law $u(t) = Kz(t) + Lw(t)$ solves the FRP for $W = \mathcal{H}$ (we call this the Setting 2).

It is remarkable that these two settings are equivalent for sup-normed spaces $\mathcal{H}$, as shown below.

Proposition 3.50. *Let $\lambda \in \mathbb{R}$ and let $\mathcal{H} \stackrel{s}{\hookrightarrow} BUC(\mathbb{R}, H)$ be a closed subspace. Then $\lambda \in sp_C(y)$ for some $y \in \mathcal{H}$ if and only if $i\lambda \in \sigma(S|_{\mathcal{H}})$.*

Proof. Let $\lambda \in sp_C(y)$ for some $y \in \mathcal{H}$ and let $S|_{\mathcal{H}_y}$ denote the restriction of $S|_{\mathcal{H}}$ to $\mathcal{H}_y$, where as usual $\mathcal{H}_y = \overline{\mathrm{span}}\{ T_S(t)|_{\mathcal{H}} y \mid t \in \mathbb{R} \}$ (closure in $\|\cdot\|_{\mathcal{H}} = \|\cdot\|_\infty$). By the results cited in Section A.1 we have that $\sigma(S|_{\mathcal{H}_y}) = isp_C(y)$. Then by the above, $i\lambda \in \sigma(S|_{\mathcal{H}_y})$. Hence by Corollary IV.2.16 in [28] $i\lambda \in \sigma(S|_{\mathcal{H}})$.

On the other hand, suppose that $i\lambda \in \sigma(S|_{\mathcal{H}})$. For each $w \in \mathcal{H}$ the map $R_w : \{ \eta \in \mathbb{C} \mid \Re(\eta) > 0 \} \to \mathcal{H} : \eta \to R(\eta, S|_{\mathcal{H}})w$ is holomorphic. The local unitary spectrum [6] $\sigma_u(S|_{\mathcal{H}}, w)$ of w is defined as the set of points $\eta \in i\mathbb{R}$ to which R_w cannot be extended holomorphically. Since $S|_{\mathcal{H}}$ generates an isometric C_0-group, by the Banach-Steinhaus Theorem, we have

$$\sigma(S|_{\mathcal{H}}) = \bigcup_{w \in \mathcal{H}} \sigma_u(S|_{\mathcal{H}}, w) \tag{3.69}$$

Hence there exists $y \in \mathcal{H}$ such that $i\lambda \in \sigma_u(S|_{\mathcal{H}}, y)$. But for $\Re(\eta) > 0$ we have

$$R(\eta, S|_{\mathcal{H}})y = \int_0^\infty e^{-\eta t} T_S(t)|_{\mathcal{H}} y\, dt = \int_0^\infty e^{-\eta t} T_S(t)|_{\mathcal{H}_y} y\, dt = R(\eta, S|_{\mathcal{H}_y})y \tag{3.70}$$

Consequently, $R(\eta, S|_{\mathcal{H}_y})y$ cannot be extended holomorphically to $i\lambda$, so that $i\lambda \in \sigma_u(S|_{\mathcal{H}_y}, y)$. Again, since $\sigma(S|_{\mathcal{H}_y}) = \cup_{w \in \mathcal{H}_y} \sigma_u(S|_{\mathcal{H}_y}, w)$, we have $i\lambda \in \sigma(S|_{\mathcal{H}_y})$. By the results of Section A.1 then $\lambda \in sp_C(y)$ and the proof is complete. $\square$

Theorem 3.51. *Assume that $\mathcal{H}_0 \stackrel{s}{\hookrightarrow} BUC(\mathbb{R}, H)$ is a given sup-norm closed subspace. Then the following assertions are equivalent:*

1. *For every individual signal $y_{ref} \in \mathcal{H}_0$ which can be regulated in the sense of the FRP it is true that the implication (3.67) is valid for the operators K, L in a regulating control law.*

2. *For every sup-norm closed subspace $\mathcal{H} \stackrel{s}{\hookrightarrow} \mathcal{H}_0$, such that for $W = \mathcal{H}$ the FRP is solvable, it is true that the implication (3.68) is valid for the operators K, L in a regulating control law.*

Proof. Assume that the first assertion is true and that for an arbitrary $W = \mathcal{H} \overset{s}{\hookrightarrow} \mathcal{H}_0$ the FRP has a solution $u(t) = Kz(t) + Lw(t)$. In order to show that the implication (3.68) is valid (for these operators K, L) we take an arbitrary $i\lambda \in \sigma(S|_{\mathcal{H}})$ and show that necessarily $H_K(i\lambda)L \neq 0$. By Proposition 3.50 there exists $y \in \mathcal{H}$ such that $\lambda \in sp_C(y)$. But since $\mathcal{H}_y = \overline{\text{span}}\{ T_S(t)|_{\mathcal{H}} y \mid t \in \mathbb{R} \} \subset \mathcal{H}$ the same control law $u(t) = Kz(t) + Lw(t)$ regulates the individual signal y, i.e. solves the FRP for $W = \mathcal{H}_y$. Since implication (3.67) is valid, it is true that $H_K(i\lambda)L \neq 0$.

Assume then that the second assertion is true and that an arbitrary given individual signal $y_{ref} \in \mathcal{H}_0$ can be regulated in the sense of the FRP, i.e. there exists a control law $u(t) = Kz(t) + Lw(t)$ which solves the FRP for $W = \mathcal{H}_{y_{ref}}$. In order to show that the implication (3.67) is valid (for these operators K, L) we take an arbitrary $\lambda \in sp_C(y_{ref})$ and show that necessarily $H_K(i\lambda)L \neq 0$. According to the results of Section A.1 $i\lambda \in \sigma(S|_{\mathcal{H}_{y_{ref}}})$. Moreover, $u(t) = Kz(t) + Lw(t)$ solves the FRP for $W = \mathcal{H}_{y_{ref}}$ according to our convention. But $\mathcal{H}_{y_{ref}}$ is a closed subspace of $\mathcal{H}_0$ and since the second assertion holds true, we must have that $H_K(i\lambda)L \neq 0$. Hence also the first assertion holds true. $\qquad\square$

Theorem 3.51 shows that the Setting 1 is equivalent to the Setting 2 in sup-normed spaces. However, we do not know if these settings are in effect in general, i.e. we do not know if the assertions of Theorem 3.51 are true for a general pivot space $\mathcal{H}_0 \subset BUC(\mathbb{R}, H)$. Based on the concluding remarks of Section 3.4, we suspect that this is not the case in general. On the other hand, in the future it would be very interesting to see what the most general conditions for $\mathcal{H}_0$ are such that the above equivalent settings are in effect. Although this problem has not been solved in the present case study section, we were able to illustrate the complex relationship between system zeros and output regulation which only exists for infinite-dimensional exosystems.

3.7 Examples of feedforward output regulation

In this section we shall present various concrete examples to illustrate the feedforward output regulation theory developed in this chapter. Throughout the section we assume, in accordance with Proposition 2.3, that $W = \mathcal{H} \hookrightarrow BUC(\mathbb{R}, H)$, $S = S|_{\mathcal{H}}$, $Q = \delta_0 \in \mathcal{L}(\mathcal{H}, H)$, $P \in \mathcal{L}(\mathcal{H}, Z)$ and $w(0) = y_{ref} \in \mathcal{H}$. Hence we are interested in the asymptotic tracking of all reference signals in some function space $\mathcal{H}$, under such disturbances which are known to have similar dynamical properties as the reference signals.

Our first example concerns finite-dimensional systems and infinite-dimensional exosystems.

Example 3.52. *Consider a finite-dimensional exponentially stable SISO-plant that is not subject to any disturbances (i.e. $P = 0$). Consider reference signals in the Sobolev space $H_{per}^{\gamma}(0, p)$, $\gamma > \frac{1}{2}$, i.e. set $\mathcal{H} = H(f_n, \omega_n)$ for $I = \mathbb{Z}$ and $f_n = \sqrt{1 + \omega_n^2}^{\gamma}$ and $\omega_n = \frac{2\pi n}{p}$ for each $n \in \mathbb{Z}$. Let N denote the relative degree of the transfer function $H(s)$ of the plant, and assume that there are no transmission zeros in the set of complex Fourier frequencies $\{ i\omega_n \mid n \in \mathbb{Z} \}$ of the reference signals.*

By the relative degree condition, $H(i\omega_n)^{-1}$ is of order $\mathcal{O}(|\omega_n|^N)$ as $n \to \pm\infty$. Let us define L as in (3.47) (with $K = 0$ since the plant is already stable and with $H_d(n) \equiv 0$ because there are no disturbances). Then for $\gamma > N + \frac{1}{2}$ the condition (3.55) holds true. This implies that for such γ, all reference signals $y_{ref} \in H_{per}^{\gamma}(0, p)$ can be asymptotically tracked using the control law $u(t) = Lw(t)$ by Corollary 3.38.

In the next example we show that there exist infinite-dimensional systems which cannot track all reference signals in $H_{per}^{\gamma}(0, p)$ for any $\gamma > \frac{1}{2}$ (in the sense of the FRP), even if there are no transmission zeros in the set of complex Fourier frequencies of the reference signals. This is in strong contrast to the finite-dimensional case, as is seen from Example 3.52 above. Moreover, Example 3.53 illustrates the fact that transmission zeros are not the only cause of trouble in output regulation problems for general bounded uniformly continuous exogenous signals: Output regulation of periodic signals is only possible if the smoothness of these signals is "compatible" with the high frequency damping rate of the plant. We refer the reader to [17, 78] for the relevant notation and definitions of the below example.

Example 3.53. *Let $f \in \mathcal{D}(\mathbb{R})$ be a test function such that $\mathrm{supp}(f) \subset [0, a]$, where $0 < a < \infty$. Let $Z = \{ g \in H^1(0, a) \mid g(a) = 0 \}$ where $H^1(0, a)$ denotes the standard Sobolev space. Since Z is the null space of a continuous linear functional, it is a closed subspace of $H^1(0, a)$. Let A be the generator of the left shift semigroup $T_A(t)$ on Z defined by $(T_A(t)g)(x) = g(x+t)$ for $x+t \leq a$, and $(T_A(t)g)(x) = 0$ otherwise, for every $g \in Z$. Clearly $T_A(t)$ is strongly continuous (cf. Example I.5.4 in [28]) and exponentially stable on Z. Let C be the point evaluation at the origin, i.e. $Cg = g(0)$ for every $g \in Z$. It is easy to show (see e.g. [55]) that $C \in \mathcal{L}(Z, \mathbb{C})$. Finally, let $Bu = fu$ for $u \in \mathbb{C}$. Then evidently $B \in \mathcal{L}(\mathbb{C}, Z)$. Moreover, in this case the system (1.1) (with $D = 0$ and $\mathcal{U}_{dist} = 0$) has $f(t)$ as its impulse response [17]. In fact, $CT_A(t)B = \left[f(x+t) \right]_{x=0} = f(t)$ for every $t \geq 0$.*

By applying Fourier transforms, we see that the transfer function $H(i\omega) = \mathcal{F}(f)(i\omega)$ is a rapidly decreasing function. Hence $\sup_{\omega \in \mathbb{R}}(1 + \omega^2)^N |H(i\omega)| < \infty$ for every $N \in \mathbb{N}$. For the purpose of output regulation, we may assume that $H(i\omega_n) = H(i\frac{2\pi n}{p}) \neq 0$ for every $n \in \mathbb{Z}$. Otherwise there would exist $m \in \mathbb{Z}$ such that the regulator equations (3.10) are not solvable in the linear span of the exponential function $t \to e^{i\omega_m t}$ (see the proof of Proposition 3.33), and hence there would exist an infinite-dimensional system which cannot track all reference signals in $H_{per}^\gamma(0, p)$ for any $\gamma > \frac{1}{2}$. But by the above, $H(i\omega_n)^{-1}$ grows faster than every polynomial in n as $n \to \pm\infty$. By Corollary 3.38 and the fact that L (if it exists in $\mathcal{L}(H_{per}^\gamma(0, p), \mathbb{C})$) is unique[8] in this case, for arbitrary $\gamma > \frac{1}{2}$ there always exists $y_{ref} \in H_{per}^\gamma(0, p)$ which this system cannot asymptotically track using the control law $u(t) = Lw(t)$. This shows that the FRP is not solvable for any $\gamma > \frac{1}{2}$, or, in other words, if the system can asymptotically track all signals in $H(\omega_n, f_n)$, then $H(\omega_n, f_n) \subset C^\infty(\mathbb{R}, \mathbb{C})$ (the space of infinitely smooth functions on $\mathbb{R}$). Moreover, the situation cannot be remedied by using an auxiliary stabilizing state feedback $Kz(t)$ by Theorem 3.40.

The following example illustrates the results of Section 3.5 for an exponentially stable infinite-dimensional system described by a delay differential equation.

Example 3.54. *Let $a > 0$, $r \neq 0$, $\tau_1 > \tau_2 > 0$ and consider the following scalar delay differential equation with control and observation [77]:*

$$\dot{x}(t) = -ax(t) - b\big[x(t - \tau_1) + x(t - \tau_2)\big] + u(t) \tag{3.71a}$$

$$y(t) = rx(t), \quad t \geq 0 \tag{3.71b}$$

Our goal is to study asymptotic tracking of the reference signals in the Sobolev spaces $H_{per}^\gamma(0, p)$, $\gamma > \frac{1}{2}$, in the disturbance-free case $(P = 0)$.

Taking the initial condition into account, the pair (3.71) can be formulated as a plant of the form (1.1) in which $D = 0$ and $\mathcal{U}_{dist} = 0$, using the techniques of Curtain and Zwart [17]. Moreover, it can be shown (see e.g. [17] Lemma 4.3.9) that the transfer function $H(s) = CR(s, A)B$ of this plant is given by

$$H(s) = \frac{r}{s + a + b(e^{-s\tau_1} + e^{-s\tau_2})} \tag{3.72}$$

for those $s \in \mathbb{C}$ at which the denominator is not equal to zero.

---

[8]because $L = \Gamma - K\Pi = \Gamma$ (since $K = 0$) and the elements $\Pi\phi_n$ and $\Gamma\phi_n$ in (3.39) and (3.40) are unique for each $n \in I$; see Remark 3.34.

The semigroup generated by A is exponentially stable if and only if $s + a + b(e^{-s\tau_1} + e^{-s\tau_2}) \neq 0$

for all $s \in \{ z \in \mathbb{C} \mid \Re(z) \geq 0 \}$ ([17] Theorem 5.1.7). Ruan and Wei [77] give a complete

characterization (in terms of a, b, τ_1 and τ_2) of those instances in which all roots of equation

$s + a + b(e^{-s\tau_1} + e^{-s\tau_2}) = 0$ have negative real parts. In their characterization, the parameter b

lies on an interval (b_0^-, b_0^+). We assume that the semigroup generated by A is exponentially stable.

By the above discussion, then $i\omega_n = i\frac{2\pi n}{p} \in \rho(A)$ and $H(i\omega_n) \neq 0$ for every $n \in \mathbb{Z}$.

It is evident that for every $\gamma > \frac{3}{2}$, $\sum_{n=-\infty}^{\infty} |H(i\omega_n)^{-1}|^2 (1 + \omega_n^2)^{-\gamma} < \infty$. Consequently, by

Corollary 3.38 the system can track all reference signals in $H_{per}^{\gamma}(0, p)$ for $\gamma > \frac{3}{2}$. The actual

control law which achieves the asymptotic tracking of $y_{ref} = \sum_{n \in \mathbb{Z}} \hat{y}(n) e^{i\omega_n \cdot} \in H_{per}^{\gamma}(0, p)$ is given

by $u_{y_{ref}} = \sum_{n \in I} H(i\omega_n)^{-1} \hat{y}(n) e^{i\omega_n \cdot}$ (see Section 3.5).

A particularly important class of infinite-dimensional SISO systems, for which the solvability of

the regulator equations (3.10) and the FRP can be readily verified for an infinite-dimensional exo-

system, is furnished by exponentially stabilizable parabolic partial differential equations. In such

cases the semigroup governing the dynamical behaviour of the system is often analytic [17], and it is

well-known (see e.g. Theorem 2.47 in [65]) that whenever A generates an analytic C_0-semigroup,

$\|R(i\omega, A)\|$ is of order $\mathcal{O}(|\omega|^{-1})$ as $\omega \to \pm\infty$. The identity

$$H_K(i\omega)^{-1} = (I - KR(i\omega, A)B)H(i\omega)^{-1} \tag{3.73}$$

obtained in Theorem 8.2 — which is valid here for all $i\omega \in \rho(A) \cap \rho(A + BK)$ if $H(i\omega) \neq 0$ —

then readily shows that we only have to estimate $|H(i\omega)|$ in order to use the methods of Section

3.5 in the solution of the output regulation problem. These methods apply for periodic exogenous

signals, but more general signals can be treated using the results of Section 8.1. The below example

illustrates periodic tracking for an exponentially stabilizable infinite-dimensional system governed

by a parabolic partial differential equation.

Example 3.55. *Consider the same disturbance-free controlled one-dimensional heat equation on*

the interval $[0, 1]$ with Neumann boundary conditions as in Example 1.1. Our goal is to study

asymptotic tracking of the reference signals in the generalized Sobolev spaces $H(f_n, \omega_n)$, in the

disturbance-free case $(P = 0)$.

It can be shown that the transfer function of this heat plant is $H(s) = \frac{2\sinh(\sqrt{s}/2)}{s\sqrt{s}\cosh(\sqrt{s}/2)}$, for $s \in \rho(A)$

[12]. Now $i\omega_n = i\frac{2\pi n}{p} \in \rho(A)$ for $n \neq 0$ [12], and $i\omega_n$ is not a transmission zero of this plant

for $n \neq 0$. Let $I = \mathbb{Z} \setminus \{0\}$, and let $f_n = \sqrt{1 + \omega_n^2}^{\gamma}$ for all $n \in I$ and some $\gamma > \frac{1}{2}$ which is to be determined. Let K be the bounded exponentially stabilizing state feedback operator for the pair (A, B) given in Example 1.1. Then by Lemma 3.32, $H_K(i\omega_n)^{-1}$ exists for $n \neq 0$. Let us define L as in (3.47) (with $P = 0$ and $D = 0$). By some elementary calculations we have that $H(i\omega_n)^{-1}$ and $H_K(i\omega_n)^{-1}$ are of order $\mathcal{O}(|\omega_n|^{\frac{3}{2}})$ as $n \to \pm\infty$. Consequently $\left(H_K(i\omega_n)^{-1} \sqrt{1 + \omega_n^2}^{-\gamma} \right)_{n \in I} \in \ell^2$ if $\gamma > 2$. By Corollary 3.38, this system is capable of asymptotically tracking those periodic reference signals in $H_{per}^{\gamma}(0, p)$, with $\gamma > 2$, that lack the constant term in the Fourier series description. More accurate information on the signals which can be asymptotically tracked could possibly be obtained by working out the explicit expression for $H_K(s)$.

As pointed out already in Example 1.1, Byrnes, Laukó, Gilliam and Shubov have thoroughly studied and simulated output regulation problems for the above heat plant in the case of constant and sinusoidal reference/disturbance signals (see Section III and Section VI of [12]). Here we extended their analysis for general periodic functions in certain Sobolev spaces.

Many infinite-dimensional SISO systems which occur e.g. as models of flexible structures have the following properties [70]:

- The state space Z is a Hilbert space.

- The actuators and sensors are collocated, i.e. $B = C^*$.

- The operator $A - BB^*$ generates a strongly stable C_0-semigroup on Z.

- $\sigma(A - BB^*) \cap i\mathbb{R} = \emptyset$.

- $\pm i\infty$ are points of accumulation for $\sigma(A - BB^*)$.

Although the last property above implies that the system is not exponentially stable, for such systems the solvability of the FRP can still be fairly easily verified whenever $S \in \mathcal{L}(W)$, using Corollary 8.6. In fact, in this case it is sufficient that $\sigma(S) \subset \rho(A)$ and $H(i\omega) = CR(i\omega, A)B + D \neq 0$ for all $i\omega$ in the compact subset $\sigma(S) \subset i\mathbb{R}$. We emphasize that here W can be infinite-dimensional, too. In particular, Lemma A.8 and Proposition A.9 provide a useful and sensible way to approximate the reference signals in order to obtain a bounded system operator S in the infinite-dimensional exosystem (2.2). In the following example we shall explicitly work out the actual control law achieving output regulation in the simpler case in which A already generates a strongly stable C_0-semigroup (i.e. no state feedback stabilization is needed):

Example 3.56. *The following is a model for the displacement in a weakly damped vibrating string of unit length, with clamped ends (see [70] and the references therein):*

$$\frac{\partial^2 v(x,t)}{\partial t^2} + M\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2}, \quad \text{for } t \geq 0 \text{ and } 0 \leq x \leq 1 \tag{3.74a}$$

$$v(0,t) = v(1,t) = 0, \quad \text{for } t \geq 0 \tag{3.74b}$$

Here M is a damping operator which will be defined shortly. We define the operator U by $Uv = -\frac{\partial^2 v}{\partial x^2}$ with $\mathcal{D}(U) = \{\, v \in H^2(0,1) \mid v(0) = v(1) = 0 \,\}$, where $H^2(0,1)$ denotes the standard Sobolev space on the unit interval. It can be shown that U has eigenvalues $\lambda_k = k^2\pi^2$, $k = 1,2,\ldots$ and the corresponding eigenvectors $\phi_k(x) = \sqrt{2}\sin(k\pi x)$ constitute an orthonormal basis in $L^2(0,1)$.

The damping operator M is defined by $Mv = \epsilon\langle g, v\rangle_{L^2(0,1)}g$ where $\epsilon > 0$ and

$$g = \sum_{k=1}^{\infty} \gamma_k\phi_k \tag{3.75}$$

with γ_k satisfying $0 < |\gamma_k| \leq \frac{m}{\sqrt{\lambda_k}}$ (for example, $\gamma_k = \frac{1}{k}$) for some $m > 0$. Next we define the Hilbert space $Z = L^2(0,1) \times L^2(0,1)$ with the natural inner product $\langle\cdot,\cdot\rangle$ and introduce

$$z(t) = \begin{pmatrix} U^{\frac{1}{2}}v(x,t) \\ \frac{\partial v(x,t)}{\partial t} \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} 0 & U^{\frac{1}{2}} \\ -U^{\frac{1}{2}} & M \end{pmatrix} \tag{3.76}$$

Then equation (3.74) can be rewritten as $\dot{z}(t) = Az(t)$, $z(0) \in Z$, and it can be shown that A generates a strongly (but not exponentially) stable C_0-semigroup $T_A(t)$ on Z. Furthermore, the eigenvalues ν_k of A satisfy

$$\nu_k = ik\pi - \frac{\epsilon}{2}\gamma_{|k|}^2 + \mathcal{O}(\frac{\epsilon^2}{k^2\pi^2}), \quad k \neq 0 \tag{3.77}$$

all of which have a negative real part. The corresponding eigenvectors $(\psi_k)_{k\neq 0}$ form a Riesz basis in Z (the biorthogonal sequence is denoted by $(\psi_k^*)_{k\neq 0}$).

Consider then the application of distributed control and observation to the system (3.74) in the following sense. For a control operator $B = \begin{pmatrix} 0 \\ b \end{pmatrix} \in \mathcal{L}(\mathbb{C}, Z)$ and the observation operator $C = B^* \in \mathcal{L}(Z, \mathbb{C})$ the (disturbance-free) SISO plant is described by the equations

$$\dot{z}(t) = Az(t) + Bu(t), \quad z(0) \in Z, \quad t \geq 0 \tag{3.78a}$$

$$y(t) = B^*z(t), \quad t \geq 0 \tag{3.78b}$$

Let $\Lambda \subset \mathbb{R}$ be a given compact set and consider the reference signals in $\mathcal{H} = \Lambda(\mathbb{R}, \mathbb{C})$ (see Chapter 2); in this case the Carleman spectrum of any reference signal is contained in the set

$\Lambda \subset \mathbb{R}$. *According to Proposition 2.3 we choose the exosystem (2.2) such that* $W = \mathcal{H}$, $S = S|_{\mathcal{H}}$, $Q = \delta_0$, $P = 0$ *and* $w(0) = y_{ref} \in \mathcal{H}$. *We assume that* $H(i\omega) = B^* R(i\omega, A)B \neq 0$ *for all* $\omega \in \Lambda$. *This condition can be verified, at least in principle, using the series representation*

$$R(\lambda, A) = \sum_n \frac{\langle \cdot, \psi_n^* \rangle \psi_n}{\lambda - \nu_n}, \quad \lambda \in \rho(A) \tag{3.79}$$

if the operator $B \in \mathcal{L}(\mathbb{C}, Z)$ *can be chosen freely.*

By Theorem 3.6 it is then sufficient to use the control law $u(t) = \Gamma T_S(t)|_{\mathcal{H}} y_{ref}$, where $\Gamma \in \mathcal{L}(\mathcal{H}, \mathbb{C})$ is one of the operators Π, Γ solving the regulator equations (3.10), for the asymptotic tracking of the reference signals $y_{ref} \in \mathcal{H}$. Proposition 8.5 together with Lemma 8.4 (for $K = 0$ and $P_n = 0$) and Theorem V.8.2 in [86] give

$$\Gamma y_{ref} = \frac{1}{2\pi i} \oint_\gamma H(\lambda)^{-1} \delta_0 R(\lambda, S|_{\mathcal{H}}) y_{ref} d\lambda, \quad \forall y_{ref} \in \mathcal{H} \tag{3.80}$$

Here γ *is a suitable contour in* $\rho(A)$ *enclosing* $\sigma(S|_{\mathcal{H}})$. *Observe that now* $S|_{\mathcal{H}} \in \mathcal{L}(\mathcal{H})$ *because* Λ *is a compact set (see Lemma A.8). Consequently, such a contour* γ *indeed exists because* $\rho(A) = \{\lambda \in \mathbb{C} \mid \inf_k |\lambda - \nu_k| > 0\}$ *(this can be shown precisely as in the proof of Theorem 2.3.5 in [17] using the Riesz basis property of the eigenvectors of* A*).*

Since $y_{ref}(t) = \delta_0 T_S(t)|_{\mathcal{H}} y_{ref}$ for all $t \in \mathbb{R}$ and all $y_{ref} \in \mathcal{H}$, the Laplace transform $\widehat{y_{ref}}(\lambda) = \delta_0 R(\lambda, S|_{\mathcal{H}}) y_{ref}$ for $\lambda \in \rho(S|_{\mathcal{H}})$ (by analytic continuation). Hence

$$\Gamma y_{ref} = \frac{1}{2\pi i} \oint_\gamma H(\lambda)^{-1} \widehat{y_{ref}}(\lambda) d\lambda, \quad \forall y_{ref} \in \mathcal{H} \tag{3.81}$$

Moreover, since by [2] p. 293 for all $\lambda \in \mathbb{C} \setminus i\mathbb{R}$ *we have that* $\widehat{T_S(t)|_{\mathcal{H}} y_{ref}}(\lambda) = \widehat{y_{ref}(\cdot + t)}(\lambda) = e^{\lambda t}\left(\widehat{y_{ref}}(\lambda) - \int_0^t e^{-\lambda s} y_{ref}(s) ds\right)$, *and since the function* $\lambda \to \int_0^t e^{-\lambda s} y_{ref}(s) ds$ *is holomorphic, the control law which achieves asymptotic tracking of* y_{ref} *is given by*

$$\Gamma T_S(t)|_{\mathcal{H}} y_{ref} = \frac{1}{2\pi i} \oint_\gamma H(\lambda)^{-1} \widehat{T_S(t)|_{\mathcal{H}} y_{ref}}(\lambda) d\lambda \tag{3.82}$$

$$= \frac{1}{2\pi i} \oint_\gamma H(\lambda)^{-1} e^{\lambda t}\left(\widehat{y_{ref}}(\lambda) - \int_0^t e^{-\lambda s} y_{ref}(s) ds\right) d\lambda \tag{3.83}$$

$$= \frac{1}{2\pi i} \oint_\gamma H(\lambda)^{-1} e^{\lambda t} \widehat{y_{ref}}(\lambda) d\lambda, \quad \forall t \geq 0 \tag{3.84}$$

Clearly if $\Lambda = \{\omega_n \mid n \in I, I$ *is finite and* $\omega_n \neq \omega_m$ *for* $n \neq m\}$, *then the above control law* $u(t) = \Gamma T_S(t)|_{\mathcal{H}} y_{ref}$ *reduces to*

$$u(t) = \sum_{n \in I} \frac{y_n}{H(i\omega_n)} e^{i\omega_n t} \tag{3.85}$$

which is analogous to the one found in Section 3.5. Here we have used the unique representation $y_{ref} = \sum_{n \in I} y_n e^{i\omega_n \cdot}$. *On the other hand, if* $\Lambda = [-n, n]$, *then the reference signals in* $\Lambda(\mathbb{R}, \mathbb{C})$ *can be constructed e.g. by taking convolutions of general bounded uniformly continuous signals with the Fejér kernel, as described in Remark 2.29 and Proposition A.9.*

In the previous examples we did not allow for any overlap between the spectra of the (stabilized) plant and the exosystem operator S at $\pm i\infty$. We conclude this section with an example illustrating the case in which such overlap exists. In this case the complex frequencies of the reference signals "mix" with those of the plant at infinity. To the author's knowledge, none of the existing methods found in the output regulation literature apply to such a situation.

Example 3.57. *Let* $p > 0$, *let* $\omega_n = \frac{2\pi n}{p}$ *for all* $n \in \mathbb{Z}$, *let* $H = \mathbb{C}$ *and let* Z *be a Hilbert space with an inner product* $\langle \cdot, \cdot \rangle$ *and an orthonormal basis* $(\psi_n)_{n \in \mathbb{Z}}$. *Consider a linear control system* (1.1) *where* $A = \sum_{n \in \mathbb{Z}} \left[-\frac{1}{|n|+1} + i\omega_n \right] \langle \cdot, \psi_n \rangle \psi_n$, *with* $\mathcal{D}(A) = \{ z \in Z \mid \sum_{n \in \mathbb{Z}} \left| -\frac{1}{|n|+1} + i\omega_n \right|^2 |\langle z, \psi_n \rangle|^2 < \infty \}$, $Bu = \psi_0 u$ *for all* $u \in \mathbb{C}$, $C = \langle \cdot, \psi_0 \rangle$ *and* $P = 0$, $D = 0$. *By [17] it is clear that* A *generates a* C_0-*semigroup* $T_A(t)$ *on* Z. *In addition,* $T_A(t)$ *is strongly stable by the Arendt-Batty-Lyubich-Vũ Theorem [28].*

We shall consider asymptotic tracking of the $p-$*periodic reference signals in the Sobolev spaces* $\mathcal{H} = H_{per}^\gamma(0, p)$, $\gamma > \frac{1}{2}$, *in the sense of the FRP using Proposition 2.3. Although rather artificial, this output regulation problem is interesting because* $\sigma(A)$ *and* $\sigma(S|_{\mathcal{H}})$ *intersect at* $\pm i\infty$.

Since A *is a Riesz spectral operator [17], clearly*

$$R(\lambda, A) = \sum_{n \in \mathbb{Z}} \frac{1}{\lambda + \frac{1}{|n|+1} - i\omega_n} \langle \cdot, \psi_n \rangle \psi_n, \quad \lambda \in \rho(A) \tag{3.86}$$

Hence we have $R(\lambda, A)B = \frac{1}{\lambda+1} \psi_0$ *and the transfer function* $H(\lambda)$ *of the plant satisfies*

$$H(i\omega_k) = CR(i\omega_k, A)B = \frac{1}{i\omega_k + 1} \neq 0, \quad \forall k \in \mathbb{Z} \tag{3.87}$$

by the orthonormality of the basis $(\psi_n)_{n \in \mathbb{Z}}$.

Using the techniques of Section 3.5 we readily see that the following operators Π, Γ *— whenever*

bounded — solve the regulator equations (3.10) (with $S = S|_{\mathcal{H}}$ and $Q = \delta_0$):

$$\Gamma y_{ref} = \sum_{n \in \mathbb{Z}} \frac{\widehat{y_{ref}}(n)}{H(i\omega_n)} = \sum_{n \in \mathbb{Z}} \widehat{y_{ref}}(n)[i\omega_n + 1], \quad \forall y_{ref} \in \mathcal{H} \tag{3.88}$$

$$\Pi y_{ref} = \sum_{n \in \mathbb{Z}} \widehat{y_{ref}}(n) R(i\omega_n, A) B \Gamma e^{i\omega_n \cdot} = \sum_{n \in \mathbb{Z}} \frac{\widehat{y_{ref}}(n)}{H(i\omega_n)} R(i\omega_n, A) B \tag{3.89}$$

$$= \sum_{n \in \mathbb{Z}} \widehat{y_{ref}}(n)[i\omega_n + 1] \frac{1}{i\omega_n + 1} \psi_0 = \psi_0 \delta_0 y_{ref}, \quad \forall y_{ref} \in \mathcal{H} \tag{3.90}$$

Here $\widehat{y_{ref}}(n)$ is the nth L^2-Fourier coefficient of $y_{ref} \in \mathcal{H}$. Hence it remains to show the boundedness of Γ for a suitable $\gamma > \frac{1}{2}$. This can be done by employing the Schwartz inequality as follows:

$$\|\Gamma y_{ref}\| \leq \sum_{n \in \mathbb{Z}} \left| \widehat{y_{ref}}(n)[i\omega_n + 1] \right| \leq \sqrt{\sum_{n \in \mathbb{Z}} |\widehat{y_{ref}}(n)|^2 (1 + \omega_n^2)^\gamma} \sqrt{\sum_{n \in \mathbb{Z}} \frac{1 + \omega_n^2}{(1 + \omega_n^2)^\gamma}} \tag{3.91}$$

So that $\|\Gamma y_{ref}\| \leq M \|y_{ref}\|_{\mathcal{H}}$ for all $y_{ref} \in \mathcal{H}$ whenever $\gamma > \frac{3}{2}$. Then by Theorem 3.6, for $\gamma > \frac{3}{2}$ the control law $u(t) = \Gamma T_S(t)|_{\mathcal{H}} y_{ref} = \sum_{n \in \mathbb{Z}} \widehat{y_{ref}}(n) e^{i\omega_n t}[i\omega_n + 1]$ achieves asymptotic tracking of an arbitrary $y_{ref} \in \mathcal{H} = H^\gamma_{per}(0, p)$.

Chapter 4

Error feedback output regulation

Although the resulting controllers are relatively simple, and hence quite appealing, the feedforward output regulation theory developed in Chapter 3 is not applicable in many practical problems. There are two principal reasons for this: The state of the plant may not be directly available for measurement — state feedback cannot be used in the stabilization of the closed loop system — and the controller does not lead to a robust (i.e. structurally stable) design. In the present chapter we shall study a more realistic output regulation problem, the error feedback regulation problem (EFRP). In the EFRP the state of the plant need not be explicitly available to us; the controller only incorporates feedback from the tracking error signal, which is directly available for measurement. Furthermore, as we shall see in Chapter 6, once we have solved the EFRP, it is often not very difficult to also achieve a degree of robustness in output regulation.

As opposed to the *static* controllers solving the FRP in Chapter 3, a solution of the EFRP involves the construction of a *dynamic* controller on some Banach state space X. The dynamic controller should appropriately stabilize the closed loop system consisting of the plant and the controller, and it should achieve the asymptotic tracking/rejection of the signals generated by the exosystem (2.2). Thus the solution of the EFRP is a similar, but a somewhat more complex, process than the solution of the FRP.

For finite-dimensional linear systems and simple reference/disturbance signals generated by systems of linear ordinary differential equations, error feedback regulation problems analogous to the EFRP were studied intensively in the 1970s. Complete solutions now exist e.g. in the work of Francis, Wonham and Davison [24, 29, 32, 93]. While Davison (with his coworkers) prima-

rily studied error feedback regulation problems using the so-called servocompensators, Francis and Wonham initiated what is nowadays known as geometric output regulation theory. The terminology here stems from the fact that these authors studied output regulation problems using geometric terms, such as subspace inclusions. The geometric approach allows for a general treatment of the output regulation problems without any regard to the choice of the controller's parameters. A particularly important result, due to Francis (see Proposition 3 of [29]), arising from the geometric output regulation theory of finite-dimensional systems is the necessary structure of an error feedback controller achieving output regulation: Under certain observability assumptions the exosystem dynamics must be embedded in the controller dynamics. A concrete version of this result was also obtained by Davison in [21].

During the past three decades several authors have generalized the work of Francis and Wonham for infinite-dimensional linear systems and finite-dimensional exosystems. Already in the 1970s Bhat [7] generalized the results of [29] with an emphasis on time-delay systems. In the early 1980s Schumacher [80] constructed finite-dimensional error feedback controllers for such infinite-dimensional plants in which the system operator A has compact resolvent and a complete set of generalized eigenfunctions. His solution of the output regulation problem is also expressed in geometric terms (cf. Theorem 3.1 in [80]). In [79] Schumacher studied the regulator problem somewhat more indirectly from the compensator design point of view; among his assumptions was discreteness of $\sigma(A)$. Several years later Byrnes et al. [12] generalized sections 1-3 of [29] for infinite-dimensional systems in such a way that the geometric conditions were explicitly replaced by the regulator equations (3.10) (with $D = 0$). These equations — which are in another form also present in the finite-dimensional work [29, 31] and in the paper [80] of Schumacher — express the geometric conditions for output regulation in an operator-theoretic way. This is particularly useful for the application of semigroup methods. Byrnes et al. [12] showed that the solvability of a feedforward regulation problem, the solvability of an error feedback regulation problem and the solvability of the regulator equations (3.10) (with $D = 0$) are all equivalent to each other provided that the plant and the finite-dimensional neutrally stable linear exogenous signal generator (2.1) have sufficient stabilizability properties. The dynamic controller of Byrnes et al. [12] solving the error feedback regulation problem is obtained from a direct generalization of the "synthesis algorithm" (SA) of Francis [29].

During the past few decades several authors have also generalized the finite-dimensional results

of Davison for infinite-dimensional systems and finite-dimensional exosystems. We mention Pohjo-lainen [73, 74], Hämäläinen and Pohjolainen [35, 34], Ukai and Iwazumi [87] and refer the reader to the monograph [33] of Timo Hämäläinen for a detailed description of the respective contributions of these (and other related) papers.

In this chapter we shall extend the error feedback regulation theory cited above for bounded uniformly continuous reference/disturbance signals generated by the exosystem (2.2). Our basic strategy is to generalize the marvelous argument of Wonham [93], Francis [29] and Byrnes et al. [12] which casts the error feedback regulation problem as a feedforward regulation problem for an *extended* system. Once we have accomplished this, the feedforward methods developed in Chapter 3 immediately yield necessary and sufficient conditions — in terms of closed loop stability and an extended set (4.3) of regulator equations — for the solvability of the EFRP. We shall also see that if the controller's parameters are chosen appropriately, then the solvability of these extended regulator equations (4.3) reduces to the solvability of the same simpler (feedforward) regulator equations (3.10) as were utilized in Chapter 3.

However, in contrast to the FRP, it turns out that if the exosystem (2.2) is infinite-dimensional, then the closed loop in an error feedback control system is notoriously difficult to stabilize expo-nentially. This is the same phenomenon that occurs in the classical repetitive control literature (see Chapter 1), and it is a consequence of the fact that the dynamical behaviour of the exosystem must — under certain assumptions — be embedded in *any* controller which solves the EFRP. As we shall show in this chapter, an infinite-dimensional exosystem (2.2) is often impossible to stabilize exponentially; hence exponential closed loop stability cannot be required in the EFRP either if W is infinite-dimensional.

In view of the above negative result it is quite fortunate that the *strong* stabilizability of closed loop (EFRP) control systems can very often in practice be achieved under certain realistic condi-tions. Hence error feedback output regulation of general bounded uniformly continuous exogenous signals is often possible in the framework of this thesis, even for such infinite-dimensional plants (1.1) for which $D = 0$. We point out that in the repetitive control scheme the asymptotic tracking of $p-$periodic reference signals cannot be achieved even for finite-dimensional plants (1.1) if there is no feedthrough, i.e. $D = 0$ [36, 92, 95, 96]. We also emphasize that as was the case with the FRP, also in the case of the EFRP we can study output regulation for prespecified spaces of reference and disturbance functions using the techniques of Chapter 2. In addition to this, in our framework

it is possible to take into account the smoothness of the regulated exogenous signals. This is crucial because we saw in Chapter 3 that in certain cases even a simpler feedforward controller can only achieve output regulation if the exogenous signals are smooth enough. The existing solutions of repetitive control problems are based on such frequency domain techniques which do not take into account the smoothness of the periodic signals to be regulated.

In the following we shall review the contents of this chapter in more detail, and we shall more precisely indicate the respective contributions of this thesis.

Section 4.1: We shall define the EFRP. This is the same error feedback regulation problem as studied in [12] except that we allow for general bounded uniformly continuous reference/disturbance signals generated by the (possibly infinite-dimensional) exosystem (2.2). Moreover, here $D \neq 0$ is possible, and we only require that the closed loop system operator, consisting of the plant and the controller, with the exogenous system detached, generates a strongly stable C_0-semigroup (exponential stability was required in [12]).

Section 4.2: We shall show in Theorem 4.4 that the following is a sufficient condition for the solvability of the EFRP: The closed loop system operator generates a strongly stable C_0-semigroup and the extended regulator equations (4.3) have a solution. A key feature in this result is that the controller structure is fixed, but the parameters F, G and J of the dynamic controller can be freely chosen as long as the above conditions are met. Theorem 4.4 generalizes Corollary to Lemma 1 in [29] and Lemma 1 in [31] for infinite-dimensional systems and infinite-dimensional exosystems. Moreover, it generalizes Theorem IV.2 in [12] where the parameters of the controller are fixed as in the synthesis algorithm of Francis [29]. The results of this section are contained in [49].

Section 4.3: Assuming that the exosystem (2.2) generates admissible reference signals, we shall first show that the solvability of the extended regulator equations (4.3) is necessary for the solvability of the EFRP *and* the regularity of an operator $\begin{pmatrix} P \\ -GQ \end{pmatrix}$ for the closed loop semigroup. If the closed loop system is also exponentially stable, then the solvability of the EFRP is equivalent to the solvability of the extended regulator equations (4.3). Here we do not need the above regularity condition for the operator $\begin{pmatrix} P \\ -GQ \end{pmatrix}$, but the exosystem (2.2) must still generate admissible reference signals. Finally, assuming that the closed loop system has been exponentially stabilized, then the solvability of the EFRP with an exponentially fast decay

rate of the tracking/rejection error is equivalent to the solvability of the extended regulator equations (4.3). In this last result we do not have to explicitly require either the admissibility of the reference signals generated by the exosystem (2.2) or the regularity of the operator $\left(\begin{smallmatrix} P \\ -GQ \end{smallmatrix} \right)$ for the closed loop semigroup.

The results of this section are based on the work in [49]; they generalize Lemma 1 in [31] and the Corollary to Lemma 1 in [29] for infinite-dimensional systems and infinite-dimensional exosystems (2.2). In addition, they generalize Theorem IV.2 of Byrnes et al. [12] for infinite-dimensional exosystems (2.2) in such a way that the parameters F, G and J of the controller (4.1) need not be fixed a priori.

Section 4.4: Under an approximate observability assumption and under the assumption that the solvability of the extended regulator equations (4.3) is necessary for the solvability of the EFRP, in this section we shall prove the following *additional* necessary condition for the solvability of the EFRP: There must exist a subspace $X_0 \subset X$ of the controller's state space X which is invariant for the controller semigroup $T_F(t)$, and there must exist a linear bijection $M : W \to X_0$ such that $M \in \mathcal{L}(W, X)$ and $T_S(t) = M^{-1} T_F(t) M$ for all $t \geq 0$. Roughly stated, this result reads: In order to achieve output regulation in the sense of the EFRP, the exosystem dynamics must be embedded in the controller. This is a generalization of Proposition 3 in [29] for infinite-dimensional systems (1.1) and infinite-dimensional exosystems (2.2); to the author's knowledge no comparable results have appeared in the state space domain for infinite-dimensional systems and finite-dimensional exosystems. The results of this section have been principally developed in [49].

Section 4.5: We shall present two dynamic controllers (4.1) which solve the EFRP under certain assumptions.

- In Subsection 4.5.1 we shall generalize the synthesis algorithm of Francis [29] for infinite-dimensional systems (1.1) and infinite-dimensional exosystems (2.2). The assumptions under which the resulting controller solves the EFRP are as follows: The regulator equations (3.10) have a solution, $D = 0$, the pair (A, B) is exponentially stabilizable, and there exists $G = \left(\begin{smallmatrix} G_1 \\ G_2 \end{smallmatrix} \right) \in \mathcal{L}(H, Z \times \mathcal{G})$ for which $\mathcal{A}_F = \left(\begin{smallmatrix} A & P \\ 0 & S \end{smallmatrix} \right) - \left(\begin{smallmatrix} G_1 \\ G_2 \end{smallmatrix} \right) \left(\begin{smallmatrix} C & -Q \end{smallmatrix} \right)$ generates a strongly stable C_0-semigroup on $Z \times W$. The main result, Theorem 4.15, generalizes Theorem IV.2 in [12] because the latter result only applies for finite-dimensional

exosystems (2.1). The results of this subsection are essentially contained in [49].

– In Subsection 4.5.2 we shall briefly introduce a generalization of Davison's dynamic state feedback controller (see e.g. [39]) for infinite-dimensional systems (1.1) and infinite-dimensional exosystems (2.2). Sufficient conditions for output regulation to occur are, however, deferred to Chapter 6 where they arise as natural consequences of a general robustness theory.

Section 4.6: We shall study the stabilization of the exogenous system (2.2) and the closed loop system resulting from the controllers of Subsection 4.5.1 and Subsection 4.5.2.

– In Subsection 4.6.1 we shall show that if $\dim(W) = \infty$, then $S + \Delta$ does not generate an exponentially stable C_0-group for any compact operator $\Delta \in \mathcal{L}(W)$. Moreover, a similar argument shows that the situation cannot be improved very much by allowing for an unbounded but $S-$compact[1] operator Δ, because in this case $S + \Delta$ can only generate an exponentially stable C_0-semigroup if S has compact resolvent. These results are well-known (Corollary 3.58 in [65] and Theorem IV.5.35 in [57]), but our proofs seem to be new. In the literature these results are proved using the theory of essential spectra, whereas we rely on an argument based on Sylvester operator equations.

– In Subsection 4.6.2 we shall present some general sufficient conditions for the strong stabilizability of the exosystem (2.2). In particular, in Theorem 4.22 we shall prove that $S|_{\mathcal{E}} - \epsilon\delta_0^*\delta_0$ generates a strongly stable C_0-semigroup on a Hilbert space $\mathcal{E} \stackrel{s}{\hookrightarrow} BUC(\mathbb{R}, E)$ for all $\epsilon > 0$. As shown in Chapter 2, $S = S|_{\mathcal{E}}$ and $Q = \delta_0$ or $P = \delta_0$ is an important special case for the exosystem (2.2). The results of this subsection are new, although they rely heavily on [61, 85, 94]. Of the results of this subsection only Lemma 4.28 has been submitted for publication in [43].

– The remainder of this section is devoted to presenting such methods which can be used to establish the strong stability of the C_0-semigroups $T_{\mathcal{A}_F}(t)$ and $T_{\mathcal{A}_{D_K}}(t)$ generated by the operators $\mathcal{A}_F$ and $\mathcal{A}_{D_K}$ introduced in Section 4.5, respectively. As regards $\mathcal{A}_F$, the results of this section are essentially contained in [49]; however the results concerning the strong stability of $T_{\mathcal{A}_{D_K}}(t)$ have not been submitted for publication.

---

[1]A linear operator $\Delta : \mathcal{D}(W) \to W$ is $S-$compact if $\Delta R(\lambda, S)$ is a compact operator for one/all $\lambda \in \rho(S)$ (see Exercise III.2.18(1) in [28]). Such an operator Δ may be unbounded.

Section 4.7: We shall present an example of error feedback output regulation for a delay-differential equation. This example is from [49], and here the plant is precisely the same as in Example 3.54 of Chapter 3.

4.1 The error feedback regulation problem EFRP

In this section we shall formulate the error feedback regulation problem EFRP. It involves the construction of such a dynamic controller for the plant (1.1) on some Banach space X which achieves strong stability of the closed loop system (consisting of the plant and the controller without exosignals) and the asymptotic tracking of the reference signals in the presence of disturbances.

Definition 4.1 (EFRP). The task in the error feedback regulation problem EFRP is to find an error feedback controller of the form

$$\dot{x}(t) = Fx(t) + Ge(t), \quad x(0) \in X, \quad t \geq 0 \tag{4.1a}$$

$$u(t) = Jx(t) \tag{4.1b}$$

(in the mild sense) on some Banach state space X where F generates a C_0-semigroup, $G \in \mathcal{L}(H, X)$ and $J \in \mathcal{L}(X, H)$. The controller (4.1) must satisfy the following requirements:

1. In the closed loop system

$$\dot{z}(t) = Az(t) + BJx(t) + Pw(t), \quad t \geq 0 \tag{4.2a}$$

$$\dot{x}(t) = GCz(t) + (F + GDJ)x(t) - GQw(t), \quad t \geq 0 \tag{4.2b}$$

$$\dot{w}(t) = Sw(t), \quad t \in \mathbb{R} \tag{4.2c}$$

$$e(t) = Cz(t) + DJx(t) - Qw(t) \tag{4.2d}$$

   the C_0-semigroup $T_{\mathcal{A}}(t)$ generated by the closed loop operator $\mathcal{A} = \begin{pmatrix} A & BJ \\ GC & F+GDJ \end{pmatrix}$, with $\mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \times \mathcal{D}(F)$, on $Z \times X$ is strongly stable.

2. The tracking error $e(t) \to 0$ as $t \to \infty$ for any initial conditions $z(0) \in Z$, $x(0) \in X$ and $w(0) \in W$.

Remark 4.2. As in the case of the FRP it is implicitly assumed in Definition 4.1 that the exosystem's free parameters W, S, P and Q are fixed, but its initial state $w(0) \in W$ may vary.

Remark 4.3. Byrnes et al. [12] considered a similar error feedback regulation problem for finite-dimensional exosystems (2.1). However, they required exponential stability of $T_{\mathcal{A}}(t)$ and $D = 0$. It turns out that in the case of an infinite-dimensional exosystem (2.2) exponential stability of $T_{\mathcal{A}}(t)$ is usually impossible to achieve in practice, because the operator F often contains a copy of S (see Section 4.4). The problems is — as we shall show in Subsection 4.6.1 — that if W is infinite-dimensional, then $S + \Delta$ does not generate an exponentially stable C_0-semigroup for any compact operator $\Delta \in \mathcal{L}(W)$ (see also Corollary 3.58 in [65]).

4.2 Sufficient conditions for the solvability of the EFRP

The following result provides sufficient conditions for the solvability of the EFRP. We point out that, as opposed to [12], in Theorem 4.4 below we do not fix the parameters F, G and J of the controller (4.1). Instead, we prove for *general operators* F, G and J that if the closed loop system operator $\mathcal{A}$ generates a strongly stable C_0-semigroup, then output regulation follows if certain extended regulator equations (4.3) are satisfied. Our proof generalizes the marvelous argument utilized by Francis and Wonham in [29, 32, 31] and Byrnes et al. in [12], in which the EFRP is formulated as an FRP for the extended system (4.2). This enables the direct use of the feedforward output regulation theory of Chapter 3.

Theorem 4.4. *Assume that F, G and J in the controller* (4.1) *have been chosen such that the closed loop operator $\mathcal{A} = \left(\begin{smallmatrix} A & BJ \\ GC & F+GDJ \end{smallmatrix} \right)$ generates a strongly stable C_0-semigroup $T_{\mathcal{A}}(t)$ on $Z \times X$. If in addition there exist $\Pi \in \mathcal{L}(W, Z)$ and $\Lambda \in \mathcal{L}(W, X)$ such that $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$ and $\Lambda(\mathcal{D}(S)) \subset \mathcal{D}(F)$, and the following extended regulator equations are satisfied*

$$A\Pi + BJ\Lambda + P = \Pi S \quad in \ \mathcal{D}(S) \tag{4.3a}$$

$$F\Lambda = \Lambda S \quad in \ \mathcal{D}(S) \tag{4.3b}$$

$$C\Pi + DJ\Lambda = Q \quad in \ W \tag{4.3c}$$

then this triplet (F, G, J) solves the EFRP.

Proof. Let $\Theta(t) = \left(\begin{smallmatrix} z(t) \\ x(t) \end{smallmatrix} \right) \in Z \times X$ and define

$$\mathcal{A} = \begin{pmatrix} A & BJ \\ GC & F + GDJ \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} P \\ -GQ \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} C & DJ \end{pmatrix}, \quad \mathcal{D} = 0 \tag{4.4}$$

with obvious domains of definition. Then write the closed loop system (4.2), with $y(t) = Cz(t) + DJx(t) = \mathcal{C}\Theta(t)$, as

$$\dot{\Theta}(t) = \mathcal{A}\Theta(t) + \mathcal{B}u(t) + \mathcal{P}w(t), \quad \Theta(0) \in Z \times X \qquad (4.5a)$$

$$\dot{w}(t) = Sw(t) \qquad (4.5b)$$

$$e(t) = \mathcal{C}\Theta(t) + \mathcal{D}u(t) - Qw(t) \qquad (4.5c)$$

Since the extended regulator equations (4.3) are satisfied, we have $\Pi S = A\Pi + BJ\Lambda + P$ and $\Lambda S = F\Lambda = GC\Pi + (F + GDJ)\Lambda - GQ$ in $\mathcal{D}(S)$. Hence

$$\begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} S = \begin{pmatrix} A & BJ \\ GC & F + GDJ \end{pmatrix} \begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} + \begin{pmatrix} P \\ -GQ \end{pmatrix} \quad \text{in } \mathcal{D}(S) \qquad (4.6)$$

$$Q = \begin{pmatrix} C & DJ \end{pmatrix} \begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} \quad \text{in } W \qquad (4.7)$$

or, using the above notation, for $\Gamma = 0 \in \mathcal{L}(W, H)$

$$\begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} S = \mathcal{A} \begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} + \mathcal{B}\Gamma + \mathcal{P} \quad \text{in } \mathcal{D}(S) \qquad (4.8)$$

$$Q = \mathcal{C} \begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} + \mathcal{D}\Gamma \quad \text{in } W \qquad (4.9)$$

Since for $\mathcal{K} = 0 \in \mathcal{L}(Z \times X, H)$ the operator $\mathcal{A} = \mathcal{A} + \mathcal{B}\mathcal{K}$ generates a strongly stable C_0−semigroup, by Theorem 3.6 the control law $u(t) = \mathcal{K}\Theta(t) + [\Gamma - \mathcal{K}\begin{pmatrix} \Pi \\ \Lambda \end{pmatrix}]w(t) \equiv 0$ solves the corresponding FRP. This means that in the system (4.2) the tracking error $\lim_{t \to \infty} e(t) = 0$ for all initial states $\Theta(0) \in Z \times X$ (i.e. for all $z(0) \in Z$ and all $x(0) \in X$) and all $w(0) \in W$. The proof is complete. □

Remark 4.5. It is clear that the solvability of the extended regulator equations (4.3) implies the solvability of the regulator equations (3.10) (take $\Gamma = J\Lambda$). On the other hand, if the regulator equations (3.10) have a solution (Π, Γ), then so do the extended regulator equations (4.3), provided that also the following regulator equations for the error feedback controller (4.1) have a solution:

$$\Lambda S = F\Lambda \quad \text{in } \mathcal{D}(S) \qquad (4.10a)$$

$$J\Lambda = \Gamma \quad \text{in } W \qquad (4.10b)$$

Remark 4.6. If $T_{\mathcal{A}}(t)$ is exponentially stable, then also the decay rate of $\|e(t)\|$ to 0 as $t \to \infty$ is exponentially fast in Theorem 4.4. In fact, under the assumptions of Theorem 4.4 we have that $e(t) = \mathcal{C}T_{\mathcal{A}}(t)\left[\left(\begin{smallmatrix} z(0) \\ x(0) \end{smallmatrix}\right) - \left(\begin{smallmatrix} \Pi w(0) \\ \Lambda w(0) \end{smallmatrix}\right)\right]$ for all $t \geq 0$. This can be proved as in Theorem 3.6.

For finite-dimensional exosystems (2.1), explicit operators F, G, J satisfying the conditions of Theorem 4.4 can be found in Theorem IV.2 of [12]. For general infinite-dimensional exosystems (2.2) such operators F, G, J will be provided in Section 4.5; as we shall see, the idea is to choose F, G and J in such a way that the solvability of the regulator equations (3.10) implies the solvability of the extended regulator equations (4.3), too[2]. However, before proceeding to the construction of F, G and J we shall first present some necessary conditions for the solvability of the EFRP in Section 4.3 below.

4.3 Necessary conditions for the solvability of the EFRP

We saw in Section 4.2 that if a controller (4.1) achieves strong stability of the C_0-semigroup $T_{\mathcal{A}}(t)$ generated by the closed loop operator $\mathcal{A}$, then the same controller also solves the EFRP provided that the extended regulator equations (4.3) have a solution. In this section we discuss a converse question: Is it necessary to be able to solve the extended regulator equations (4.3) in order to be able to solve the EFRP?

Since we can cast the EFRP as an FRP for the extended system (4.2), based on the results of Section 3.3 the reader should not be very surprised to learn that a complete characterization of the solvability of the EFRP in terms of the extended regulator equations (4.3) alone is, in general, rather difficult unless additional assumptions are made e.g. about the reference signals. This is because the extended regulator equations (4.3) play the role of the regulator equations (3.10) for the extended system (4.2). Consequently, their solvability may imply such state space behaviour of the closed loop system which cannot be guaranteed by strong stability and asymptotic tracking/rejection alone. Of course, we can employ regularity (cf. Definition 3.9) and Lemma 3.12 to overcome this difficulty, but some additional care is necessary here: The uniform boundedness of $T_{\mathcal{A}}(t)$ and the (unique) solvability of the operator equation $\Pi S = \mathcal{A}\Pi + \Delta$ for certain operators $\Delta \in \mathcal{L}(W, Z)$ sometimes already *necessitate* exponential stability of the closed loop semigroup $T_{\mathcal{A}}(t)$ (see e.g. Theorem 4.2 in [90]). This may result in a contradictory situation if $\dim(W) = \infty$,

---

[2]We remind the reader that the solvability of the regulator equations (3.10) is treated in Chapter 8.

because exponential closed loop stability is then often impossible to achieve in practice. On the other hand, we shall see that if the closed loop system *is* exponentially stable, then complete characterizations for the solvability of the EFRP in terms of the extended regulator equations (4.3) can be proved. In this section shall extensively employ the notation and definitions of Section 3.3.

Theorem 4.7. *Assume that the exogenous system* (2.2) *generates admissible reference signals. If the EFRP is solvable for some triplet* (F, G, J) *and if the operator* $\mathcal{P} = \left(\begin{smallmatrix} P \\ -GQ \end{smallmatrix} \right)$ *is regular for the semigroup* $T_{\mathcal{A}}(t)$ *generated by* $\mathcal{A} = \left(\begin{smallmatrix} A & BJ \\ GC & F+GDJ \end{smallmatrix} \right)$ *on* $Z \times X$, *then there exist* $\Pi \in \mathcal{L}(W, Z)$ *such that* $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$ *and* $\Lambda \in \mathcal{L}(W, X)$ *such that* $\Lambda(\mathcal{D}(S)) \subset \mathcal{D}(F)$ *which satisfy the extended regulator equations* (4.3).

Proof. Assume that the EFRP is solvable using a controller of the form (4.1). Let $\Theta(t) = \left(\begin{smallmatrix} z(t) \\ x(t) \end{smallmatrix} \right) \in Z \times X$ and consider the closed loop system (4.2) in the form (4.5), with the relevant operators defined in (4.4). Let $\mathcal{K} \in \mathcal{L}(Z \times X, H)$ and $\mathcal{L} \in \mathcal{L}(W, H)$ be arbitrary. Then since $\mathcal{B} = 0$ and $\mathcal{D} = 0$, the control law $u(t) = \mathcal{K}\Theta(t) + \mathcal{L}w(t)$ solves the FRP for the system (4.5). Moreover, by our assumption $\mathcal{P} = \mathcal{B}\mathcal{L} + \mathcal{P}$ is regular for the semigroup generated by $\mathcal{A} + \mathcal{B}\mathcal{K} = \mathcal{A}$. Hence by Theorem 3.16 there exist $\left(\begin{smallmatrix} \Pi \\ \Lambda \end{smallmatrix} \right) \in \mathcal{L}(W, Z \times X)$ and $\Gamma \in \mathcal{L}(W, H)$ such that $\left(\begin{smallmatrix} \Pi \\ \Lambda \end{smallmatrix} \right)(\mathcal{D}(S)) \subset \mathcal{D}(\mathcal{A})$ and

$$\left(\begin{smallmatrix} \Pi \\ \Lambda \end{smallmatrix} \right) S = \mathcal{A}\left(\begin{smallmatrix} \Pi \\ \Lambda \end{smallmatrix} \right) + \mathcal{B}\Gamma + \mathcal{P} \quad \text{in } \mathcal{D}(S) \tag{4.11}$$

$$Q = \mathcal{C}\left(\begin{smallmatrix} \Pi \\ \Lambda \end{smallmatrix} \right) + \mathcal{D}\Gamma \quad \text{in } W \tag{4.12}$$

Expanding these equations gives

$$\begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} S = \begin{pmatrix} A & BJ \\ GC & F+GDJ \end{pmatrix} \begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Gamma + \begin{pmatrix} P \\ -GQ \end{pmatrix} \tag{4.13}$$

$$= \begin{pmatrix} A & BJ \\ GC & F+GDJ \end{pmatrix} \begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} + \begin{pmatrix} P \\ -GQ \end{pmatrix} \tag{4.14}$$

$$= \begin{pmatrix} A & BJ \\ 0 & F \end{pmatrix} \begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} + \begin{pmatrix} P \\ 0 \end{pmatrix} \quad \text{in } \mathcal{D}(S) \tag{4.15}$$

$$Q = \begin{pmatrix} C & DJ \end{pmatrix} \begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} + 0\Gamma = \begin{pmatrix} C & DJ \end{pmatrix} \begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} \quad \text{in } W \tag{4.16}$$

We now immediately see that the extended regulator equations (4.3) are satisfied for Π and Λ as in the above. $\qquad \square$

Corollary 4.8. *Assume that the exogenous system* (2.2) *generates admissible reference signals and that the controller* (4.1) *has been chosen in such a way that $\mathcal{A}$ generates a strongly stable C_0−semigroup on $Z \times X$. Then the same controller also solves the EFRP and $\mathcal{P} = \left(\begin{smallmatrix} P \\ -GQ \end{smallmatrix} \right)$ is regular for $T_{\mathcal{A}}(t)$ if and only if there exist $\Pi \in \mathcal{L}(W, Z)$ such that $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$ and $\Lambda \in \mathcal{L}(W, X)$ such that $\Lambda(\mathcal{D}(S)) \subset \mathcal{D}(F)$ which satisfy the extended regulator equations* (4.3).

Proof. To prove sufficiency, we observe that if Π and Λ satisfy the extended regulator equations (4.3), then the EFRP is solvable according to Theorem 4.4. Moreover,

$$\begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} S = \begin{pmatrix} A & BJ \\ GC & F + GDJ \end{pmatrix} \begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} + \begin{pmatrix} P \\ -GQ \end{pmatrix} \quad \text{in } \mathcal{D}(S) \tag{4.17}$$

or $\left(\begin{smallmatrix} \Pi \\ \Lambda \end{smallmatrix} \right) S = \mathcal{A} \left(\begin{smallmatrix} \Pi \\ \Lambda \end{smallmatrix} \right) + \mathcal{P}$ in $\mathcal{D}(S)$. Hence $\mathcal{P}$ is regular for $T_{\mathcal{A}}(t)$ by Lemma 3.12.

The necessity part of the result follows directly from Theorem 4.7. □

Below we shall employ the fact that good enough stability properties for $T_{\mathcal{A}}(t)$ automatically imply regularity of *every* operator $\mathcal{P} \in \mathcal{L}(W, Z \times X)$ for $T_{\mathcal{A}}(t)$.

Theorem 4.9. *Assume that the exogenous system* (2.2) *generates admissible reference signals and assume that the operators F, G and J in* (4.1) *are chosen such that $\mathcal{A}$ generates an exponentially stable C_0-semigroup on $Z \times X$. Then the same controller solves the EFRP if and only if there exist $\Pi \in \mathcal{L}(W, Z)$ such that $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$ and $\Lambda \in \mathcal{L}(W, X)$ such that $\Lambda(\mathcal{D}(S)) \subset \mathcal{D}(F)$ which satisfy the extended regulator equations* (4.3).

Proof. Sufficiency part of the result is contained in Theorem 4.4. In order to prove necessity, let $\Theta(t) = \left(\begin{smallmatrix} z(t) \\ x(t) \end{smallmatrix} \right) \in Z \times X$ and write the closed loop system (4.2) in the form (4.5), with the relevant operators defined in (4.4). Let $\mathcal{K} \in \mathcal{L}(Z \times X, H)$ and $\mathcal{L} \in \mathcal{L}(W, H)$ be arbitrary. Then the control law $u(t) = \mathcal{K}\Theta(t) + \mathcal{L}w(t)$ solves the FRP for the system (4.5) and $\mathcal{A} = \mathcal{A} + \mathcal{B}\mathcal{K}$ generates an exponentially stable C_0−semigroup. The result now follows from Theorem 3.19 as in the proof of Theorem 4.7. □

In analogy with the FRP, also here it is possible to dispense with the above requirement for admissibility of the reference signals if both $\|T_{\mathcal{A}}(t)\|$ and $\|e(t)\|$ decay exponentially as $t \to \infty$:

Theorem 4.10. *Assume that the controller* (4.1) *is chosen such that $\mathcal{A}$ generates an exponentially stable C_0-semigroup on $Z \times X$. Then the same controller solves the EFRP in such a way that*

$\|e(t)\| \leq Me^{-\omega t}[\|z(0)\| + \|x(0)\| + \|w(0)\|]$ *for all $t \geq 0$ and some $M, \omega > 0$ which do not depend on the initial conditions $z(0) \in Z$, $x(0) \in X$, and $w(0) \in W$, if and only if there exists $\Pi \in \mathcal{L}(W, Z)$ and $\Lambda \in \mathcal{L}(W, X)$ such that $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$ and $\Lambda(\mathcal{D}(S)) \subset \mathcal{D}(F)$ and the extended regulator equations (4.3) are satisfied.*

Proof. This result follows from Theorem 3.20 using the above methods and Remark 4.6. We omit the details. $\qquad\square$

Remark 4.11. In Theorem 4.10 we have endowed the product space $Z \times X$ with the $1-$norm. Obviously we could also have used the $2-$norm or the $\infty-$norm without altering the result (the value of the constant M would have been altered, though).

It is possible that Theorem 4.9 and Theorem 4.10 may only be applicable when W is finite-dimensional because exponential closed loop stability may be impossible to achieve for an infinite-dimensional W. We shall return to this matter in Subsection 4.6.1, but we emphasize that these results are new even for finite-dimensional spaces W.

4.4 On the necessity of reduplication of S in F

In the previous two sections we have derived general necessary and sufficient conditions for the solvability of the EFRP in terms of strong closed loop stability and the extended regulator equations (4.3). In this section we shall further investigate the necessary structure of error feedback controllers (4.1) that solve the EFRP. In particular, we shall generalize Proposition 3 in [29] for infinite-dimensional systems and exosystems (2.2). For finite-dimensional plants and exosystems Proposition 3 in [29] establishes, using a commutative diagram, the fact that under certain assumptions any controller solving the EFRP must incorporate the exosystem dynamics.

Theorem 4.12. *Let the pair $\left(\left(\begin{smallmatrix} A & P \\ 0 & S \end{smallmatrix} \right), \left(\begin{smallmatrix} C & -Q \end{smallmatrix} \right) \right)$ be approximately observable[3]. Moreover, assume that the extended regulator equations (4.3) have a solution. Then there exists a subspace $X_0 \subset X$ which is invariant for the semigroup $T_F(t)$ generated by F on X, and there exists a linear bijection $M : W \to X_0$ such that $M \in \mathcal{L}(W, X)$ and $T_S(t) = M^{-1} T_F(t) M$.*

---

[3]By definition this means that $\left(\begin{smallmatrix} C & -Q \end{smallmatrix} \right) T(t) \left(\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix} \right) = 0$ for all $t \geq 0$ implies $z_0 = 0 \in Z$ and $w_0 = 0 \in W$ [17]. Here $T(t)$ is the C_0-semigroup generated by $\left(\begin{smallmatrix} A & P \\ 0 & S \end{smallmatrix} \right)$ on $Z \times W$.

Proof. By our assumptions there exist bounded linear operators Π and Λ such that

$$\Pi S = A\Pi + BJ\Lambda + P \quad \text{in } \mathcal{D}(S) \tag{4.18a}$$

$$\Lambda S = F\Lambda \quad \text{in } \mathcal{D}(S) \tag{4.18b}$$

$$Q = C\Pi + DJ\Lambda \quad \text{in } W \tag{4.18c}$$

Since $\Lambda(\mathcal{D}(S)) \subset \mathcal{D}(F)$, from (4.18b) we see that for every $w \in \mathcal{D}(S)$ and every $t \geq 0$

$$\Lambda T_S(t)w - T_F(t)\Lambda w = \Big|_0^t T_F(t-\tau)\Lambda T_S(\tau)w = \int_0^t \frac{d}{d\tau} T_F(t-\tau)\Lambda T_S(\tau)w d\tau \tag{4.19}$$

$$= \int_0^t T_F(t-\tau)[\Lambda S - F\Lambda]T_S(\tau)w d\tau = 0 \tag{4.20}$$

A suitable denseness argument (see Lemma 3.5) shows that $\Lambda T_S(t)w - T_F(t)\Lambda w = 0$ for every $w \in W$ and every $t \geq 0$.

We now show that $\Lambda : W \to \text{ran}(\Lambda)$ is injective. If $\Lambda w = 0$ for some $w \neq 0$, then by the above $BJ\Lambda T_S(t)w = BJT_F(t)\Lambda w = 0$ and $DJ\Lambda T_S(t)w = DJT_F(t)\Lambda w = 0$ for every $t \geq 0$. Consider the semigroup

$$T(t) = \begin{pmatrix} T_A(t) & \int_0^t T_A(t-s)PT_S(s)ds \\ 0 & T_S(t) \end{pmatrix} \tag{4.21}$$

generated by $\left(\begin{smallmatrix} A & P \\ 0 & S \end{smallmatrix}\right)$ on $Z \times W$. We immediately see by (4.18a) and Lemma 3.5 that

$$T(t)\left(\begin{smallmatrix} \Pi w \\ w \end{smallmatrix}\right) = \begin{pmatrix} T_A(t)\Pi w + \int_0^t T_A(t-s)(BJ\Lambda + P)T_S(s)w ds \\ T_S(t)w \end{pmatrix} \tag{4.22}$$

$$= \begin{pmatrix} T_A(t)\Pi w + \Pi T_S(t)w - T_A(t)\Pi w \\ T_S(t)w \end{pmatrix} \tag{4.23}$$

$$= \begin{pmatrix} \Pi T_S(t)w \\ T_S(t)w \end{pmatrix} \tag{4.24}$$

Hence by (4.18c) we have $(C - Q)T(t)\left(\begin{smallmatrix} \Pi w \\ w \end{smallmatrix}\right) = (C\Pi + DJ\Lambda - Q)T_S(t)w = 0$ for every $t \geq 0$. This violates the approximate observability assumption. Hence Λ is injective.

By the fact that $\Lambda T_S(t) = T_F(t)\Lambda$ in W, it is evident that $X_0 = \text{ran}(\Lambda)$ is a $T_F(t)$-invariant subspace of X. Consequently $M = \Lambda$ is a bounded linear bijection $W \to X_0$ and $T_S(t) = M^{-1}T_F(t)M$, as was claimed. $\qquad\square$

Remark 4.13. In finite dimensions the (approximate) observability of the pair $\left(\left(\begin{smallmatrix} A & P \\ 0 & S \end{smallmatrix} \right), \left(C \ -Q \right) \right)$ implies the exponential detectability of the pair $\left(\left(\begin{smallmatrix} A & P \\ 0 & S \end{smallmatrix} \right), \left(C \ -Q \right) \right)$; the latter was also assumed in Proposition 3 of [29] which the above result generalizes.

4.5 Some explicit controllers solving the EFRP

Thus far in this chapter we have only presented some necessary and sufficient conditions for the solvability of the EFRP without any regard to the actual choice of the parameters F, G and J in the controller (4.1). In this section we shall construct two dynamic controllers (4.1) which solve the EFRP under certain assumptions. These controllers employ infinite-dimensional generalizations of some classical constructions of Francis and Davison.

4.5.1 A generalization of the synthesis algorithm of Francis

In [29] Francis presented a synthesis algorithm (SA) for the construction of a dynamic controller (4.1) which solves the EFRP if both the plant (1.1) and the exosystem (2.1) are finite-dimensional. Subsequently Byrnes et al. [12] have generalized this procedure for infinite-dimensional plants. In the present subsection we shall generalize the SA for infinite-dimensional systems (1.1) and infinite-dimensional exosystems (2.2). Throughout this subsection we make the following standing assumptions:

1. There is no feedthrough, i.e. $D = 0$.

2. There exists $G = \left(\begin{smallmatrix} G_1 \\ G_2 \end{smallmatrix} \right) \in \mathcal{L}(H, Z \times W)$ for which $\mathcal{A}_F = \left(\begin{smallmatrix} A & P \\ 0 & S \end{smallmatrix} \right) - \left(\begin{smallmatrix} G_1 \\ G_2 \end{smallmatrix} \right) \left(C \ -Q \right)$ generates a strongly stable C_0-semigroup $T_{\mathcal{A}_F}(t)$ on $Z \times W$.

3. There exists $K \in \mathcal{L}(Z, H)$ such that $A + BK$ generates an exponentially stable C_0-semigroup.

4. There exist $\Pi \in \mathcal{L}(W, Z)$, such that $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$, and $\Gamma \in \mathcal{L}(W, H)$ which satisfy the regulator equations (3.10).

Remark 4.14. In Subsection 4.6.3 we shall discuss how the assumption 2 above can be satisfied, whereas in Chapter 8 we shall study solvability of the regulator equations (3.10) (i.e. we show how the assumption 4 above can be satisfied). The assumptions 1 and 3 above are often comparably easy to verify in applications.

It has been shown in [12, 29] for finite-dimensional exosystems that the above assumptions and *exponential* stability of $T_{\mathcal{A}_F}(t)$ make it possible to solve the error feedback regulation problem using an observer-based construction. The idea is to use an observer to generate an estimate $\Theta(t) = \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix}$ of the state $\begin{pmatrix} z(t) \\ w(t) \end{pmatrix}$ of the system

$$\begin{pmatrix} \dot{z}(t) \\ \dot{w}(t) \end{pmatrix} = \begin{pmatrix} A & P \\ 0 & S \end{pmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u(t) \tag{4.25a}$$

$$e(t) = \begin{pmatrix} C & -Q \end{pmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} \tag{4.25b}$$

and then apply the control $u(t) = J\Theta(t) = J_1\theta_1(t) + J_2\theta_2(t)$ where $J_1 \in \mathcal{L}(Z, H)$ and $J_2 \in \mathcal{L}(W, H)$ are chosen so that they solve a corresponding FRP. This amounts to choosing $J_1 = K$ and $J_2 = \Gamma - K\Pi$ (see [29] for more details).

As will be demonstrated in Subsection 4.6.1, if W is infinite-dimensional, then $S + \Delta$ cannot generate an exponentially stable C_0-semigroup for any compact operator $\Delta \in \mathcal{L}(W)$. Consequently, in the general setup of this thesis we cannot assume exponential stability of $T_{\mathcal{A}_F}(t)$ as was done in [12, 29]. However, the following result shows that we do not need exponential stability of $T_{\mathcal{A}_F}(t)$ to solve the EFRP.

Theorem 4.15. *Let the assumptions $1-4$ above hold. Then the dynamic controller* (4.1) *given on the state space $X = Z \times W$ by*

$$F = \begin{pmatrix} A + BK - G_1C & P + B(\Gamma - K\Pi) + G_1Q \\ -G_2C & S + G_2Q \end{pmatrix}, \quad G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} \quad and \quad J = \begin{pmatrix} K & \Gamma - K\Pi \end{pmatrix} \tag{4.26}$$

solves the EFRP.

Proof. It is easy to see that the operators Π and $\Lambda = \begin{pmatrix} \Pi \\ I \end{pmatrix} \in \mathcal{L}(W, Z \times W)$ satisfy the extended regulator equations (4.3). On the other hand, the closed loop system operator is given by

$$\mathcal{A} = \begin{pmatrix} A & BJ \\ GC & F \end{pmatrix} = \begin{pmatrix} A & BK & B(\Gamma - K\Pi) \\ G_1C & A + BK - G_1C & P + B(\Gamma - K\Pi) + G_1Q \\ G_2C & -G_2C & S + G_2Q \end{pmatrix} \tag{4.27}$$

If we can establish that $\mathcal{A}$ generates a strongly stable C_0-semigroup $T_{\mathcal{A}}(t)$, then the error feedback controller (4.26) solves the EFRP by Theorem 4.4.

Applying a similarity transform U given as

$$U = \begin{pmatrix} I & 0 & 0 \\ I & -I & 0 \\ 0 & 0 & -I \end{pmatrix} \tag{4.28}$$

on $Z \times Z \times W$ to $\mathcal{A}$ we see that $\mathcal{A}$ is similar to the operator $\widetilde{\mathcal{A}} = U\mathcal{A}U$ having the expression

$$\widetilde{\mathcal{A}} = \begin{pmatrix} A + BK & -BK & -B(\Gamma - K\Pi) \\ 0 & A - G_1 C & P + G_1 Q \\ 0 & -G_2 C & S + G_2 Q \end{pmatrix} = \begin{pmatrix} A + BK & M \\ 0 & \mathcal{A}_F \end{pmatrix} \tag{4.29}$$

for $M = \begin{pmatrix} -BK & -B(\Gamma - K\Pi) \end{pmatrix}$. By our assumption $\mathcal{A}_F$ generates a strongly stable C_0-semigroup on X and $A+BK$ generates an exponentially stable C_0-semigroup on Z. Clearly the C_0-semigroup generated by $\widetilde{\mathcal{A}}$ on $Z \times X$ is given by

$$T_{\widetilde{\mathcal{A}}}(t) = \begin{pmatrix} T_{A+BK}(t) & \int_0^t T_{A+BK}(t-s) M T_{\mathcal{A}_F}(s) ds \\ 0 & T_{\mathcal{A}_F}(t) \end{pmatrix} \tag{4.30}$$

Consequently $T_{\widetilde{\mathcal{A}}}(t)$ is strongly stable if

$$\lim_{t \to \infty} \int_0^t T_{A+BK}(t-s) M T_{\mathcal{A}_F}(s) x ds = 0 \quad \forall x \in X \tag{4.31}$$

But (4.31) holds by Proposition 5.6.1 in [2]. This proves that also $T_{\mathcal{A}}(t)$ is strongly stable. $\quad\square$

Although Theorem 4.15 provides a solution to the EFRP without assuming exponential stability of $T_{\mathcal{A}_F}(t)$, we point out that it may not be easy to verify the strong stability of $T_{\mathcal{A}_F}(t)$ in practice either. Before discussing the verification of strong stability of $T_{\mathcal{A}_F}(t)$ in Section 4.6, we shall first introduce another explicit solution of the EFRP.

4.5.2 A generalization of Davison's dynamic state feedback controller

A generalization of Davison's dynamic state feedback controller (see [39]) is described (in the mild sense) by the equations

$$\dot{x}(t) = Sx(t) + G_0 e(t), \quad x(0) \in W \tag{4.32a}$$

$$u(t) = K_1 z(t) + K_2 x(t) \tag{4.32b}$$

on the state space $X = W$. Here the parameters $K_1 \in \mathcal{L}(Z, H)$, $K_2 \in \mathcal{L}(W, H)$ and $G_0 \in \mathcal{L}(H, W)$ should be chosen such that the closed loop system operator

$$\mathcal{A} = \mathcal{A}_{D_K} = \begin{pmatrix} A + BK_1 & BK_2 \\ G_0(C + DK_1) & S + G_0 DK_2 \end{pmatrix} \tag{4.33}$$

generates a strongly stable C_0−semigroup on $Z \times W$ and asymptotic tracking/rejection of the exogenous signals generated by the exosystem (2.2) occurs.

The controller (4.32) is clearly simpler than the one with parameters as in (4.26) in the sense that the state space X is smaller. Moreover, here it is also possible to have $D \neq 0$. However, in order to achieve sufficient closed loop stability, we cannot in general assume that $K_1 \neq 0$, i.e. state feedback must be allowed [39].

It turns out that under certain assumptions[4] on the operators S, K_1, K_2 and G_0 strong closed loop stability, i.e. strong stability of $T_{\mathcal{A}_{D_K}}(t)$, already *implies* output regulation in the sense that the dynamic controller

$$\dot{x}(t) = Sx(t) + G_0 e(t), \quad x(0) \in W \tag{4.34a}$$

$$u(t) = K_2 x(t) \tag{4.34b}$$

(which does not employ state feedback) solves the EFRP for a plant in which A is replaced by $A + BK_1$ and C is replaced by $C + DK_1$. We choose to defer the proof of this result to Chapter 6 because it arises as a natural consequence of the general robustness theory employing the so called internal model structure. Moreover, in Chapter 6 we shall be able to derive additional conditions under which the use of direct state feedback can be avoided. Strong stability of the C_0−semigroup $T_{\mathcal{A}_{D_K}}(t)$ generated by $\mathcal{A}_{D_K}$ will be discussed in Subsection 4.6.4.

We point out that in [35] Hämäläinen and Pohjolainen generalized (using frequency domain techniques) some finite-dimensional output regulation results of Davison for stable plants in the Callier-Desoer algebra[5] and for certain finite-dimensional exogenous systems which also allow for polynomial reference signals. In the case of trigonometric polynomial reference/disturbance signals their controller has the transfer function $H_C(s) = \sum_{k=0}^{2n} \frac{\epsilon K_{k_0}}{s - i\omega_k}$, for $s \neq i\omega_k$ and for certain $\epsilon > 0$,

---

[4]See Subsection 6.5.2.

[5]The systems studied in [35] exclude e.g. that in Example 3.56. However, the systems studied in [35] can also have a more general impulse response than $CT_A(t)B + D\delta(t)$ which is the impulse response of the plant (1.1). We refer the reader to [17] for more details.

matrices K_{k_0} and some fixed $n \in \mathbb{N}$. Here ω_k are the frequencies of the exogenous signals. If $H = \mathbb{C}^N$ and if in (4.34) we can take $W = \mathcal{H} = H_{AP}(\mathbb{C}^N, f_n, \omega_n)$ for some $N \in \mathbb{N}$, $(f_n)_{n \in I} \subset \mathbb{R}$ and $(\omega_n)_{n \in I} \subset \mathbb{R}$, with $S = S|_{\mathcal{H}}$, $P \in \mathcal{L}(\mathcal{H}, Z)$, $Q = \delta_0 \in \mathcal{L}(\mathcal{H}, \mathbb{C}^N)$ and $w(0) = y_{ref} \in \mathcal{H}$ as in Proposition 2.3, then the transfer function of the controller (4.34) is given by the strongly convergent series

$$H_C(s) = K_2 R(s, S|_{\mathcal{H}}) G_0 = \sum_{n \in I} \frac{K_2 P_n G_0}{s - i\omega_n}, \quad s \in \rho(S|_{\mathcal{H}}) \tag{4.35}$$

where the bounded linear operator $P_n : \mathcal{H} \to \mathcal{H}$, $n \in I$, is defined[6] by $P_n y = \hat{y}(n) e^{i\omega_n \cdot}$ for every $y = \sum_{n \in I} \hat{y}(n) e^{i\omega_n \cdot} \in \mathcal{H}$. Observe that the series (4.35) of operators indeed converges strongly, because $P_n G_0 e = (\widehat{G_0 e})(n) e^{i\omega_n \cdot} \in \mathcal{H}$ and $\sum_{n \in I} f_n^2 \|(\widehat{G_0 e})(n)\|^2 < \infty$ for all $e \in H$. Moreover, since $S|_{\mathcal{H}} P_n y = i\omega_n P_n y$ for all $y \in \mathcal{H}$, clearly $R(s, S|_{\mathcal{H}}) P_n y = \frac{1}{s - i\omega_n} P_n y$ for all $y \in \mathcal{H}$ and all $s \in \rho(S|_{\mathcal{H}})$. We emphasize that although the transfer function in (4.35) is more general than the corresponding one in [35], at this stage we have no guarantee of output regulation; conditions for this to occur will be provided in Chapter 6.

4.6 On the stabilization of the closed loop system

One of the most delicate issues in the solution of the EFRP for an infinite-dimensional exogenous system (2.2) turns out to be appropriate stabilization of the closed loop system. We saw in Section 4.5 that the operator

$$\mathcal{A}_F = \begin{pmatrix} A & P \\ 0 & S \end{pmatrix} - \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} \begin{pmatrix} C & -Q \end{pmatrix} \tag{4.36}$$

should generate a strongly stable C_0-semigroup on $Z \times W$, if a controller with parameters as in (4.26) is to be used in the solution of the EFRP. On the other hand, if state feedback from the plant is allowed, then we need the strong stability of the semigroup generated by the operator

$$\mathcal{A}_{D_K} = \begin{pmatrix} A & 0 \\ G_0 C & S \end{pmatrix} + \begin{pmatrix} B \\ G_0 D \end{pmatrix} \begin{pmatrix} K_1 & K_2 \end{pmatrix} \tag{4.37}$$

on $Z \times W$, in order to use the controller (4.32) for output regulation purposes.

---

[6] Since $H_{AP}(\mathbb{C}^N, f_n, \omega_n) \subset AP(\mathbb{R}, \mathbb{C}^N)$ we can obviously use the Fourier-Bohr transform to define the operators P_n [63]. If $\{\omega_n \mid n \in I\}$ is also a discrete set, then they can also be defined via the spectral projections corresponding to the isolated points $i\omega_n$.

Noteworthy in the above operators $\mathcal{A}_F$ and $\mathcal{A}_{D_K}$ is that they both incorporate a copy of the exosystem's operator S. This feature is a source of severe stabilizability problems if $\dim(W) = \infty$. In fact, in Subsection 4.6.1 we will prove in a novel way the well-known result that whenever $\dim(W) = \infty$, the operator S cannot be compactly additively perturbed to obtain a generator of an exponentially stable C_0-semigroup on W. Moreover, the situation is not much better if we allow for a degree of unboundedness in the perturbation; S must have compact resolvent in order that $S-$compact perturbations can exponentially stabilize it. In Subsection 4.6.2 we shall provide general sufficient conditions that $S + \Delta$ generates a strongly stable C_0-semigroup for certain $\Delta \in \mathcal{L}(W)$. The remainder of this section is then devoted to methods which can be used to establish the strong stability of the semigroups generated by $\mathcal{A}_F$ and $\mathcal{A}_{D_K}$. We hasten to emphasize that although some of the results of this section may seem difficult to apply in practice, the robustness results of Chapter 6 show their relatively wide applicability.

4.6.1 The lack of exponential stabilizability of the exosystem

In this subsection we shall prove the following negative results about the *lack* of exponential stabilizability of the exogenous system (2.2):

1. Assuming that $\Delta \in \mathcal{L}(W)$ is compact, the operator $S+\Delta$ can only generate an exponentially stable C_0-semigroup if $\dim(W) < \infty$.

2. Assuming that Δ is $S-$compact, the operator $S + \Delta$ can only generate an exponentially stable C_0-semigroup if $R(\lambda, S)$ is compact for one/all $\lambda \in \rho(S)$.

Here, as before, S generates an isometric C_0-group on a Banach space W. These results are well-known (see Corollary 3.58 in [65] and Theorem IV.5.35 in [57]), but our method of proof is new. While the earlier proofs of these results employ conservation of the so called essential spectrum under (relatively) compact perturbations, our proof here relies on a direct operator equation method.

We shall begin with two lemmata. In what follows $\mathcal{K}(W)$ denotes the Banach space of compact operators in $\mathcal{L}(W)$.

Lemma 4.16. *Let E and F generate C_0-groups $T_E(t)$ and $T_F(t)$ on a Banach space W, and let*

$G \in \mathcal{K}(W)$. *Then the operator*

$$w \to \int_0^t T_E(s)GT_F(-s)wds, \quad \forall w \in W \tag{4.38}$$

is compact for each (fixed) $t > 0$.

Proof. We imitate the proof of Theorem 3.53 in [65]. Let $\Omega_1 = \{\, T_F(-s)w \mid w \in W, \|w\| \leq 1, 0 \leq s \leq t \,\}$, which is a bounded subset of W. Then $\Omega_2 = G(\Omega_1)$ is precompact, since G is a compact operator. We show that $\Omega_3 = \{\, T_E(s)w \mid w \in \Omega_2, 0 \leq s \leq t \,\}$ is also precompact. Let $C > 0$ be a constant such that $\|T_E(s)\| \leq C$ for all $s \in [0, t]$. Let $\epsilon > 0$. Then there exist $w_1, w_2, \ldots, w_n \in \Omega_2$ such that whenever $w \in \Omega_2$ it is true that $\|w - w_i\| < \frac{\epsilon}{2C}$ for some $1 \leq i \leq n$. Since $T_E(t)$ is strongly continuous, there exist $s_1^i, s_2^i, \ldots, s_k^i \in [0, t]$ such that if $s \in [0, t]$ then $\|T_E(s_j^i)w_i - T_E(s)w_i\| < \frac{\epsilon}{2}$ for some index j. Hence for every $w \in \Omega_2$ and every $0 \leq s \leq t$ there exist indices i and j such that

$$\|T_E(s)w - T_E(s_j^i)w_i\| \leq \|T_E(s)w - T_E(s)w_i\| + \|T_E(s)w_i - T_E(s_j^i)w_i\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{4.39}$$

and so Ω_3 is precompact. Then by Mazur's Theorem (cf. [27]), the closed convex hull $\overline{co}(\overline{\Omega_3})$ is a compact set.

Now, it is well known (see p. 48 of [26]) that whenever Ω is a closed and convex set in W, ν is a positive Borel measure on $[0, \infty)$, $B \subset [0, \infty)$ is $\nu-$measurable such that $0 < \nu(B) < \infty$, and $f : B \to W$ is Bochner integrable with respect to ν such that $f(s) \in \Omega$ for almost every $s \in B$, then

$$\frac{1}{\nu(B)} \int_B f d\nu \in \Omega \tag{4.40}$$

Since for every $w \in W$ such that $\|w\| \leq 1$ the (continuous and hence Bochner integrable) function $s : [0, t] \to T_E(s)GT_F(-s)w \in \overline{co}(\overline{\Omega_3})$, by the above, $\int_0^t T_E(s)GT_F(-s)wds \in t\overline{co}(\overline{\Omega_3})$. Consequently the bounded linear operator $w \to \int_0^t T_E(s)GT_F(-s)wds$ is compact for all $t > 0$. $\qquad\square$

Lemma 4.17. *Let E and F generate C_0-groups $T_E(t)$ and $T_F(t)$ on a Banach space W, and let $G \in \mathcal{K}(W)$. For all $t \geq 0$ define the families of operators $\mathcal{Q}_{Y,t}$ and $\mathcal{R}_t$ as follows:*

$$\mathcal{Q}_{Y,t}w = T_E(t)YT_F(-t)w, \quad \forall Y \in \mathcal{L}(W), \quad \forall w \in W \tag{4.41}$$

$$\mathcal{R}_t w = \int_0^t \mathcal{Q}_{G,s}wds = \int_0^t T_E(s)GT_F(-s)wds, \quad \forall w \in W \tag{4.42}$$

If for all $Y \in \mathcal{L}(W)$ we have $\lim_{t\to\infty} \mathcal{Q}_{Y,t} = 0$ in the uniform operator topology and if there exists $X \in \mathcal{L}(W)$ such that $X(\mathcal{D}(F)) \subset \mathcal{D}(E)$ and $EX - XF = G$ in $\mathcal{D}(F)$, then $\lim_{t\to\infty} \mathcal{R}_t = -X$ in the uniform operator topology.

Proof. As in Lemma 3.5 it is easy to show that

$$\mathcal{R}_t w = \int_0^t T_E(s)[EX - XF]T_F(-s)w\,ds = T_E(t)XT_F(-t)w - Xw = \mathcal{Q}_{X,t}w - Xw \qquad (4.43)$$

for all $w \in W$ and all $t > 0$. Hence $\lim_{t \to \infty} \mathcal{R}_t = -X$ in the uniform operator topology. $\square$

The following two theorems are the main results of this subsection.

Theorem 4.18. *Let $\Delta \in \mathcal{K}(W)$. If $S + \Delta$ generates an exponentially stable C_0-semigroup, then* $\dim(W) < \infty$.

Proof. First observe that since $\Delta \in \mathcal{L}(W)$, the operator $S + \Delta$ also generates an exponentially stable C_0-group. It is clear that $X = I$ solves the operator equation $(S + \Delta)X - XS = \Delta$ in $\mathcal{D}(S)$. Moreover, since $T_S(t)$ is an isometry for all $t \in \mathbb{R}$ and since $T_{S+\Delta}(t)$ is exponentially stable, we have for some $M, \omega > 0$ that

$$\|T_{S+\Delta}(t)YT_S(-t)\| \le Me^{-\omega t}\|Y\|, \quad \forall t \ge 0 \qquad (4.44)$$

so that $\lim_{t \to \infty} T_{S+\Delta}(t)YT_S(-t) = 0$ in the uniform operator topology for all $Y \in \mathcal{L}(W)$. By Lemma 4.17 we have that

$$\lim_{t \to \infty} \int_0^t T_{S+\Delta}(s)\Delta T_S(-s)w\,ds = -Iw = -w \quad \forall w \in W \qquad (4.45)$$

in the uniform operator topology. But the operators $w \to \int_0^t T_{S+\Delta}(s)\Delta T_S(-s)w\,ds$ are compact for every $t > 0$ by Lemma 4.16. Since $\mathcal{K}(W)$ is closed with respect to the uniform operator norm, the identity operator $I \in \mathcal{K}(W)$. This is possible only if $\dim(W) < \infty$. $\square$

Theorem 4.19. *Let Δ be $S-$compact. If $S + \Delta$ generates an exponentially stable C_0-semigroup, then $R(\lambda, S)$ is compact for one/all $\lambda \in \rho(S)$.*

Proof. First observe that since Δ is $S-$compact, so is $-\Delta$. Hence the operator $-S - \Delta$ also generates a C_0-semigroup, and so $S + \Delta$ generates an exponentially stable C_0-group. Since Δ is $S-$compact, the operator $\Delta R(\lambda, S) \in \mathcal{K}(W)$ for some $\lambda \in \rho(S)$. It is clear that $X = R(\lambda, S)$ solves the operator equation $(S + \Delta)X - XS = \Delta R(\lambda, S)$ in $\mathcal{D}(S)$. Moreover, as in the above we have for some $M, \omega > 0$ that

$$\|T_{S+\Delta}(t)YT_S(-t)\| \le Me^{-\omega t}\|Y\|, \quad \forall t \ge 0 \qquad (4.46)$$

for all $Y \in \mathcal{L}(W)$ so that $\lim_{t\to\infty} T_{S+\Delta}(t)YT_S(-t) = 0$ in the uniform operator topology. By Lemma 4.17 we have that

$$\lim_{t\to\infty} \int_0^t T_{S+\Delta}(s)\Delta R(\lambda,S)T_S(-s)w\,ds = -R(\lambda,S)w \quad \forall w \in W \tag{4.47}$$

in the uniform operator topology. But the operators $w \to \int_0^t T_{S+\Delta}(s)\Delta R(\lambda,S)T_S(-s)w\,ds$ are compact for every $t > 0$ by Lemma 4.16. Since $\mathcal{K}(W)$ is closed with respect to the uniform operator norm, the resolvent operator $R(\lambda,S) \in \mathcal{K}(W)$. That $R(\mu,S)$ is compact for all $\mu \in \rho(S)$ now follows easily from the resolvent identity [28]. □

In view of the above negative results it is interesting to observe that for every $\epsilon > 0$ the bounded (but noncompact) additive perturbation $\Delta = -\epsilon I$ to S results in the generator of an exponentially stable C_0-semigroup. Hence, although a compact perturbation never stabilizes S exponentially, exponential stabilization of S can be accomplished by a bounded perturbation whose norm is arbitrarily small.

Remark 4.20. If the exogenous system (2.2) represents some *physical system*, then its exponential stabilization can sometimes be achieved under (more or less) realistic assumptions. Consider, for example, the following one-dimensional wave equation with *distributed damping*:

$$\frac{\partial^2}{\partial t^2}w(x,t) = \frac{\partial^2}{\partial x^2}w(x,t) - d(x)\frac{\partial}{\partial t}w(x,t), \quad 0 < x < L, \quad t \in \mathbb{R} \tag{4.48a}$$

$$w(0,t) = \frac{\partial}{\partial x}w(L,t) = 0 \tag{4.48b}$$

$$w(x,0) = \phi(x), \quad 0 \le x \le L \tag{4.48c}$$

where $d(x) \ge 0$, $d(x) > d_0$ on a subinterval of $(0,L)$, and $d(\cdot)$ is bounded and continuous. It has been shown by Chen et al. [14] that the partial differential equation (4.48) can be described by an abstract Cauchy problem $\dot{w}(t) = Sw(t) + \Delta w(t)$ on a certain Hilbert space W. Here S represents the "wave equation" part of (4.48) and it generates an isometric C_0-group on W — hence it is suitable for the exosystem (2.2). On the other hand, $\Delta \in \mathcal{L}(W)$ represents the "distributed damping" part of (4.48) and it is dissipative, i.e. $\Re\langle \Delta w, w \rangle \le 0$ for all $w \in W$. It has been shown in [14] that $S + \Delta$ generates an exponentially stable C_0-semigroup on W. However, an obvious drawback in the use of this operator S in the exosystem (2.2) is that it may not be possible to obtain a useful description of the signals that can be generated as in Chapter 2.

4.6.2 On the strong stabilizability of the exosystem

A decisive conclusion that can be made based on the results of Subsection 4.6.1 is that in practice it is often impossible to stabilize the exogenous system (i.e. the operator S) exponentially unless $R(\lambda, S)$ is compact for some/all $\lambda \in \rho(S)$. Unfortunately, since $S-$compact additive perturbations to S do not alter the essential spectrum of S (see [57] Theorem IV.5.35, p. 244), also the strong stabilizability of the exosystem seems to be difficult to verify in many interesting cases. This is because all nonisolated boundary points of $\rho(S)$ — which in our case are precisely the nonisolated points of $\sigma(S)$ — belong to the essential spectrum of S (see [57] Problem IV.5.37, p. 244). The problem is illustrated in the following example.

Example 4.21. *The celebrated Arendt-Batty-Lyubich-Vũ (ABLV) Theorem ([28] Theorem V.5.21) is one of the most powerful tools available for the verification of strong stability of a given Banach space C_0-semigroup. However, in order to use it, we need the countability of the imaginary spectrum of the generator. The fact that the essential spectrum of S is invariant under $S-$compact perturbations renders the ABLV Theorem useless in some of our most general setups. For example, let $\mathcal{E} = AP(\mathbb{R}, E)$ for some Banach space E and consider the left translation C_0-group $T_S(t)|_{\mathcal{E}}$ generated by $S|_{\mathcal{E}}$ on $\mathcal{E}$. Since the functions $t \to \phi(t) = ae^{i\omega t}$, $a \in E$, are in $\mathcal{E}$ for all $\omega \in \mathbb{R}$, and $S|_{\mathcal{E}}\phi = i\omega\phi$, we have that $i\mathbb{R} = \sigma(S|_{\mathcal{E}})$. Hence for all $S-$compact perturbations Δ we have $\sigma(S + \Delta) = i\mathbb{R}$, which is uncountable. Consequently, the ABLV Theorem does not apply. However, we point out that this does not imply that $S + \Delta$ cannot generate a strongly stable C_0-semigroup.*

Fortunately, the above problems only appear to arise in the Banach space setup: If W is a Hilbert space, then we have the powerful decomposition of contractive C_0-semigroups due to Szőkefalvi-Nagy and Foias[7] at our disposal [85]. The Szőkefalvi-Nagy-Foias theory enables us to prove the following genuinely positive result, which shows that any generator of an isometric translation C_0-group on a separable Hilbert function space can be strongly stabilized by feedback operators employing point evaluations and their adjoints only. This result is very useful for our considerations, because, as we saw in Chapter 2, in output regulation problems the exosystem operator S is often chosen such that it generates the isometric left translation C_0-group on some prespecified function space $\mathcal{E} \stackrel{s}{\hookrightarrow} BUC(\mathbb{R}, E)$, and the operators P and/or Q are constructed using suitable point evaluation operators δ_0.

---

[7]The abbreviation Sz.Nagy-Foias is often used in the literature.

Theorem 4.22. *Let E and $\mathcal{E} \stackrel{s}{\hookrightarrow} BUC(\mathbb{R}, E)$ be (separable complex) Hilbert spaces. Consider the generator $S|_{\mathcal{E}}$ of the isometric left translation C_0-group $T_S(t)|_{\mathcal{E}}$ on $\mathcal{E}$, the point evaluation operator $\delta_0 \in \mathcal{L}(\mathcal{E}, E)$ and its adjoint $\delta_0^* \in \mathcal{L}(E, \mathcal{E})$. Then $S|_{\mathcal{E}} - \epsilon \delta_0^* \delta_0$ generates a strongly stable C_0-group on $\mathcal{E}$ for all $\epsilon > 0$.*

Proof. Let $\epsilon > 0$. By Stone's Theorem (Theorem 2.32 in [65]) and Definition 2.6.3 in [17] the adjoint $S|_{\mathcal{E}}^* = -S|_{\mathcal{E}}$ so that $\mathcal{D}(S|_{\mathcal{E}}^*) = \mathcal{D}(S|_{\mathcal{E}})$ and $T_S(t)|_{\mathcal{E}}^* = T_S(-t)|_{\mathcal{E}}$. Moreover, since $T_S(t)|_{\mathcal{E}}$ is an isometric (i.e. unitary) group, by Theorem 3.2 in [61] $S|_{\mathcal{E}} - \epsilon \delta_0^* \delta_0$ also generates a contraction group on $\mathcal{E}$.

Since $T_S(t)|_{\mathcal{E}}$ is the translation group, it is clear that the unitary space $H_u(T_{S|_{\mathcal{E}}}) = \{ f \in \mathcal{E} \mid \|T_S(t)|_{\mathcal{E}} f\| = \|f\| = \|T_S(t)|_{\mathcal{E}}^* f\|, t \geq 0 \}$ of the Szőkefalvi-Nagy-Foias canonical decomposition [85] is the entire space $\mathcal{E}$. By Theorem 3.2 in [61] we then have for the unitary space corresponding to the semigroup generated by $S|_{\mathcal{E}} - \epsilon \delta_0^* \delta_0$ that $H_u(T_{S|_{\mathcal{E}} - \epsilon \delta_0^* \delta_0}) = H_u(T_{S|_{\mathcal{E}} - (\sqrt{\epsilon}\delta_0)^*(\sqrt{\epsilon}\delta_0)}) \subset \ker(\sqrt{\epsilon}\delta_0)$. But the unitary space $H_u(T_{S|_{\mathcal{E}} - \epsilon \delta_0^* \delta_0})$ also reduces[8] $T_S(t)|_{\mathcal{E}}$ (see p. 724 of [62] or the proof of Theorem 3.2 in [61]) so that by Lemma 2.1 *(ii)* in [61] it must be true that

$$H_u(T_{S|_{\mathcal{E}} - \epsilon \delta_0^* \delta_0}) \subset \left[\cap_{t \geq 0} \ker(\sqrt{\epsilon}\delta_0 T_S(t)|_{\mathcal{E}}^*) \right] \cap \left[\cap_{t \geq 0} \ker(\sqrt{\epsilon}\delta_0 T_S(t)|_{\mathcal{E}}) \right] \tag{4.49}$$

$$= \left[\cap_{t \geq 0} \ker(\delta_0 T_S(t)|_{\mathcal{E}}^*) \right] \cap \left[\cap_{t \geq 0} \ker(\delta_0 T_S(t)|_{\mathcal{E}}) \right] \tag{4.50}$$

$$= \left[\cap_{t \geq 0} \{ f \in \mathcal{E} \mid \delta_0 T_S(-t)|_{\mathcal{E}} f = 0 \} \right] \cap \left[\cap_{t \geq 0} \{ f \in \mathcal{E} \mid \delta_0 T_S(t)|_{\mathcal{E}} f = 0 \} \right] \tag{4.51}$$

$$= \cap_{t \in \mathbb{R}} \{ f \in \mathcal{E} \mid \delta_0 T_S(t)|_{\mathcal{E}} f = 0 \} \tag{4.52}$$

$$= \{ f \in \mathcal{E} \mid f(t) = 0 \;\; \forall t \in \mathbb{R} \} = \{0\} \tag{4.53}$$

Hence according to the canonical decomposition of contraction semigroups [85], $T_{S|_{\mathcal{E}} - \epsilon \delta_0^* \delta_0}(t)$ is completely nonunitary, i.e. the completely nonunitary space $H_{cnu}(T_{S|_{\mathcal{E}} - \epsilon \delta_0^* \delta_0}) = \mathcal{E}$. It remains to show that $T_{S|_{\mathcal{E}} - \epsilon \delta_0^* \delta_0}(t)$ is strongly stable on $H_{cnu}(T_{S|_{\mathcal{E}} - \epsilon \delta_0^* \delta_0})$. But this follows directly from Lemma 4.1 in [61] because $T_S(t)|_{\mathcal{E}}$ and $T_S(t)|_{\mathcal{E}}^*$ are strongly stable on the completely nonunitary space $H_{cnu}(T_{S|_{\mathcal{E}}}) = \{0\}$. $\qquad \square$

Remark 4.23. For an arbitrary Banach space E and for an arbitrary uniformly bounded C_0-group $T(t)$ generated by Y on E we can always describe $T_Y(t)e$, $e \in E$, by the (strongly continuous and isometric) left translation group $T_S(t)|_{\mathcal{E}}$ on $\mathcal{E} = \overline{\text{span}}\{ T_Y(\cdot)e \mid e \in E \} \subset BUC(\mathbb{R}, E)$ as

---

[8] A subspace $M \subset \mathcal{E}$ reduces $T_S(t)|_{\mathcal{E}}$ if it is invariant for both $T_S(t)|_{\mathcal{E}}$ and $T_S(t)|_{\mathcal{E}}^*$ [61].

$T_Y(t)e = \delta_0 T_S(t)|_{\mathcal{E}}[T_Y(\cdot)e]$. The above result suggests that in order to be able to strongly stabilize the general operator Y, it is sufficient to strongly stabilize $S|_{\mathcal{E}}$. In certain cases this can be done using the feedback $-\epsilon\delta_0^*\delta_0$ as in Theorem 4.22.

Remark 4.24. Theorem 4.22 shows that the norm of a strongly stabilizing feedback for the pair $(S|_{\mathcal{E}}, \delta_0^*)$ can be made arbitrarily small.

Remark 4.25. If $\sigma(S|_{\mathcal{E}})$ is countable, then the result of Theorem 4.22 also follows from Theorem 14 of [5] using a more elementary reasoning.

Corollary 4.26. *Let E be a (separable and complex) Hilbert space and consider the generalized Sobolev space $\mathcal{E} = H_{AP}(E, f_n, \omega_n)$ introduced in Chapter 2. Then $S|_{\mathcal{E}} - \epsilon\delta_0^*\delta_0$ generates a strongly stable C_0-semigroup on $\mathcal{E}$ for all $\epsilon > 0$.*

The following auxiliary result provides sufficient conditions that the operator $S|_{\mathcal{E}}$ (and hence also $S|_{\mathcal{E}} - \epsilon\delta_0^*\delta_0$ by [28] p. 159) in Corollary 4.26 has compact resolvent. This information turns out to be quite useful in the strong stabilization of any closed loop system containing a copy of $S|_{\mathcal{E}}$ (see e.g. Subsection 4.6.3 and Section 6.7).

Proposition 4.27. *Let E be a finite-dimensional space and consider the generalized Sobolev space $\mathcal{E} = H_{AP}(E, f_n, \omega_n)$ introduced in Chapter 2. If $\left(\frac{1}{\lambda - i\omega_n}\right)_{n \in I} \in \ell^2$ for some $\lambda \in \rho(S|_{\mathcal{E}})$, then $S|_{\mathcal{E}}$ has compact resolvent. This is the case, in particular, if $(\omega_n)_{n \in I} \subset \{ \frac{2\pi n}{p} \mid n \in \mathbb{Z} \}$, i.e. $\mathcal{E} \hookrightarrow P_p(\mathbb{R}, E)$, for some $p > 0$.*

Proof. Let us define (using e.g. the Fourier-Bohr transformation [63]) the family $(P_n)_{n \in I} \subset \mathcal{L}(\mathcal{E})$ of finite rank operators by

$$P_n y = \hat{y}(n)e^{i\omega_n \cdot} \quad \text{for each } y = \sum_{n \in I} \hat{y}(n)e^{i\omega_n \cdot} \in \mathcal{E} \tag{4.54}$$

Since $S|_{\mathcal{E}} P_n y = i\omega_n P_n y$ for all $y \in \mathcal{E}$, it is evident that $R(\lambda, S|_{\mathcal{E}})P_n y = \frac{1}{\lambda - i\omega_n} P_n y$ for all $\lambda \in \rho(S|_{\mathcal{E}})$ and all $y \in \mathcal{E}$. As a result, we have

$$R(\lambda, S|_{\mathcal{E}})y = R(\lambda, S|_{\mathcal{E}})\sum_{n \in I} P_n y = \sum_{n \in I} \frac{1}{\lambda - i\omega_n} P_n y, \quad \forall y \in \mathcal{E}, \lambda \in \rho(S|_{\mathcal{E}}) \tag{4.55}$$

We show that $R(\lambda, S|_{\mathcal{E}})$ is the uniform limit of the finite rank operators $\sum_{|n| \leq N} \frac{1}{\lambda - i\omega_n} P_n$, hence compact. To this end, let $\lambda \in \rho(S|_{\mathcal{E}})$ be such that $\left(\frac{1}{\lambda - i\omega_n}\right)_{n \in I} \in \ell^2$. Then, by the Schwartz

inequality, for all $y \in \mathcal{E}$ we have

$$\left\| R(\lambda, S|_{\mathcal{E}})y - \sum_{|n| \leq N} \frac{1}{\lambda - i\omega_n} P_n y \right\|_{\mathcal{E}} = \left\| \sum_{|n| > N} \frac{1}{\lambda - i\omega_n} \hat{y}(n) e^{i\omega_n \cdot} \right\|_{\mathcal{E}} \tag{4.56}$$

$$\leq \sum_{|n| > N} \left\| \frac{1}{\lambda - i\omega_n} \hat{y}(n) e^{i\omega_n \cdot} \right\|_{\mathcal{E}} \tag{4.57}$$

$$= \sum_{|n| > N} \left| \frac{1}{\lambda - i\omega_n} \right| f_n \|\hat{y}(n)\|_E \tag{4.58}$$

$$\leq \sqrt{\sum_{|n| > N} \left| \frac{1}{\lambda - i\omega_n} \right|^2} \sqrt{\sum_{|n| > N} f_n^2 \|\hat{y}(n)\|_E^2} \tag{4.59}$$

$$\leq \sqrt{\sum_{|n| > N} \left| \frac{1}{\lambda - i\omega_n} \right|^2} \sqrt{\sum_{n \in I} f_n^2 \|\hat{y}(n)\|_E^2} \tag{4.60}$$

$$= M(N) \|y\|_{\mathcal{E}} \tag{4.61}$$

where $M(N) = \sqrt{\sum_{|n| > N} |\frac{1}{\lambda - i\omega_n}|^2} \to 0$ as $N \to \infty$. Hence $R(\lambda, S|_{\mathcal{E}})$, being the uniform limit of the finite rank operators $\sum_{|n| \leq N} \frac{1}{\lambda - i\omega_n} P_n$, is compact. $\qquad \square$

Many authors have also solved the strong stabilization problem for bounded C_0-semigroups using so-called pole-placement techniques (see e.g. [94] and the references therein). In the case of the exosystem operator S, the idea behind such techniques is to design a perturbation operator Δ such that $S + \Delta$ has a prespecified spectrum contained in the left half of the complex plane. Although such techniques generally result in more complicated feedback operators than the remarkably simple one provided by Theorem 4.22 and Corollary 4.26, and although such techniques often only work under rather restricted assumptions on S, W and H, they have the advantage that the growth rate of the norm $\|R(i\omega, S + \Delta)\|$ as $|\omega| \to \infty$ can often be easily estimated. We conclude this subsection by illustrating this very useful feature in a result, which plays an important role in the robustness considerations of certain repetitive control applications later on in this thesis (see Section 6.7):

Proposition 4.28. *Let $W = \mathcal{G} = H_{per}^{\alpha}(0, p)$ for some $\alpha > \frac{1}{2}$ and let $S = S|_{\mathcal{G}}$, $Q = \delta_0 \in \mathcal{L}(\mathcal{G}, \mathbb{C})$, in accordance with Proposition 2.3. Let $\gamma > \alpha + \frac{1}{2}$. Then there exists $L \in \mathcal{L}(\mathbb{C}, \mathcal{G})$ such that*

1. *$S|_{\mathcal{G}} + L\delta_0$ generates a strongly stable C_0-semigroup on $\mathcal{G}$,*

2. *The resolvent satisfies $\|R(i\omega_n, S|_{\mathcal{G}} + L\delta_0)\| \leq C'\sqrt{1 + \omega_n^2}^{\gamma}$ for some $C' > 0$ and every $n \in \mathbb{Z}$,*

3. *There exists a unique $l \in \mathcal{G}$ such that $Lu = lu$ for every $u \in \mathbb{C}$ and $\langle l, \phi_n \rangle_{\mathcal{G}} \neq 0$ for every*

   $n \in \mathbb{Z}$.

Here $\omega_n = \frac{2\pi n}{p}$ for all $n \in \mathbb{Z}$ and $(\phi_n)_{n \in \mathbb{Z}}$ denotes the orthonormal basis of (weighted) exponentials $c_n e^{i\omega_n \cdot} = \frac{e^{i\omega_n \cdot}}{\sqrt{1 + \omega_n^2}^{\alpha}}$, $n \in \mathbb{Z}$, for $\mathcal{G}$, which are also the eigenvectors of $S|_{\mathcal{G}}$ corresponding to the eigenvalues $i\omega_n$.

Proof.   1. We shall first employ the theory of Xu and Sallet [94] to find $L \in \mathcal{L}(\mathbb{C}, \mathcal{G})$ such that $S|_{\mathcal{G}}^* + \delta_0^* L^*$ generates a strongly stable C_0−semigroup[9]. To this end, we have to show that the standing assumptions $H1 - H3$ in [94] are satisfied.

First of all, $H1$ amounts to verifying that $S|_{\mathcal{G}}^*$ has compact resolvent and simple spectrum. This follows easily from the equality $S|_{\mathcal{G}}^* = -S|_{\mathcal{G}}$ (cf. Stone's Theorem [65]), Proposition 4.27 and the fact that $\sigma(S|_{\mathcal{G}}) = \{ i\omega_n \mid n \in \mathbb{Z} \}$, where each eigenvalue $i\omega_n$ is simple: The range of any spectral projection $P_{i\omega_n}^{S|_{\mathcal{G}}}$ is just $\{ a e^{i\omega_n \cdot} \mid a \in \mathbb{C} \}$, which is one-dimensional. Thus, hypothesis $H1$ in [94] is satisfied.

Secondly, $H2$ amounts to showing that $\mathcal{D}(S|_{\mathcal{G}})$ is a Hilbert space in the graph norm and that δ_0^* (when interpreted as an input element) satisfies $\delta_0^* \in \mathcal{D}(S|_{\mathcal{G}})'$ (the topological dual of $\mathcal{D}(S|_{\mathcal{G}})$). The first assertion is evident because by Section II.5.a of [28] $\mathcal{D}(S|_{\mathcal{G}})$ with the graph norm is just the Sobolev space $H_{per}^{\alpha+1}(0, p)$. The second assertion follows from the fact that $\delta_0^* \in \mathcal{L}(\mathbb{C}, \mathcal{G})$; then $\delta_0^* u \in H_{per}^{\alpha}(0, p)$ for all $u \in \mathbb{C}$.

Finally, in order to meet assumption $H3$ in [94] we first point out that the eigenvectors ϕ_n of $S|_{\mathcal{G}}^* = -S|_{\mathcal{G}}$ constitute an orthonormal basis in $\mathcal{G}$. Then for $H3$ to hold it suffices to show that $\langle \phi_n, \delta_0^* \rangle_{\mathcal{G}} \neq 0$ for each $n \in \mathbb{Z}$ and that there exists a positive constant M such that[10]

$$\sum_{n \in \mathbb{Z}} \left| \frac{\langle \phi_n, \delta_0^* \rangle}{\lambda + i\omega_n} \right|^2 \leq M \quad \text{and} \quad \sup_{m \in \mathbb{Z}} \sum_{n \neq m} \left| \frac{\langle \phi_n, \delta_0^* \rangle}{-i\omega_m + i\omega_n} \right|^2 \leq M \tag{4.62}$$

whenever the distance from λ to $\sigma(S|_{\mathcal{G}}^*)$ is at least $\frac{2\pi}{3p}$. Since $\phi_n(\cdot) = c_n e^{i\omega_n \cdot}$, with $c_n = \frac{1}{\sqrt{1 + \omega_n^2}^{\alpha}}$, for each $f \in \mathcal{G}$ we have $\delta_0 f = f(0) = \sum_{n \in \mathbb{Z}} \langle f, \phi_n \rangle_{\mathcal{G}} \phi_n(0) = \sum_{n \in \mathbb{Z}} c_n \langle f, \phi_n \rangle_{\mathcal{G}} = \langle f, \sum_{n \in \mathbb{Z}} c_n \phi_n \rangle_{\mathcal{G}}$, so that the adjoint of δ_0 satisfies $\delta_0^* u = u \sum_{n \in \mathbb{Z}} c_n \phi_n$ for every $u \in \mathbb{C}$.

---

[9]Here and elsewhere the superscript * denotes the operator adjoint; recall that $\mathcal{G}$ is a Hilbert space.

[10]The fact that in [94] Xu and Sallet employ summation over positive integers does not play any role; only the elements that are summed matter because we can bijectively transform the summation such that it is carried out over positive integers.

Obviously then $\langle \phi_n, \delta_0^* \rangle_{\mathcal{G}} = c_n \neq 0$ for every $n \in \mathbb{Z}$. Moreover, for λ as in the above, we have (since $\alpha > \frac{1}{2}$)

$$\sum_{n \in \mathbb{Z}} \left| \frac{\langle \phi_n, \delta_0^* \rangle_{\mathcal{G}}}{\lambda + i\omega_n} \right|^2 \leq \frac{9p^2}{4\pi^2} \sum_{n \in \mathbb{Z}} |c_n|^2 = \frac{9p^2}{4\pi^2} \sum_{n \in \mathbb{Z}} (1 + \omega_n^2)^{-\alpha} = M < \infty \qquad (4.63)$$

and

$$\sup_{m \in \mathbb{Z}} \sum_{n \neq m} \left| \frac{\langle \phi_n, \delta_0^* \rangle_{\mathcal{G}}}{-i\omega_m + i\omega_n} \right|^2 = \sup_{m \in \mathbb{Z}} \sum_{n \neq m} \left| \frac{c_n}{\frac{2\pi i}{p}(n-m)} \right|^2 \leq \frac{p^2}{4\pi^2} \sum_{n \in \mathbb{Z}} (1 + \omega_n^2)^{-\alpha} < M \qquad (4.64)$$

In conclusion, the assumptions $H1 - H3$ in [94] are satisfied.

Let us now define $\mu_n = -i\omega_n - \frac{1}{\sqrt{1+\omega_n^2}^\gamma}$ for every $n \in \mathbb{Z}$. We next design a feedback $L^* \in \mathcal{L}(\mathcal{G}, \mathbb{C})$ such that it assigns the spectrum $\sigma(S|_{\mathcal{G}}^* + \delta_0^* L^*) = \{ \mu_n \mid n \in \mathbb{Z} \} \subset \{ z \in \mathbb{C} \mid \Re(z) < 0 \}$. If we can accomplish this, then $S|_{\mathcal{G}}^* + \delta_0^* L^*$ generates a strongly stable C_0−semigroup on $\mathcal{G}$. In fact, the operator $S|_{\mathcal{G}}^* + \delta_0^* L^*$ is regular spectral [94], i.e. it has compact resolvent and its eigenvectors $(\theta_n)_{n \in \mathbb{Z}}$, with $(S|_{\mathcal{G}}^* + \delta_0^* L^*)\theta_n = \mu_n \theta_n$ for every $n \in \mathbb{Z}$, constitute a Riesz basis in $\mathcal{G}$. Hence the C_0−semigroup $T_{S|_{\mathcal{G}}^* + \delta_0^* L^*}(t)$ is uniformly bounded; strong stability follows from the Arendt-Batty-Lyubich-Vũ Theorem (see e.g. Theorem V.2.21 in [28]) because $S|_{\mathcal{G}}^* + \delta_0^* L^*$ has no spectrum on $i\mathbb{R}$. In order to show that such an L^* indeed exists, we aim to apply Theorem 1 in [94]. To this end, we observe that the assigned spectral points μ_n satisfy the necessary and sufficient condition (3) on p. 522 of [94]: $\sum_{n \in \mathbb{Z}} \left| \frac{\mu_n + i\omega_n}{\langle \phi_n, \delta_0^* \rangle_{\mathcal{G}}} \right|^2 = \sum_{n \in \mathbb{Z}} |\frac{1}{c_n} \frac{1}{\sqrt{1+\omega_n^2}^\gamma}|^2 = \sum_{n \in \mathbb{Z}} |\sqrt{1+\omega_n^2}^{\alpha-\gamma}|^2 = \sum_{n \in \mathbb{Z}} (1 + \omega_n^2)^{\alpha-\gamma} < \infty$ because $\gamma > \alpha + \frac{1}{2}$. Theorem 1 in [94] now implies that the claim follows for the particular choice $L^* = \langle \cdot, l \rangle$ where $l \in \mathcal{G}$ is given by

$$l = \sum_{n \in \mathbb{Z}} l_n \phi_n \quad \text{where} \quad \overline{l_n} = -\frac{\sqrt{1+\omega_n^2}^\alpha}{\sqrt{1+\omega_n^2}^\gamma} \prod_{k=-\infty, k \neq n}^{\infty} \frac{-i\omega_n + i\omega_k + \sqrt{1+\omega_k^2}^{-\gamma}}{-i\omega_n + i\omega_k} \qquad (4.65)$$

The convergence of this infinite product is explained on p. 524 of [94]. Our final task in item 1 is to show that the adjoint $S|_{\mathcal{G}} + L\delta_0$ is also the generator of a strongly stable C_0−semigroup on $\mathcal{G}$. Clearly $S|_{\mathcal{G}} + L\delta_0$ generates at least a weakly stable C_0−semigroup on $\mathcal{G}$. Since $S|_{\mathcal{G}}$ has compact resolvent, so has $S|_{\mathcal{G}} + L\delta_0$ (see [28] p. 159). For such semigroup generators weak stability implies strong stability (see Proposition 3.21 in [65]).

2. According to Theorem 1 in [94] the operator $S|_{\mathcal{G}}^* + \delta_0^* L^*$ above is regular spectral, i.e. it has compact resolvent and its eigenvectors $(\theta_n)_{n \in \mathbb{Z}}$, with $(S|_{\mathcal{G}}^* + \delta_0^* L^*)\theta_n = \mu_n \theta_n$ for every $n \in \mathbb{Z}$,

constitute a Riesz basis in $\mathcal{G}$. We remark that the eigenvalues $(\mu_n)_{n\in\mathbb{Z}}$ of $S|_{\mathcal{G}}^* + \delta_0^* L^*$ need not be simple in general. Let $(\psi_n)_{n\in\mathbb{Z}}$ denote the sequence in $\mathcal{G}$ which is biorthogonal to $(\theta_n)_{n\in\mathbb{Z}}$ [17]. Whenever $\lambda \neq \mu_n$ for each $n \in \mathbb{Z}$ we have $(\lambda I - S|_{\mathcal{G}}^* - \delta_0^* L^*)\theta_n = (\lambda - \mu_n)\theta_n$, whence $R(\lambda, S|_{\mathcal{G}}^* + \delta_0^* L^*)\theta_n = \frac{1}{\lambda-\mu_n}\theta_n$, and so for every such $\lambda \in \mathbb{C}$

$$R(\lambda, S|_{\mathcal{G}}^* + \delta_0^* L^*)f = \sum_{n\in\mathbb{Z}} \langle f, \psi_n \rangle_{\mathcal{G}} R(\lambda, S|_{\mathcal{G}}^* + \delta_0^* L^*)\theta_n = \sum_{n\in\mathbb{Z}} \frac{\langle f, \psi_n \rangle_{\mathcal{G}} \theta_n}{\lambda - \mu_n} \quad \forall f \in \mathcal{G} \qquad (4.66)$$

For every $m \in \mathbb{Z}$ and each $f \in \mathcal{G}$ we may now estimate using well-known properties of Riesz bases [17] as follows:

$$\|R(-i\omega_m, S|_{\mathcal{G}}^* + \delta_0^* L^*)f\|^2 = \left\| \sum_{n\in\mathbb{Z}} \frac{\langle f, \psi_n \rangle_{\mathcal{G}} \theta_n}{-i\omega_m - \mu_n} \right\|^2 \leq D' \sum_{n\in\mathbb{Z}} \left| \frac{\langle f, \psi_n \rangle_{\mathcal{G}}}{-i\omega_m - \mu_n} \right|^2 \qquad (4.67)$$

$$= D' \left[\sum_{n\in I_m} \left| \frac{\langle f, \psi_n \rangle_{\mathcal{G}}}{-i\omega_m - \mu_m} \right|^2 + \sum_{n\notin I_m} \left| \frac{\langle f, \psi_n \rangle_{\mathcal{G}}}{-i\omega_m - \mu_n} \right|^2 \right] \qquad (4.68)$$

$$\leq D' \left[(1+\omega_m^2)^\gamma \sum_{n\in\mathbb{Z}} |\langle f, \psi_n \rangle_{\mathcal{G}}|^2 + c' \sum_{n\in\mathbb{Z}} |\langle f, \psi_n \rangle_{\mathcal{G}}|^2 \right] \qquad (4.69)$$

$$\leq C'^2 (1+\omega_m^2)^\gamma \|f\|^2 \qquad (4.70)$$

where I_m is the finite (cf. Corollary IV.1.19 in [28]) multiplicity of the eigenvalue μ_m, and c', D', C' are positive constants. This shows that $\|R(-i\omega_n, S|_{\mathcal{G}}^* + \delta_0^* L^*)\| \leq C'\sqrt{1+\omega_n^2}^\gamma$ for each $n \in \mathbb{Z}$. But according to Lemma A.3.65 in [17] and Lemma A.3.60 in [17], $\|R(i\omega_n, S|_{\mathcal{G}} + L\delta_0)\| = \|R(-i\omega_n, S|_{\mathcal{G}}^* + \delta_0^* L^*)\| \leq C'\sqrt{1+\omega_n^2}^\gamma$ for each $n \in \mathbb{Z}$.

3. If we choose L^* according to (4.65), then evidently $Lu = lu$ for the unique $l \in \mathcal{G}$. Moreover, $\langle \phi_n, l \rangle_{\mathcal{G}} = \overline{l_n}$ for each $n \in \mathbb{Z}$. Now by Theorem 1 in [94] we have

$$\lim_{M,N\to\infty} \frac{\mu_n + i\omega_n}{c_n} \prod_{k=-M, k\neq n}^{N} \frac{-i\omega_n + i\omega_k + \frac{1}{\sqrt{1+\omega_k^2}^\gamma}}{-i\omega_n + i\omega_k} \qquad (4.71)$$

$$= \frac{\mu_n + i\omega_n}{c_n} \lim_{M,N\to\infty} \prod_{k=-M, k\neq n}^{N} \left[1 + \frac{1}{i(\omega_k - \omega_n)\sqrt{1+\omega_k^2}^\gamma} \right]$$

$$= \overline{l_n} \qquad (4.72)$$

Since $\left| \prod_{k=-M, k\neq n}^{N} \left[1 + \frac{1}{i(\omega_k - \omega_n)\sqrt{1+\omega_k^2}^\gamma} \right] \right| = \prod_{k=-M, k\neq n}^{N} \left| 1 - \frac{i}{(\omega_k - \omega_n)\sqrt{1+\omega_k^2}^\gamma} \right| > 1$ for every $M, N \in \mathbb{N}$, the above infinite product cannot converge to 0. Hence we have $\overline{l_n} \neq 0$ for each $n \in \mathbb{Z}$.

$\square$

4.6.3 Strong stability of the C_0-semigroup generated by $\mathcal{A}_F$

In this subsection we shall discuss the strong detectability of the pair $(\mathcal{A}_0, \mathcal{C})$ where $\mathcal{A}_0 = \left(\begin{smallmatrix} A & P \\ 0 & S \end{smallmatrix} \right)$ and $\mathcal{C} = \left(\begin{smallmatrix} C & -Q \end{smallmatrix} \right)$. In other words, we want to find $\mathcal{G} = \left(\begin{smallmatrix} G_1 \\ G_2 \end{smallmatrix} \right) \in \mathcal{L}(H, Z \times W)$ such that $\mathcal{A}_F = \mathcal{A}_0 - \mathcal{G}\mathcal{C}$ generates a strongly stable C_0-semigroup $T_{\mathcal{A}_F}(t)$ on $Z \times W$.

Algebraic Riccati equations provide a useful direct method for the study of the strong detectability of the pair $(\mathcal{A}_0, \mathcal{C})$ [18]. Using this approach, in Proposition 4.29 below we present sufficient conditions for the existence of a strongly stabilizing output injection operator $\mathcal{G}$; the operator itself can be found by solving a suitable Riccati equation. Recall that if H is a Hilbert space then $U \in \mathcal{L}(H)$ is coercive provided $\langle Uh, h \rangle \geq \epsilon \|h\|^2$ for some $\epsilon > 0$ and all $h \in H$ [18]. In the following $\mathcal{A}_0^*$ denotes the adjoint operator of $\mathcal{A}_0$.

Proposition 4.29. *Assume the following.*

1. *H and $Z \times W$ are (separable) Hilbert spaces.*

2. *$\mathcal{A}_0 = \left(\begin{smallmatrix} A & P \\ 0 & S \end{smallmatrix} \right)$ generates a contraction C_0-semigroup on $Z \times W$.*

3. *The pair $(\mathcal{A}_0^*, \mathcal{C})$ is approximately observable (here $\mathcal{C} = \left(\begin{smallmatrix} C & -Q \end{smallmatrix} \right)$).*

4. *$\mathcal{A}_0$ has compact resolvent.*

Then for any coercive operators $U = U^* \in \mathcal{L}(H)$ and $R = R^* \in \mathcal{L}(H)$ the Riccati equation

$$\mathcal{A}_0 \Delta z + \Delta \mathcal{A}_0^* z - \Delta \mathcal{C}^* R^{-1} \mathcal{C} \Delta z + \mathcal{C}^* U^{-1} \mathcal{C} z = 0 \quad \forall z \in \mathcal{D}(\mathcal{A}_0^*) \tag{4.73}$$

has a unique self-adjoint solution $\Delta \in \mathcal{L}(Z \times W)$ such that $\mathcal{A}_F = \mathcal{A}_0 - \Delta \mathcal{C}^* R^{-1} \mathcal{C}$ generates a strongly stable C_0-semigroup.

Proof. It is evident that since $\mathcal{A}_0$ has compact resolvent, so has $\mathcal{A}_0^*$ because for real $\alpha \in \rho(\mathcal{A}_0)$ we have $R(\alpha, \mathcal{A}_0^*) = R(\alpha, \mathcal{A}_0)^*$ (cf. Lemma A.3.65 in [17], Theorem 7.3 in [86] and equality (A.3.15) in [17]). Furthermore, also $\mathcal{A}_0^*$ generates a contraction semigroup on $Z \times W$ because $\|T_{\mathcal{A}_0}(t)\| = \|T_{\mathcal{A}_0^*}(t)\| \leq 1$ for each $t \geq 0$. By Theorem 4 in [18], the algebraic Riccati equation (4.73) has a unique self-adjoint solution $\Delta \in \mathcal{L}(Z \times W)$. By Corollary 5 in [18] the operator $\mathcal{A}_F^* = \mathcal{A}_0^* - \mathcal{C}^* R^{-1} \mathcal{C} \Delta$ generates a strongly stable C_0-semigroup. Consequently its adjoint $\mathcal{A}_F = \mathcal{A}_0 - \Delta \mathcal{C}^* (R^{-1})^* \mathcal{C} = \mathcal{A}_0 - \Delta \mathcal{C}^* R^{-1} \mathcal{C}$ generates a weakly stable C_0-semigroup. But since $\mathcal{A}_0$ has compact resolvent, the boundedly perturbed operator $\mathcal{A}_F$ also has compact resolvent (cf. [28] p.

159). This implies that the semigroup generated by $\mathcal{A}_F$ is in fact strongly stable (see e.g. [18] Lemma 1 or [65] Proposition 3.21). $\qquad\square$

Remark 4.30. If the assumptions of Proposition 4.29 are met, then we may choose the strongly stabilizing output injection operator $\mathcal{G}$ for the pair $(\mathcal{A}_0, \mathcal{C})$ as $\mathcal{G} = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = \Delta \mathcal{C}^* R^{-1}$.

Although neither exponential stabilizability nor exponential detectability is required of the plant, the assumptions of Proposition 4.29 may seem quite restrictive from the practical point of view. However, its assumption 1 can be met, for example, if Z is a Hilbert space and if we study asymptotic tracking of the reference signals in the Sobolev spaces $W = H_{AP}(\mathbb{C}^N, f_n, \omega_n)$, $N \in \mathbb{N}$, using Proposition 2.3. The product space $Z \times W$ can then be endowed with the natural inner product to obtain a Hilbert space. The assumptions 2 and 3 in Proposition 4.29 are of technical nature, but they are easy to verify if, say, we may let $P = 0$ (which often is the case once we incorporate robustness; see Chapter 6 and in particular Subsection 6.5.1). Observe that S always generates an isometric — hence contractive — C_0−group. Finally, $\mathcal{A}_0$ has compact resolvent if S and A have compact resolvents. Proposition 4.27 presents fairly general conditions under which S has compact resolvent in output regulation applications. Moreover quite often in practice A is also a differential operator with compact resolvent [28]. Hence the assumptions of Proposition 4.29 can be met in many important special cases.

It turns out that sometimes it is not necessary to solve the Riccati equation (4.73) in order to achieve the strong stability of $T_{\mathcal{A}_F}(t)$:

Proposition 4.31. *Assume the following.*

    1. H and $Z \times W$ are (separable) Hilbert spaces.

    2. $\mathcal{A}_0 = \begin{pmatrix} A & P \\ 0 & S \end{pmatrix}$ generates a contraction C_0−semigroup on $Z \times W$.

    3. $\ker(\mu I - \mathcal{A}_0^*) \cap \ker(\mathcal{C}) = \{0\}$ for all $\mu \in i\mathbb{R} \cap \sigma_P(\mathcal{A}_0^*)$ (here $\mathcal{C} = \begin{pmatrix} C & -Q \end{pmatrix}$).

    4. $\mathcal{A}_0$ has compact resolvent.

then $\mathcal{A}_F = \mathcal{A}_0 - \mathcal{C}^*\mathcal{C}$ generates a strongly stable C_0−semigroup on $Z \times W$.

Proof. It suffices to apply Theorem VI.8.28 in [28] to the strong stabilization of the pair $(\mathcal{A}_0, \mathcal{C}^*)$. $\qquad\square$

We again point out that the robustness theory of Chapter 6 often allows us to *choose* the operators P and Q which are contained in the operator $\mathcal{A}_F$ (see Subsection 6.5.1 for more details). In this case the assumptions of Proposition 4.31 are fairly straightforward to verify.

In the following result we demonstrate that good stabilizability properties of the plant can compensate the difficult stabilizability of the exosystem (and vice versa) in the stabilization of $T_{\mathcal{A}_F}(t)$.

Theorem 4.32. *Assume that* either

- *there exists $G_1 \in \mathcal{L}(H, Z)$ such that $A - G_1 C$ generates an exponentially stable C_0-semigroup on Z and there exist $G_2 \in \mathcal{L}(H, W)$ and $L \in \mathcal{L}(W, H)$ such that $S + G_2 L$ generates a strongly stable C_0-semigroup on W,*

or

- *there exists $G_1 \in \mathcal{L}(H, Z)$ such that $A - G_1 C$ generates a strongly stable C_0-semigroup on Z and there exist $G_2 \in \mathcal{L}(H, W)$ and $L \in \mathcal{L}(W, H)$ such that $S + G_2 L$ generates an exponentially stable C_0-semigroup on W.*

If in addition there exists $Y \in \mathcal{L}(W, Z)$ such that $Y(\mathcal{D}(S)) \subset \mathcal{D}(A)$ and the following operator equations are satisfied

$$YS = AY - G_1 L - P \quad in\ \mathcal{D}(S) \tag{4.74a}$$

$$CY = L - Q \quad in\ W \tag{4.74b}$$

then $\mathcal{A}_F = \mathcal{A} - \mathcal{G}\mathcal{C}$ generates a strongly stable C_0-semigroup on $Z \times W$ whenever

$$\mathcal{G} = \begin{pmatrix} G_1 - YG_2 \\ G_2 \end{pmatrix} \tag{4.75}$$

Proof. It is a straightforward calculation to show using the operator equations (4.74) that

$$\begin{pmatrix} I & Y \\ 0 & I \end{pmatrix} \left[\begin{pmatrix} A & P \\ 0 & S \end{pmatrix} - \begin{pmatrix} G_1 - YG_2 \\ G_2 \end{pmatrix} \begin{pmatrix} C & -Q \end{pmatrix} \right] \begin{pmatrix} I & -Y \\ 0 & I \end{pmatrix} = \tag{4.76}$$

$$\begin{pmatrix} A - G_1 C & 0 \\ -G_2 C & S + G_2 L \end{pmatrix} = \mathcal{A}^s \tag{4.77}$$

so that $\mathcal{A}_F$ is similar to the operator $\mathcal{A}^s$ in (4.77). The semigroup $T_{\mathcal{A}^s}(t)$ generated by $\mathcal{A}^s$ on $Z \times W$ is given by

$$T_{\mathcal{A}^s}(t) = \begin{pmatrix} T_{A-G_1C}(t) & 0 \\ -\int_0^t T_{S+G_2L}(t-s)G_2CT_{A-G_1C}(s)ds & T_{S+G_2L}(t) \end{pmatrix}, \quad t \geq 0 \qquad (4.78)$$

where $T_{S+G_2L}(t)$ is the strongly (exponentially) stable C_0-semigroup generated by $S + G_2L$ on W. Since by our assumption $T_{A-G_1C}(t)$ is exponentially (strongly) stable on Z, we only need to show that

$$\lim_{t\to\infty} \int_0^t T_{S+G_2L}(t-s)G_2CT_{A-G_1C}(s)wds = 0 \quad \forall w \in W \qquad (4.79)$$

to ensure that $\mathcal{A}^s$ (and hence also $\mathcal{A}_F$) generates a strongly stable C_0-semigroup. That (4.79) holds in both of the above cases for each $w \in W$ follows immediately from Theorem 5.1.2 and Proposition 5.6.4 in [2]. □

Remark 4.33. The operator equations (4.74) are of the same form as the regulator equations (3.10). Their solution is discussed in Chapter 8.

Remark 4.34. If A generates an exponentially stable C_0-semigroup and if we may let $P = 0$ (which is often the case if robustness is present), then we may let $G_1 = 0$ and take $Y = 0$ in Theorem 4.32. Hence in this case it is sufficient to find $G_2 \in \mathcal{L}(H,W)$ such that $S + G_2Q$ generates a strongly stable C_0-semigroup on W. Sufficient conditions for the existence of such G_2 were presented in Section 4.6.2.

Remark 4.35. If under the assumptions of Theorem 4.32 both $A - G_1C$ and $S + G_2L$ generate exponentially stable C_0-semigroups, then so does $T_{\mathcal{A}_F}(t)$.

Remark 4.36. In Chapter 6 we shall derive conditions under which output regulation is robust with respect to the choice of P and Q in the operator $\mathcal{A}_F$. Under such conditions these operators P and Q need not coincide with those in the exosystem (2.2); they can be regarded as design parameters in Proposition 4.29, Proposition 4.31 and Theorem 4.32.

4.6.4 Strong stability of the C_0-semigroup generated by $\mathcal{A}_{D_K}$

Since the strong stability of $T_{\mathcal{A}_{D_K}}(t)$ means the existence of $G_0 \in \mathcal{L}(H,W)$ and a strongly stabilizing feedback $\mathcal{K} = \begin{pmatrix} K_1 & K_2 \end{pmatrix}$ for the pair $(\mathcal{A}_0, \mathcal{B})$, where $\mathcal{A}_0 = \begin{pmatrix} A & 0 \\ G_0C & S \end{pmatrix}$ and $\mathcal{B} = \begin{pmatrix} B \\ G_0D \end{pmatrix}$, it is not

very surprising that methods "dual" to those presented in Subsection 4.6.3 apply directly here. As regards the method employing Riccati equations we have:

Proposition 4.37. *Assume the following.*

1. *H and $Z \times W$ are (separable) Hilbert spaces.*

2. *$\mathcal{A}_0 = \left(\begin{smallmatrix} A & 0 \\ G_0 C & S \end{smallmatrix} \right)$ generates a contraction C_0-semigroup on $Z \times W$.*

3. *The pair $(\mathcal{A}_0, \mathcal{B}^*)$ is approximately observable (here $\mathcal{B} = \left(\begin{smallmatrix} B \\ G_0 D \end{smallmatrix} \right)$).*

4. *$\mathcal{A}_0$ has compact resolvent.*

Then for any coercive operators $U = U^* \in \mathcal{L}(H)$ and $R = R^* \in \mathcal{L}(H)$ the Riccati equation

$$\mathcal{A}_0^* \Delta z + \Delta \mathcal{A}_0 z - \Delta \mathcal{B} R^{-1} \mathcal{B}^* \Delta z + \mathcal{B} U^{-1} \mathcal{B}^* z = 0 \quad \forall z \in \mathcal{D}(\mathcal{A}_0) \tag{4.80}$$

has a unique self-adjoint solution $\Delta \in \mathcal{L}(Z \times W)$ such that $\mathcal{A}_{D_K} = \mathcal{A}_0 - \mathcal{B} R^{-1} \mathcal{B}^* \Delta$ generates a strongly stable C_0-semigroup.

Proof. This is essentially the same result as in Proposition 4.29. It also follows directly from [18]. We omit the details. □

Remark 4.38. If the assumptions of Proposition 4.37 are met, then we may choose as the strongly stabilizing feedback for the pair $(\mathcal{A}_0, \mathcal{B})$ the operator $\mathcal{K} = \left(\begin{smallmatrix} K_1 & K_2 \end{smallmatrix} \right) = -R^{-1} \mathcal{B}^* \Delta \in \mathcal{L}(Z \times W, H)$.

The dual version of Proposition 4.31 is given next.

Proposition 4.39. *Assume the following.*

1. *H and $Z \times W$ are (separable) Hilbert spaces.*

2. *$\mathcal{A}_0 = \left(\begin{smallmatrix} A & 0 \\ G_0 C & S \end{smallmatrix} \right)$ generates a contraction C_0-semigroup on $Z \times W$.*

3. *$\ker(\mu I - \mathcal{A}_0^*) \cap \ker(\mathcal{B}^*) = \{0\}$ for all $\mu \in i\mathbb{R} \cap \sigma_P(\mathcal{A}_0^*)$ (here $\mathcal{B} = \left(\begin{smallmatrix} B \\ G_0 D \end{smallmatrix} \right)$).*

4. *$\mathcal{A}_0$ has compact resolvent.*

then $\mathcal{A}_{D_K} = \mathcal{A}_0 - \mathcal{B} \mathcal{B}^*$ generates a strongly stable C_0-semigroup on $Z \times W$.

Proof. This result is just Theorem VI.8.28 in [28]. □

Finally, we arrive at a dual version of Theorem 4.32 below.

Theorem 4.40. *Assume that there exist $K \in \mathcal{L}(Z, H)$, $K_2 \in \mathcal{L}(W, H)$ and $L \in \mathcal{L}(H, W)$ such that* either

- $A + BK$ *generates an exponentially stable C_0−semigroup on Z and $S + LK_2$ generates a strongly stable C_0−semigroup on W,*

or

- $A + BK$ *generates a strongly stable C_0−semigroup on Z and $S + LK_2$ generates an exponentially stable C_0−semigroup on W.*

If there exist $G_0 \in \mathcal{L}(H, W)$ and $Y \in \mathcal{L}(Z, W)$ such that $Y(\mathcal{D}(A)) \subset \mathcal{D}(S)$ and

$$SY = Y(A + BK) + G_0(C + DK) \quad in \ \mathcal{D}(A) \tag{4.81a}$$

$$L = YB + G_0D \quad in \ H \tag{4.81b}$$

then for $K_1 = K + K_2Y$ the operator $\mathcal{A}_{D_K} = \begin{pmatrix} A+BK_1 & BK_2 \\ G_0(C+DK_1) & S+G_0DK_2 \end{pmatrix}$ generates a strongly stable C_0-semigroup on $Z \times W$.

Proof. With the above choices we have $\mathcal{A}_{D_K} = \begin{pmatrix} A+B(K+K_2Y) & BK_2 \\ G_0(C+D[K+K_2Y]) & S+G_0DK_2 \end{pmatrix}$. Then a direct calculation shows that

$$\begin{pmatrix} I & 0 \\ Y & I \end{pmatrix} \begin{pmatrix} A + B(K + K_2Y) & BK_2 \\ G_0(C + D[K + K_2Y]) & S + G_0DK_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ -Y & I \end{pmatrix} = \begin{pmatrix} A + BK & BK_2 \\ 0 & S + LK_2 \end{pmatrix} \tag{4.82}$$

which shows (see Theorem 5.1.2 and Proposition 5.6.4 in [2]) that $\mathcal{A}_{D_K}$ generates a strongly stable C_0−semigroup. $\square$

Remark 4.41. If under the assumptions of Theorem 4.40 both $A+BK$ and $S+LK_2$ also generate exponentially stable C_0−semigroups, then so does the operator $\mathcal{A}_{D_K}$ on $Z \times W$.

Remark 4.42. Strictly speaking, the operator equations (4.81) are not of the same form as the regulator equations (3.10). However, if we work in Hilbert spaces, then we can take adjoints of the equations (4.81) and solve them for Y^* and G_0^* using precisely the same methods which apply for the regulator equations (3.10) (see Chapter 8).

4.7   An example of error feedback output regulation

In this section we shall present an example to illustrate the error feedback output regulation theory developed in this chapter. More specifically, we study the same output regulation problem as in Example 3.54, but we want to use *error feedback* to achieve the asymptotic tracking of $p-$periodic reference signals in certain Sobolev spaces whenever there are no disturbances. The reader will observe, in particular, that in the example below we solve a repetitive control problem for such an infinite-dimensional system which does not have a direct feedthrough term (i.e. $D = 0$). This would not be possible — even for *finite-dimensional* systems — using the error feedback controllers of the classical repetitive control literature [36, 96].

Example 4.43. *Let $a > 0$, $r \neq 0$, $\tau_1 > \tau_2 > 0$ and consider the disturbance-free scalar delay differential equation*

$$\dot{x}(t) = -ax(t) - b\big[x(t - \tau_1) + x(t - \tau_2)\big] + u(t) \tag{4.83a}$$

$$y(t) = rx(t), \quad t \geq 0 \tag{4.83b}$$

of Example 3.54. Our goal is to build, using Theorem 4.15, a dynamic controller (4.1) which solves the EFRP for this plant and an exosystem (2.2) which is constructed using Proposition 2.3 for $W = \mathcal{H} = H^\alpha_{per}(0, p)$, where $\alpha > \frac{1}{2}$ is to be fixed, with $Q = \delta_0$, $P = 0$ and $w(0) = y_{ref} \in \mathcal{H}$. We assume that the system operator of (4.83) generates an exponentially stable C_0-semigroup as in Example 3.54. Then there are no transmission zeros of the plant in the set $\{\frac{2\pi ni}{p} \mid n \in \mathbb{Z}\}$.

By Example 3.54, for $\alpha > \frac{3}{2}$ we can solve the regulator equations (3.10) for bounded operators Π and Γ. Using the methods of Section 3.5 we obtain

$$\Gamma y_{ref} = \sum_{n \in \mathbb{Z}} \frac{\widehat{y_{ref}}(n)}{H(i\omega_n)}, \quad \forall y_{ref} \in \mathcal{H} \tag{4.84}$$

$$\Pi y_{ref} = \sum_{n \in \mathbb{Z}} \widehat{y_{ref}}(n) R(i\omega_n, A) B\Gamma\phi_n, \quad \forall y_{ref} \in \mathcal{H} \tag{4.85}$$

where $(\phi_n)_{n \in \mathbb{Z}} = (e^{i\omega_n \cdot})_{n \in \mathbb{Z}}$ is the natural orthogonal basis for $\mathcal{H}$ and $\widehat{y_{ref}}(n)$ is the nth L^2 Fourier coefficient of $y_{ref} \in \mathcal{H} \subset L^2(0, p)$.

By Theorem 4.22 $S|_{\mathcal{H}} - \delta_0^* \delta_0$ generates a strongly stable C_0-semigroup on $W = \mathcal{H} = H^\alpha_{per}(0, p)$ (with $\alpha > \frac{3}{2}$). Hence we may use Theorem 4.15 and Remark 4.34 to deduce that an error feedback

controller (4.1) *with*

$$F = \begin{pmatrix} A & B\Gamma \\ \delta_0^* C & S|_{\mathcal{H}} - \delta_0^* \delta_0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & \Gamma \end{pmatrix} \quad and \quad G = \begin{pmatrix} 0 \\ -\delta_0^* \end{pmatrix} \tag{4.86}$$

solves the EFRP in question.

A concrete example of the generalization (4.32) of Davison's dynamic state feedback controller [39] (and its extension which does not employ state feedback) will be provided in Chapter 6. We cannot present these examples here because we have not yet provided a proof of output regulation. In Chapter 6, examples robust error feedback output regulation for systems described by partial differential equations will also be provided.

Chapter 5

A feedforward-error feedback controller

In Chapter 3 and Chapter 4 we have studied the existence and construction of feedforward and error feedback controllers for the regulation of bounded uniformly continuous signals generated by the exosystem (2.2). Unfortunately there are some practical issues which may sometimes limit the applicability of these controllers.

First of all, although the feedforward controllers resulting from Theorem 3.6 are very simple (and hence quite appealing), they employ direct feedback from the state of the plant; this may be unrealistic in many applications. Moreover, under various assumptions the results of Section 3.3 show that in the case of the FRP it is actually also *necessary* to use a controller $u(t) = Kz(t) + (\Gamma - K\Pi)w(t)$, where K stabilizes the pair (A, B) strongly and Π and Γ solve the regulator equations (3.10). Since the operator $\Gamma - K\Pi$ in this controller in general depends on both the plant data and the stabilizing feedback K, this open loop controller does not provide robust (i.e. structurally stable) output regulation — even with respect to K.

On the other hand, the error feedback controller (4.1) does not utilize state feedback from the plant directly, and it is perhaps the more realistic one in applications. Moreover, in the finite-dimensional case it is well-known that error feedback controllers can provide robustness in output regulation [24, 29, 32, 60]. However, for an infinite-dimensional exosystem (2.2) designing error feedback controllers is not a very easy task — in particular, sufficient closed loop stability can be

difficult to achieve in practice. As we have seen in Section 4.6, this is chiefly because the system operator S of the exosystem is often in some form embedded in the system operator F of the controller, and because S is difficult to stabilize by compact feedback.

In order to overcome the above problems and also to illustrate the wide applicability of methods based on the regulator equations, in the present chapter we shall design a two-degrees-of-freedom (2-DOF) hybrid controller, employing *both* error feedback *and* feedforward control, for output regulation purposes. In our design the stabilizing state feedback $Kz(t)$ of an FRP controller is replaced by an output from a stabilizing dynamic controller, while the feedforward part of the controller is again tuned using the regulator equations (3.10). This procedure essentially results in a controller with two degrees of freedom, because it turns out that we can resolve the feedback and feedforward parts of the controller *independently* of each other. However, the resulting controller is not a 2-DOF controller in the conventional sense (see e.g. [69] p. 26), because 2-DOF controllers are commonly utilized to handle those cases where disturbance signal dynamics is different from the reference signal dynamics.

A key feature in our design is that we deliberately avoid the inclusion of the exosystem generator S — which causes the stabilizability problems — in the controller generator F. It turns out that, even if the exosystem (2.2) is infinite-dimensional, in this case exponential closed loop stability can often be achieved (as opposed to the error feedback controllers of Chapter 4). Moreover, our construction always guarantees robustness with respect to the stabilizing feedback part (as opposed to the feedforward controllers of Chapter 3).

In the following we shall review the contents of this chapter in more detail, and we shall more precisely indicate the respective contributions of this thesis.

Section 5.1: We shall define the feedforward-feedback regulation problem FFRP. This is a combination of the FRP and the EFRP; both error feedback and feedforward control are employed to achive output regulation. To our knowledge, this problem has not been explicitly studied before.

Section 5.2: We shall present sufficient conditions for the solvability of the FFRP. In particular, we shall show that if the operators F, G and J of the dynamic part of the controller can be chosen such that the closed loop system is strongly stable, then the operator Γ in the static feedforward part of this controller can be chosen according to solutions of the regulator equations (3.10), without any regard to the choice of the operators F, G or J.

Section 5.3: We shall present necessary conditions for the solvability of the FFRP. In particular, if the exogenous system (2.2) generates admissible reference signals (see Definition 3.14), and if some operators F, G, J and Γ solve the FFRP in such a way that $\sigma(F) \cap \sigma(S) = \emptyset$ and the operator $\mathcal{P}_\Gamma = \left(\begin{smallmatrix} P+B\Gamma \\ G(D\Gamma-Q) \end{smallmatrix} \right) \in \mathcal{L}(W, Z \times X)$ is regular for the closed loop semigroup, then there exists $\Pi \in \mathcal{L}(W, Z)$, with $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$, such that Π and Γ satisfy the regulator equations (3.10). In particular, in this case it is necessary to tune the feedforward part of the controller using solutions of the regulator equations (3.10).

Section 5.4: We shall present an example of feedforward-feedback output regulation.

The results of this chapter rely heavily on our earlier constructions in Chapter 3 and Chapter 4. They are based on those in [40].

5.1 The feedforward-feedback regulation problem FFRP

In this section we shall formulate the feedforward-feedback output regulation problem FFRP. It involves the construction of a dynamic controller on some Banach space X, such that also direct feedforward control from the exosystem (2.2) is permitted.

Definition 5.1 (FFRP). The task in the FFRP is to find a controller of the form

$$\dot{x}(t) = Fx(t) + G(y(t) - y_{ref}(t)), \quad x(0) \in X, \quad t \geq 0 \tag{5.1a}$$

$$u(t) = Jx(t) + \Gamma w(t) \tag{5.1b}$$

on some Banach state space X where F generates a C_0-semigroup, $G \in \mathcal{L}(H, X)$, $J \in \mathcal{L}(X, H)$ and $\Gamma \in \mathcal{L}(W, H)$. We require the following.

1. In the closed loop system

$$\dot{z}(t) = Az(t) + BJx(t) + (B\Gamma + P)w(t), \quad t \geq 0 \tag{5.2a}$$

$$\dot{x}(t) = GCz(t) + (F + GDJ)x(t) + G(D\Gamma - Q)w(t), \quad t \geq 0 \tag{5.2b}$$

$$\dot{w}(t) = Sw(t), \quad t \in \mathbb{R} \tag{5.2c}$$

$$e(t) = Cz(t) + DJx(t) + (D\Gamma - Q)w(t), \quad t \geq 0 \tag{5.2d}$$

the semigroup $T_\mathcal{A}(t)$ generated by the closed loop operator $\mathcal{A} = \left(\begin{smallmatrix} A & BJ \\ GC & F+GDJ \end{smallmatrix} \right)$, with $\mathcal{D}(\mathcal{A}) \subset Z \times X$, on $Z \times X$ is strongly stable.

2. In the closed loop system (5.2) the tracking error $e(t) = y(t) - y_{ref}(t) \to 0$ as $t \to \infty$ regardless of the initial conditions $z(0) \in Z$, $x(0) \in X$ and $w(0) \in W$.

Remark 5.2. The FFRP defined above is clearly a mixture of the EFRP and the FRP. The key difference between the FFRP and the FRP is the use of dynamic control to achieve closed loop stability. On the other hand, the key difference between the FFRP and the EFRP is the additional use of static feedforward control $\Gamma w(t)$ to enhance asymptotic tracking and disturbance rejection.

5.2 Sufficient conditions for the solvability of the FFRP

In Theorem 5.3 we shall prove that the solvability of the regulator equations (3.10) provides a sufficient condition for the solvability of the FFRP under the assumption of strong closed loop stability.

Theorem 5.3. *Assume that $\mathcal{A} = \left(\begin{smallmatrix} A & BJ \\ GC & F+GDJ \end{smallmatrix} \right)$ generates a strongly stable semigroup on $Z \times X$. If $\Gamma \in \mathcal{L}(W, H)$ can be chosen such that there exists $\Pi \in \mathcal{L}(W, Z)$, with $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$, and Π, Γ satisfy the regulator equations (3.10), then a controller (5.1) with these parameters F, G, J and Γ solves the FFRP.*

Proof. Let $\Theta(t) = \left(\begin{smallmatrix} z(t) \\ x(t) \end{smallmatrix} \right) \in Z \times X$ and define

$$\mathcal{A} = \begin{pmatrix} A & BJ \\ GC & F+GDJ \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathcal{P}_\Gamma = \begin{pmatrix} P+B\Gamma \\ G(D\Gamma - Q) \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} C & DJ \end{pmatrix}, \quad \mathcal{D} = 0$$

(5.3)

with obvious domains of definition. Then write the closed loop system (5.2) as

$$\dot{\Theta}(t) = \mathcal{A}\Theta(t) + \mathcal{B}u(t) + \mathcal{P}_\Gamma w(t), \quad \Theta(0) \in Z \times X \tag{5.4a}$$

$$\dot{w}(t) = Sw(t) \tag{5.4b}$$

$$e(t) = \mathcal{C}\Theta(t) + \mathcal{D}u(t) + [D\Gamma - Q]w(t) \tag{5.4c}$$

For appropriate operators this closed loop system is precisely of the same form as in the EFRP. Thus it is clear by Theorem 4.4 that we only have to verify the existence of $\Pi_0 \in \mathcal{L}(W, Z)$ and $\Lambda \in \mathcal{L}(W, X)$ such that $\Pi_0(\mathcal{D}(S)) \subset \mathcal{D}(A)$ and $\Lambda(\mathcal{D}(S)) \subset \mathcal{D}(F)$, and the following extended

regulator equations are satisfied

$$A\Pi_0 + BJ\Lambda + P + B\Gamma = \Pi_0 S \quad \text{in } \mathcal{D}(S) \tag{5.5a}$$

$$F\Lambda = \Lambda S \quad \text{in } \mathcal{D}(S) \tag{5.5b}$$

$$C\Pi_0 + DJ\Lambda = Q - D\Gamma \quad \text{in } W \tag{5.5c}$$

Clearly $\Pi_0 = \Pi$ and $\Lambda = 0$ is a possible choice since Π and Γ satisfy the regulator equations (3.10). $\qquad\square$

Remark 5.4. If $T_{\mathcal{A}}(t)$ is exponentially stable, then also the decay of $\|e(t)\|$ to 0 as $t \to \infty$ is exponentially fast in Theorem 5.3 (see also Remark 4.6). In fact, under the assumptions of Theorem 5.3 we have that $e(t) = \begin{pmatrix} C & DJ \end{pmatrix} T_{\mathcal{A}}(t) \left[\begin{pmatrix} z(0) \\ x(0) \end{pmatrix} - \begin{pmatrix} \Pi w(0) \\ 0 \end{pmatrix} \right]$ for all $t \geq 0$. This can be proved as in Theorem 3.6.

We point out that the proof of Theorem 5.3 is based on the solution of the extended regulator equations (4.3) in such a way that $\Lambda = 0$. This means that, in contrast to the controllers of Section 4.5, S need not be reduplicated in F. This feature of the controller can drastically simplify the stabilization of the closed loop system; as opposed to the EFRP, exponential closed loop stability is often possible in the FFRP even if the exosystem (2.2) is infinite-dimensional. However, we shall see in Chapter 6 that it is precisely the reduplication of S in F which makes *robust* output regulation possible in certain cases.

Another remarkable feature in Theorem 5.3 above is that the feedforward part (i.e. Γ) and the feedback part (i.e. F, G and J) of the controller (5.1) can be designed completely independently of each other. In fact, the operator Γ is chosen so that the regulator equations (3.10) are satisfied for some Π. These equations do not depend on the parameters F, G and J of the feedback controller, and hence it is possible to use *any* F, G and J in (5.1) which yield a generator $\mathcal{A}$ of a strongly stable semigroup. Consequently, Theorem 5.3 above is particularly suitable for the design of add-on controllers (see e.g. [83]), in which the stabilizing feedback part may have to be designed with also other constraints in mind.

5.3 Necessary conditions for the solvability of the FFRP

We next show that if the closed loop operator $\mathcal{A}$ generates a strongly stable semigroup on $Z \times X$ and if the spectra of S and F are disjoint, then the above structure for the feedforward part

$v(t) = \Gamma w(t)$ of a controller solving the FFRP is actually also necessary in the same practically important cases as in Chapter 3 and Chapter 4.

Theorem 5.5. *Assume that the exogenous system* (2.2) *generates admissible reference signals (see Definition 3.14). If some operators F, G, J and Γ solve the FFRP in such a way that $\sigma(F) \cap \sigma(S) = \emptyset$ and the operator $\mathcal{P}_\Gamma = \left(\begin{smallmatrix} P+B\Gamma \\ G(D\Gamma-Q) \end{smallmatrix} \right) \in \mathcal{L}(W, Z \times X)$ is regular (see Definition 3.9) for $T_{\mathcal{A}}(t)$, then there exists $\Pi \in \mathcal{L}(W, Z)$, with $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$, such that Π and Γ satisfy the regulator equations* (3.10).

Proof. According to the equations (5.4), the closed loop FFRP system (5.2) represents such a closed loop EFRP system where P is replaced by $P + B\Gamma$ and Q is replaced by $Q - D\Gamma$. Hence by Theorem 4.7 there exists $\Pi \in \mathcal{L}(W, Z)$ such that $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$ and $\Lambda \in \mathcal{L}(W, X)$ such that $\Lambda(\mathcal{D}(S)) \subset \mathcal{D}(F)$ satisfying

$$\Pi S = A\Pi + BJ\Lambda + P + B\Gamma \quad \text{in } \mathcal{D}(S) \tag{5.6a}$$

$$\Lambda S = F\Lambda \quad \text{in } \mathcal{D}(S) \tag{5.6b}$$

$$Q - D\Gamma = C\Pi + DJ\Lambda \quad \text{in } W \tag{5.6c}$$

It remains to show that $\Lambda = 0$. By Lemma A.8 there exists a sequence $(W_n)_{n \in \mathbb{N}} \subset W$ of closed $T_S(t)-$invariant subspaces such that $W_n \subset W_{n+1}$ for every $n \in \mathbb{N}$, $\sigma(S|_{W_n}) \subset \sigma(S)$, $S_n = S|_{W_n} \in \mathcal{L}(W_n)$, and $\overline{\cup_{n \in \mathbb{N}} W_n} = W$. Consequently for an arbitrary $n \in \mathbb{N}$ we have $\Lambda S f = \Lambda S_n f = F\Lambda f$ for each $f \in W_n$, so that $\Lambda S_n = F\Lambda$ in $\mathcal{D}(S_n) = W_n$. But $\sigma(S_n) \cap \sigma(F) \subset \sigma(S) \cap \sigma(F) = \emptyset$, so that the solution of this operator equation is unique (cf. Section 2 in [90]). Consequently $\Lambda = 0$ in W_n for each $n \in \mathbb{N}$. By continuity of Λ and denseness of $\cup_{n \in \mathbb{N}} W_n$, we must have that $\Lambda = 0$ in W. This shows that Π and Γ satisfy the regulator equations (3.10). $\square$

Corollary 5.6. *Assume that the exogenous system* (2.2) *generates admissible reference signals. If some operators F, G, J and Γ solve the FFRP in such a way that F and $\mathcal{A}$ generate exponentially stable C_0-semigroups on X and $Z \times X$ respectively. Then there exists $\Pi \in \mathcal{L}(W, Z)$, with $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$, such that Π and Γ satisfy the regulator equations* (3.10).

Proof. If F generates an exponentially stable C_0-semigroup, then $\sigma(F) \cap \sigma(S) \subset \sigma(F) \cap i\mathbb{R} = \emptyset$. If $\mathcal{A}$ generates an exponentially stable C_0-semigroup, then every operator $\mathcal{P}_\Gamma \in \mathcal{L}(W, Z \times X)$ is regular for $T_{\mathcal{A}}(t)$. The result follows by the above. $\square$

5.4 An example of feedforward-feedback output regulation

In this section we shall present a fairly simple example to illustrate the solution of the FFRP. A noteworthy feature in the example is that the stabilizing dynamic feedback part of the controller (5.1) is finite-dimensional, although the exosystem (2.2) is infinite-dimensional.

Example 5.7. *Let $x_1, x_2 \in (0,1)$, let $\epsilon > 0$ and consider the SISO process*

$$\frac{\partial z(x,t)}{\partial t} = \frac{\partial^2 z(x,t)}{\partial x^2} + 2\pi^2 z(x,t) + b_\epsilon(x)u(t), \quad 0 \le x \le 1, \quad t \ge 0 \tag{5.7a}$$

$$z(0,t) = z(1,t) = 0 \tag{5.7b}$$

$$y(t) = \int_0^1 c_\epsilon(x)z(x,t)dx \tag{5.7c}$$

Here $b_\epsilon(x) = \frac{1}{\epsilon}$ for $x_1 - \epsilon \le x \le x_1$ and $b_\epsilon(x) = 0$ otherwise, and $c_\epsilon(x) = \frac{1}{\epsilon}$ for $x_2 - \epsilon \le x \le x_2$ and $c_\epsilon(x) = 0$ otherwise. Our goal is to construct a feedforward-feedback controller (5.1) for the asymptotic tracking of $p-$periodic reference signals in certain generalized Sobolev spaces $\mathcal{H} = H_{AP}(\mathbb{C}, f_n, \frac{2\pi n}{p}) = H(f_n, \omega_n)$ (see Definition 2.18), assuming that there are no disturbances. To this end, we shall construct the exosystem (2.2) as in Proposition 2.3, with $W = \mathcal{H}$, $S = S|_{\mathcal{H}}$, $P = 0$ and $Q = \delta_0 \in \mathcal{L}(\mathcal{H}, \mathbb{C})$.

Let $Z = L^2(0,1)$ and define $Af = \frac{d^2}{dx^2}f + 2\pi^2 f$ with $\mathcal{D}(A) = \{ f \in Z \mid \frac{d^2}{dx^2}f \in Z, f(0) = f(1) = 0 \}$. Also define $B \in \mathcal{L}(\mathbb{C}, Z)$ such that $Bu = b_\epsilon u$ for every $u \in \mathbb{C}$, and $C \in \mathcal{L}(Z, \mathbb{C})$ such that $Cf = \int_0^1 c_\epsilon(x)f(x)dx$ for each $f \in Z$. With these operators A, B and C, and $D = 0$ the system (5.7) takes the form (1.1) with $\mathcal{U}_{dist}(t) = 0$ for each t.

It is easy to see that A has a self-adjoint compact inverse, and that the eigenvalues of A are $\lambda_n = (2 - n^2)\pi^2$, $n \in \mathbb{N}$. These eigenvalues are real and the corresponding eigenvectors constitute an orthonormal basis in Z. Moreover, A has only one eigenvalue in the closed right half plane, namely $\lambda = \pi^2$. The corresponding eigenfunction is $\phi(x) = \sqrt{2}\sin(\pi x)$.

Assume that x_1, x_2 and $\epsilon > 0$ have been chosen such that $b = \int_0^1 b_\epsilon(x)\phi(x)dx \ne 0$ and $c = \int_0^1 c_\epsilon(x)\phi(x)dx \ne 0$. Then the state feedback operator $J = \langle \cdot, -\frac{3\pi^2}{b}\phi \rangle \in \mathcal{L}(Z, \mathbb{C})$ exponentially stabilizes the pair (A, B). Moreover, the output injection operator $L \in \mathcal{L}(\mathbb{C}, Z)$ defined by $Ly = y\frac{-5\pi^2}{c}\phi$ for each $y \in \mathbb{C}$ is such that $A + LC$ generates an exponentially stable C_0-semigroup. Consequently, as we let $X = Z$, we see that the closed loop operator

$$\mathcal{A} = \begin{pmatrix} A & BJ \\ -LC & A + LC + BJ \end{pmatrix} \tag{5.8}$$

generates an exponentially stable C_0-semigroup in $Z \times X$ so that we may choose $G = -L$ and $F = A + LC + BJ$ in the controller (5.1).

Having constructed the operators F, G and J of the controller (5.1), we may now proceed to the construction of the feedforward part of the controller as demonstrated in Section 3.5 and Chapter 8. We assume that the transfer function $H(s) = CR(s, A)B$ of the plant does not vanish at the points $i\omega_n$ for any $n \in I$. Let $\phi_n(x) = e^{i\omega_n x}$ for each $n \in I$ and $x \in \mathbb{R}$. We then immediately see that the following operators Γ, Π solve the regulator equations (3.10) whenever they are in $\mathcal{L}(\mathcal{H}, \mathbb{C})$ and $\mathcal{L}(\mathcal{H}, Z)$ respectively:

$$\Gamma y_{ref} = \sum_{n \in I} \frac{\widehat{y_{ref}}(n)}{H(i\omega_n)}, \quad \forall y_{ref} \in \mathcal{H} \tag{5.9}$$

$$\Pi y_{ref} = \sum_{n \in I} \widehat{y_{ref}}(n) R(i\omega_n, A) B \Gamma \phi_n = \sum_{n \in I} \frac{\widehat{y_{ref}}(n)}{H(i\omega_n)} R(i\omega_n, A) B, \quad \forall y_{ref} \in \mathcal{H} \tag{5.10}$$

Here $\omega_n = \frac{2\pi n}{p}$, $n \in I$, and $\widehat{y_{ref}}(n)$ is the nth L^2-Fourier coefficient of $y_{ref} \in \mathcal{H}$. For a suitable weighting sequence $(f_n)_{n \in I}$ in the space $\mathcal{H}$ the coefficients $\widehat{y_{ref}}(n)$ tend to 0 so fast that the operators Π and Γ above are indeed bounded; for example, by the Schwartz inequality

$$\|\Gamma y_{ref}\| \leq \sum_{n \in I} \left| \frac{\widehat{y_{ref}}(n)}{H(i\omega_n)} \right| \leq \sqrt{\sum_{n \in I} |\widehat{y_{ref}}(n)|^2 |f_n|^2} \sqrt{\sum_{n \in I} \frac{1}{|f_n|^2 |H(i\omega_n)|^2}} = M \|y_{ref}\|_{\mathcal{H}} \tag{5.11}$$

for all $y_{ref} \in \mathcal{H}$ whenever $\sum_{n \in I} |f_n|^{-2} |H(i\omega_n)|^{-2} = M^2 < \infty$. As a matter of fact, the condition $(f_n^{-1} H(i\omega_n)^{-1})_{n \in I} \in \ell^2$ for the sequence $(f_n)_{n \in I}$ also guarantees the boundedness of Π because clearly $\|R(i\omega_n, A)B\|$ is uniformly bounded in n (use e.g. the eigenfunction expansion of A, hence $T_A(t)$ and $R(i\omega_n, A)$, to see this).

For the sake of a numerical example, we let $x_1 = 1, x_2 = \frac{1}{2}, \epsilon = \frac{1}{2}$ (this is obviously a valid choice as regards the constants b and c above). Using the method in [17] (pp. 184-186) it is straightforward to verify that in this case the transfer function $H(s)$ of the plant is given by

$$H(s) = \frac{4\left[\cosh(\frac{\sqrt{s - 2\pi^2}}{2}) - 1\right]^2}{(s - 2\pi^2)^{\frac{3}{2}} \sinh(\sqrt{s - 2\pi^2})}, \quad s \in \rho(A) \tag{5.12}$$

so that indeed $H(i\omega_n) \neq 0$ for all $n \in I$. Since the term

$$\left| \frac{\sinh(\sqrt{i\omega - 2\pi^2})}{\left[\cosh(\frac{\sqrt{i\omega - 2\pi^2}}{2}) - 1\right]^2} \right| \tag{5.13}$$

is uniformly bounded in $\omega \in \mathbb{R}$, we have that $|H(i\omega)|^{-2}$ is of order $\mathcal{O}(|\omega|^3)$ as $|\omega| \to \infty$. This shows that it is sufficient to choose the sequence $(f_n)_{n \in I}$ such that $f_n \geq 1$ for all $n \in I$ and — if

the index set I is not finite — such that $(f_n^{-1})_{n \in I} \in \ell^2$ and f_n^2 grows faster than $|\omega_n|^{4+\gamma}$ for some (arbitrary) $\gamma > 0$ as $|\omega_n| \to \infty$.

Whenever the space $\mathcal{H}$ (i.e. the sequence $(f_n)_{n \in I}$) is chosen such that the operators Π and Γ above are bounded, for any given reference function $y_{ref} \in \mathcal{H}$ we should choose the feedforward part $\Gamma w(t)$ of the controller (5.1) as

$$\Gamma w(t) = \Gamma T_S(t)|_{\mathcal{H}} y_{ref} = \sum_{n \in I} \frac{\widehat{y_{ref}}(n)}{H(i\omega_n)} e^{i\omega_n t}, \quad y_{ref} \in \mathcal{H} \tag{5.14}$$

We point out that this feedforward control law is similar — but not equal — to the feedforward control law used in Section 3.5 where knowledge of the transfer function of the stabilized *plant is in general required. Here we only need knowledge of the transfer function of the original plant because the stabilizing feedback part of the controller can be designed independently of the feedforward part of the controller.*

Chapter 6

Robustness and the internal model structure

In the previous chapters we have designed feedforward controllers, error feedback controllers and feedforward-feedback controllers for output regulation purposes. In practice, however, it is also desirable to achieve a degree of robustness in output regulation. In very general terms robustness means *tolerance for uncertainty*. In this thesis robustness is understood in the sense that perturbations to some of the parameters of the plant (1.1), the employed controller and the exogenous signal generator (2.2) should not affect the closed loop stability and the asymptotic tracking/rejection of the exogenous signals (see Definition 6.4). This type of robustness is often referred to as *structural stability* in the literature [29, 32, 93]; it covers e.g. small modelling errors but it does not cover e.g. the effect of small time delays on output regulation [64, 76].

Because of their practical importance it is not at all surprising that robust controllers of the above type have been subject to a vast amount of research during the past three decades. The problem of robust output regulation was solved for finite-dimensional linear systems and exosystems by Davison [24], Davison and Goldenberg [19], Francis [29], Francis and Wonham [32], Sebakhy and Wonham [82], Wonham [93] and others in the 1970s. It is now well known that error feedback must be utilized in order to achieve robust output regulation. In fact, we saw in Chapter 3 and Chapter 5 respectively that feedforward controllers do not in general achieve robust output regulation, and that a feedforward-feedback controller is often only guaranteed to be robust with respect to the

stabilizing part of the controller.

According to Kwatny and Kalnitsky [60], the finite-dimensional linear error feedback control methods which achieve robust output regulation can be quite generally grouped into two distinct categories — those employing estimates of (possibly artificial) disturbance states (e.g. [29]), and those employing dynamic error augmentation (e.g. [24]). Moreover, these two categories arise as special instances of the general robust output regulation paradigm for finite-dimensional systems, the celebrated Internal Model Principle due to Francis and Wonham [32] (see also [24]). This principle asserts that an error feedback controller which stabilizes the closed loop system also achieves robust output regulation if and only if the controller utilizes a suitably reduplicated copy of the maximal cyclic component of the exogenous system matrix S. For a survey of these (and related) results, with an emphasis on the differences between the above two categories, the reader should see [60].

During the past several decades many authors have also extended portions of the above finite-dimensional robust output regulation theory for infinite-dimensional plants and finite-dimensional exogenous systems. The finite-dimensional results of Davison and his coworkers have been generalized by Pohjolainen [73, 74], Hämäläinen and Pohjolainen [34, 35], Ukai and Iwazumi [87] and others (see [33] for more detailed information). On the other hand, the finite-dimensional results of Francis [29] have been generalized to this setting by Bhat [7] who focused on applications in time-delay systems. However, it is interesting to observe that although the papers [12, 80] do generalize the error feedback output regulation theory of Francis [29] and Wonham [93] for certain infinite-dimensional systems, they do not address the issue of robustness.

To the author's knowledge so far no one has generalized the Internal Model Principle for infinite-dimensional systems in the *state space domain*. Yamamoto and Hara have proved in Theorem 4.10 of [96] a *frequency domain* analogue of the Internal Model Principle for systems having a pseudorational impulse response (e.g. repetitive control systems): Under the hypothesis of internal closed loop stability[1], the existence of an internal model in the controller is equivalent to stable tracking. However, the theory of Yamamoto and Hara [96] seems to suffer from relatively narrow applicability because the hypothesis of internal closed loop stability is impossible to meet in many applications (see Chapter 1 for more details). Moreover, the plant in the repetitive control

---

[1]Yamamoto and Hara show in Theorem 3.5 of [96] that for suitably observable realizations of their class of systems this stability notion is equivalent to exponential stability.

applications [36, 92, 95, 96] is invariably finite-dimensional.

The lack of a general Internal Model Principle for infinite-dimensional state space systems may be a consequence of the fact that the principle, when formulated precisely in the state space domain, employs purely finite-dimensional concepts such as minimal polynomials and rational canonical decompositions of matrices [32]. Thus, although there is evidence that the Internal Model Principle also holds for infinite-dimensional state space systems [96], it is not at all trivial what a reasonable formulation — and interpretation — of this principle for such systems is in the state space domain.

In the present chapter we shall develop a robustness theory for error feedback (EFRP) output regulation of infinite-dimensional systems (1.1) and (possibly infinite-dimensional) exosystems (2.2). In particular, we shall present such an operator-theoretic state space generalization for the Internal Model Principle which does not employ any purely finite-dimensional concepts. The core of this infinite-dimensional Internal Model Principle lies in the observation that *any* closed loop error feedback (EFRP) control system, when appropriately stabilized, in a sense already contains the error zeroing dynamics. A suitable choice of the controller's parameters F and G resulting in the so called internal model structure (see Definition 6.19) then realizes the desired dynamical behaviour of robust output regulation. In fact, error feedback controllers which have the internal model structure also have the crucial property that sufficient closed loop stability already *implies* the solvability of the extended regulator equations (4.3); this has been shown to guarantee output regulation in Theorem 4.4.

Before reviewing the contents of this chapter in more detail it is appropriate to precisely define the robustness concepts that we are going to employ. The following assumption holds throughout this chapter.

Assumption 6.1 (Perturbations)**.** *All perturbations to the parameters of the plant, the controller and the exosystem are bounded, linear and additive. Moreover, all perturbations are independent of each other in the sense that no perturbation to any given parameter affects the other parameters of the closed loop system. Perturbations are denoted by Δ_M where M is the parameter that is subject to perturbation[2].*

Remark 6.2. The reason why we restrict our attention to bounded perturbations only is their

---

[2]For example, A can be subject to a perturbation $\Delta_A \in \mathcal{L}(Z)$ such that the perturbed operator is $A + \Delta_A$.

simplicity. The so-called relatively bounded perturbations are also important e.g. for differential operators [28, 57], but the theory developed in this chapter does not apply to them.

Quite often we shall employ the following self-explanatory notion of smallness in perturbations:

Definition 6.3 (Small perturbations)**.** A perturbation Δ_M to a parameter M of the plant, the controller or the exosystem is called small (enough) if $\|\Delta_M\|$ is small (enough).

Definition 6.4 (Robust output regulation)**.** Let Ω be a class of perturbations to the parameters of the plant, the controller and/or the exosystem[3]. Then output regulation is

- robust with respect to Ω if all perturbations in Ω, when applied to the corresponding parts of the closed loop control system, preserve strong closed loop stability and asymptotic tracking/disturbance rejection for all initial states of the plant, the controller and the exosystem;

- conditionally robust with respect to Ω if all perturbations in Ω, when applied to the corresponding parts of the closed loop control system, have the following property: *If* they preserve strong closed loop stability, *then* they also preserve asymptotic tracking/disturbance rejection for all initial states of the plant, the controller and the exosystem.

Conditional robustness in the above sense is a rephrasement of the widely used finite-dimensional robustness criterion: Closed loop stability implies output regulation. Clearly any purely finite-dimensional control system having this property achieves a degree of robustness in output regulation, with respect to certain sufficiently small perturbations. However, we point out that in infinite dimensions strong stability of a C_0-semigroup can sometimes be destroyed by arbitrarily small bounded perturbations to its generator — hence the study of conditional robustness and robustness separately is reasonable for infinite-dimensional systems:

Example 6.5. *Let Z be a Hilbert space with an orthonormal basis $(\phi_n)_{n\geq 1}$ and consider the bounded diagonal operator A on Z defined by $Az = \sum_{n\geq 1}(-\frac{1}{n})\langle z, \phi_n\rangle\phi_n$ for all $z \in Z$. Clearly A generates a strongly stable C_0-semigroup on Z by the Arendt-Batty-Lyubich-Vũ Theorem (Theorem V.2.21 in [28]), but the semigroup generated by $A + \epsilon I$ is unbounded for all $\epsilon > 0$.*

Remark 6.6. A potentially useful result in the study of conditional robustness is the following one due to Casarino and Piazzera [13]: Assuming that an operator E generates a strongly stable

---

[3]For example, if A is subject to perturbations, one possibility is $\Omega = \{\,\Delta_A \in \mathcal{L}(Z) \mid \|\Delta_A\| < \epsilon\,\}$ for some $\epsilon > 0$.

C_0-semigroup $T_E(t)$ on a Banach space Y, a sufficient condition that the perturbed operator $E + \Delta_E$, where $\Delta_E \in \mathcal{L}(Y)^4$, still generates a strongly stable C_0-semigroup on Y is

$$\sup_{t \geq 0} \int_0^t \|\Delta_E T_E(s)y\| ds \leq C\|y\|, \quad \forall y \in \mathcal{D}(E) \tag{6.1}$$

for some $0 < C < 1$.

Based on the above discussion we choose as our basic strategy in this chapter to establish conditional robustness results for output regulation; these results then immediately yield robustness results if the closed loop system is also exponentially stable and if we only consider small enough perturbations. Moreover, our focus in the present chapter will be on controllers solving the EFRP of Chapter 4, because they do not incorporate feedforward control. This makes (conditionally) robust output regulation possible.

We shall now review the contents of this chapter in more detail, and we shall more precisely indicate the respective contributions of this thesis.

Section 6.1: We shall prove general conditional robustness results for dynamic controllers (4.1) which solve the EFRP. Here generality refers to the fact that we do not fix the parameters F, G and J of the controller (4.1) in any specific way, as opposed to e.g. [7, 33, 87]. The conditional robustness results are obtained by studying the persistence of the unique solvability of the extended regulator equations (4.3) under perturbations to some of the parameters A, B, C, D, P, Q, G and J of the plant (1.1), the exosystem (2.2) and the controller (4.1) (observe that S and F cannot be perturbed). These results, together with Theorem 4.4, readily imply conditional robustness of output regulation. Our proofs employ operator-theoretic generalizations of some finite-dimensional methods of Francis [29]. However, they seem to be new even for finite-dimensional systems. In fact, although the conditions of the main result (Theorem 6.16) resemble those found in the finite-dimensional control literature, it generalizes the work of Francis [29] and Francis and Wonham [32] by allowing for perturbations also in C, D and Q. Moreover, it generalizes the results of Davison [24] by allowing for the parameters F, G and J of the controller to be free modulo strong closed loop stability and the unique solvability of the extended regulator equations (4.3). The results of this section are contained in [45].

---

[4]In [13] the additive perturbation Δ_E may actually be of the more general Miyadera-Voigt type.

Section 6.2: We shall define the internal model structure, and we shall prove (cf. Theorem 6.20) that under exponential closed loop stability this structure is both necessary and sufficient for any dynamic controller (4.1) which achieves robust output regulation in the sense of the EFRP. This result is an infinite-dimensional state space generalization of the Internal Model Principle of Francis and Wonham [32]. By allowing for conditional robustness we shall also extend its sufficiency part for such control systems for which the closed loop operator $\mathcal{A}$ only generates a strongly stable C_0-semigroup (cf. Theorem 6.25 and Theorem 6.32). In this case we shall need an additional assumption about the unique solvability of a Sylvester operator equation; it turns out that this can often be verified *without explicitly calculating the solution operator* (cf. Corollary 6.26 and Corollary 6.33). The results of this section are essentially contained in [43, 45]; to the author's knowledge they are new even for finite-dimensional exosystems.

Section 6.3: In order to gain a deeper understanding of controllers having the internal model structure we shall derive general characterizations for this structure using geometric methods, spectral theory and an interplay of ergodic theory and the recent theory of implemented semigroups due to Alber and Kühnemund [1, 59]. Although the results of this section are rather abstract from the application point of view, they show, in particular, that the finite-dimensional structurally stable synthesis algorithm of Francis [29], whenever applied to a finite-dimensional EFRP with S as in Proposition 2.3, results in a dynamic controller having the internal model structure (cf. Example 6.36). These results are new even for finite-dimensional systems; they are contained in [45].

Section 6.4: We shall continue the study of controllers which have the internal model structure by considering the special case where F generates an isometric C_0-group such that $\sigma(S) = \sigma(F)$. It turns out (cf. Section 6.5) that in practice the verification of the internal model structure can often be reduced to this special case; it then corresponds to the finite-dimensional procedure of embedding suitable reduplications of the maximal cyclic component of S in F [29]. Theorem 6.46 shows how the operator G in (4.1) should be chosen to obtain the internal model structure if nothing extra is known about $\sigma(S) = \sigma(F)$ but X and W are Hilbert spaces, and S and F are bounded. Theorem 6.47, on the other hand, shows that

whenever $\sigma(S) = \sigma(F)$ are also discrete[5] sets the controller has the internal model structure if $P_{i\omega}^F G : H \to \operatorname{ran}(P_{i\omega}^F)$ is injective for all $i\omega \in \sigma(F)$. Here $P_{i\omega}^F$ is the spectral projection corresponding to an isolated point $i\omega \in \sigma(F)$, and neither S nor F need be bounded. An example shows a special case in which this injectivity condition is equivalent to the approximate controllability of the pair (F, G). Theorem 6.47 is contained in [43] while Theorem 6.46 is new even for finite-dimensional systems.

Section 6.5: In this section we shall prove conditional robustness results for the controllers of Section 4.5 which generalize certain finite-dimensional controllers of Davison (cf. [39]) and Francis [29]. Our standing assumptions are mostly the same as those in Section 4.5; however a fundamental difference is that the operator S in (4.26) and (4.32) is replaced by an *auxiliary operator* S_a defined on another Banach space W_a, such that in a sense S_a resembles S. As will be shown, this replacement is quite benefical, because it allows us to design the controllers in such a way that they have the internal model structure, and because it is then possible to regard the operators P and Q in the controller (4.26) as *design parameters* — a feature which is very convenient in the (otherwise quite difficult) process of stabilization of the closed loop system. In Subsection 6.5.1 we shall prove conditional robustness results for the Francis-type controllers of Subsection 4.5.1. The Davison-type controllers of Subsection 4.5.2 are then treated in Subsection 6.5.2 where the dynamic state feedback controller (4.32) is also generalized so that the use of state feedback is not necessary. The results of this section are mostly contained in [43]; they generalize the finite-dimensional robust output regulation theory, e.g. [24, 19, 29, 32], for infinite dimensional systems. Moreover, they provide a robustness theory for the controllers utilized in [12].

Section 6.6: In some applications it may be sensible to *trade* perfect output regulation without guaranteed robustness to almost perfect output regulation with guaranteed robustness. In this case study section we shall illustrate how this can be done in the case that the reference signals are in some infinite-dimensional Sobolev space $\mathcal{G} = H_{AP}(\mathbb{C}^M, f_n, \omega_n)$ with fixed sequences $(\omega_n)_{n \in I}$ and $(f_n)_{n \in I}$ (see Chapter 2). In Theorem 6.65 we propose a controller which achieves robust approximate output regulation of every $y_{ref} \in \mathcal{G}$ in the sense that $\limsup_{t \to \infty} \|e(t)\| \leq \epsilon \|y_{ref}\|_{\mathcal{G}}$ where $\epsilon > 0$ is a prespecified accuracy. A remarkable feature of the controller is

---

[5] A discrete spectrum consists of isolated points only [57].

that it is guaranteed to work under the assumptions of nonexistence of transmission zeros on a certain finite subset of $\{\, i\omega_n \mid n \in I \,\}$, $D = 0$, the exponential stabilizability of the pair (A, B) and the exponential detectability of the pair (A, C). These requirements may be simpler to verify in practice than those utilized in [12], which are also applicable here (but do not guarantee robustness of output regulation).

Section 6.7: In this case study section we shall focus on a repetitive control application for exponentially stable SISO plants. We shall prove that in our framework it is possible to conditionally robustly regulate p−periodic signals with an infinite number of distinct frequency components even if $D = 0$ in the plant. In the conventional repetitive control scheme [36, 96] this is not possible, because internal (i.e. exponential) closed loop stability — which implies output regulation — can only be attained if the finite-dimensional plant is *not* strictly proper (cf. Section V of [96] or Chapter 1). Theorem 6.70 proposes a controller which achieves strong stability of the closed loop semigroup $T_{\mathcal{A}}(t)$ by utilizing a stabilized copy of the generator of the p−periodic translation group on a *suitably larger* space than on which output regulation is required. Moreover, if the reference signals are *smooth enough* with respect to the above copy, then output regulation is conditionally robust with respect to certain perturbations $\Delta_{\mathcal{A}}$ to the closed loop operator $\mathcal{A}$ satisfying $\sup_{n \in \mathbb{Z}} \|\Delta_{\mathcal{A}} R(\frac{2\pi i n}{p}, \mathcal{A})\| < 1$. The results of this section are mostly contained in [43].

Section 6.8: In this section we shall provide some simple but concrete examples of (conditionally) robust output regulation for infinite-dimensional systems.

6.1 General conditional robustness results for error feedback controllers

In this section we shall present general conditional robustness results for dynamic controllers (4.1) solving the EFRP. Here the term "general" refers to the fact that the only restrictions for the choice of the parameters F, G and J of a controller (4.1) solving the EFRP are strong closed loop stability and the *unique* solvability of the extended regulator equations (4.3) for certain operators P and Q. We emphasize that this section only contains sufficient conditions for conditionally robust output regulation; that these conditions can (and sometimes *must*) be satisfied in practice is the topic of

the subsequent sections.

We begin by introducing a number of auxiliary operators.

Definition 6.7. Let the Banach space X and the operators F, G and J be as in the controller (4.1)[6]. Then

- The linear Sylvester operator $\mathcal{T}_{A,S}$ is defined on a subspace of $\mathcal{L}(W, Z)$ by

$$\mathcal{D}(\mathcal{T}_{A,S}) = \{\, \Pi \in \mathcal{L}(W, Z) \mid \Pi(\mathcal{D}(S)) \subset \mathcal{D}(A), \exists Y \in \mathcal{L}(W, Z) :$$

$$Y w = A \Pi w - \Pi S w \ \forall w \in \mathcal{D}(S) \,\} \tag{6.2}$$

$$\mathcal{T}_{A,S} \Pi = Y \tag{6.3}$$

- The linear Sylvester operator $\mathcal{T}_{F,S}$ is defined on a subspace of $\mathcal{L}(W, X)$ by

$$\mathcal{D}(\mathcal{T}_{F,S}) = \{\, \Lambda \in \mathcal{L}(W, X) \mid \Lambda(\mathcal{D}(S)) \subset \mathcal{D}(F), \exists Y \in \mathcal{L}(W, X) :$$

$$Y w = F \Lambda w - \Lambda S w \ \forall w \in \mathcal{D}(S) \,\} \tag{6.4}$$

$$\mathcal{T}_{F,S} \Lambda = Y \tag{6.5}$$

- The linear multiplication operator $\mathbf{C} : \mathcal{L}(W, Z) \to \mathcal{L}(W, H)$ is defined by $\mathbf{C}\Pi = C\Pi$ for each $\Pi \in \mathcal{L}(W, Z)$.

- The linear multiplication operator $\mathbf{B} : \mathcal{L}(W, X) \to \mathcal{L}(W, Z)$ is defined by $\mathbf{B}\Lambda = BJ\Lambda$ for each $\Lambda \in \mathcal{L}(W, X)$.

- The linear multiplication operator $\mathbf{D} : \mathcal{L}(W, X) \to \mathcal{L}(W, H)$ is defined by $\mathbf{D}\Lambda = DJ\Lambda$ for each $\Lambda \in \mathcal{L}(W, X)$.

From [1, 3, 59] it follows that the Sylvester operators $\mathcal{T}_{A,S}$ and $\mathcal{T}_{F,S}$ in Definition 6.7 are closed on $\mathcal{L}(W, Z)$ and $\mathcal{L}(W, X)$ respectively. Furthermore, it is easy to see that $\mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ are bounded operators because B, C, D and J are bounded operators. We shall need the following combined operators:

Definition 6.8. Let $\mathcal{T}_{A,S}, \mathcal{T}_{F,S}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ be as in Definition 6.7. Then

- the linear operator $[\mathcal{T}_{A,S} \& \mathbf{B}] : \mathcal{D}(\mathcal{T}_{A,S}) \times \mathcal{L}(W, X) \to \mathcal{L}(W, Z)$ is defined by

$$[\mathcal{T}_{A,S} \& \mathbf{B}]\left(\begin{smallmatrix} \Pi \\ \Lambda \end{smallmatrix}\right) = \mathcal{T}_{A,S}\Pi + \mathbf{B}\Lambda \quad \forall \Pi \in \mathcal{D}(\mathcal{T}_{A,S}) \text{ and } \forall \Lambda \in \mathcal{L}(W, X) \tag{6.6}$$

---

[6]At this point we do not assume that the controller (4.1) solves any output regulation problem.

- the linear operator $[\&\mathcal{T}_{F,S}] : \mathcal{L}(W,Z) \times \mathcal{D}(\mathcal{T}_{F,S}) \to \mathcal{L}(W,X)$ is defined by

$$[\&\mathcal{T}_{F,S}]\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right) = \mathcal{T}_{F,S}\Lambda \quad \forall\Pi \in \mathcal{L}(W,Z) \text{ and } \forall\Lambda \in \mathcal{D}(\mathcal{T}_{F,S}) \tag{6.7}$$

- the linear operator $[\mathbf{C}\&\mathbf{D}] : \mathcal{L}(W,Z) \times \mathcal{L}(W,X) \to \mathcal{L}(W,H)$ is defined by $[\mathbf{C}\&\mathbf{D}]\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right) = \mathbf{C}\Pi + \mathbf{D}\Lambda$ for each $\Pi \in \mathcal{L}(W,Z)$ and every $\Lambda \in \mathcal{L}(W,X)$.

It is straightforward to show that $[\mathcal{T}_{A,S}\&\mathbf{B}]$ and $[\&\mathcal{T}_{F,S}]$ are closed operators on $\mathcal{L}(W,Z) \times \mathcal{L}(W,X)$. Furthermore, since $\mathbf{C}$ and $\mathbf{D}$ are bounded operators, so is $[\mathbf{C}\&\mathbf{D}]$.

We are now ready to turn to the conditional robustness results of this section. The first one is an elementary sufficient condition that the EFRP is solvable regardless of the disturbance operator P in the exosystem (2.2).

Proposition 6.9. *Assume that C has a right inverse $C^- \in \mathcal{L}(H,Z)$ such that $C^-Q \in \mathcal{D}(\mathcal{T}_{A,S})$. If F, G and J in (4.1) are chosen so that the closed loop system operator $\mathcal{A} = \left(\begin{smallmatrix} A & BJ \\ GC & F+GDJ \end{smallmatrix}\right)$ generates a strongly stable C_0-semigroup on $Z \times X$ and if for every $P \in \mathcal{L}(W,Z)$ there exist $\Pi \in \mathcal{L}(W,Z)$ and $\Lambda \in \mathcal{L}(W,X)$ such that $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$ and $\Lambda(\mathcal{D}(S)) \subset \mathcal{D}(F)$, and the following extended regulator equations are satisfied*

$$A\Pi + BJ\Lambda + P = \Pi S \quad in \ \mathcal{D}(S) \tag{6.8a}$$

$$F\Lambda = \Lambda S \quad in \ \mathcal{D}(S) \tag{6.8b}$$

$$C\Pi + DJ\Lambda = 0 \quad in \ W \tag{6.8c}$$

then with this triplet (F,G,J) the EFRP is solvable regardless of P in the exosystem (2.2).

Proof. Let $P \in \mathcal{L}(W,Z)$ be arbitrary and set $M = P + \mathcal{T}_{A,S}C^-Q$. Since $M \in \mathcal{L}(W,Z)$, by our assumption there exist operators $\Pi \in \mathcal{L}(W,Z)$ and $\Lambda \in \mathcal{L}(W,X)$ such that $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$ and $\Lambda(\mathcal{D}(S)) \subset \mathcal{D}(F)$, and

$$A\Pi + BJ\Lambda + M = \Pi S \quad in \ \mathcal{D}(S) \tag{6.9a}$$

$$F\Lambda = \Lambda S \quad in \ \mathcal{D}(S) \tag{6.9b}$$

$$C\Pi + DJ\Lambda = 0 \quad in \ W \tag{6.9c}$$

Here equation (6.9a) reads $A\Pi + BJ\Lambda + P + \mathcal{T}_{A,S}C^-Q = A\Pi + BJ\Lambda + AC^-Q - C^-QS + P = \Pi S$

in $\mathcal{D}(S)$. Consequently we have that

$$A(\Pi + C^- Q) + BJ\Lambda + P = (\Pi + C^- Q)S \quad \text{in } \mathcal{D}(S) \tag{6.10a}$$

$$F\Lambda = \Lambda S \quad \text{in } \mathcal{D}(S) \tag{6.10b}$$

$$C(\Pi + C^- Q) + DJ\Lambda = Q \quad \text{in } W \tag{6.10c}$$

This shows that the conditions of Theorem 4.4 hold, and hence the EFRP is solvable for every P. $\qquad\qquad\square$

In the finite-dimensional case the existence of a right inverse C^- for $C \in \mathcal{L}(Z, H)$ is equivalent to C being surjective — a standard assumption [24, 29, 32]. Moreover, in the finite-dimensional setting the Sylvester operator $\mathcal{T}_{A,S}$ is bounded (and hence everywhere defined). Consequently the technical assumptions of Proposition 6.9 about C^- and $C^- Q$ involve no loss of generality if the spaces H, W, X and Z are finite-dimensional. An example of an infinite-dimensional setup in which the technical assumptions of Proposition 6.9 are satisfied is given next. If the reader is not familiar with bi-continuous and implemented semigroups, it may be worthwhile to look at Appendix A.3 at this point.

Example 6.10. *Let $H = \mathbb{C}$, $W = \mathcal{H} \xhookrightarrow{s} BUC(\mathbb{R}, H)$ and assume that $S = S|_\mathcal{H} \in \mathcal{L}(\mathcal{H})$, $Q = \delta_0 \in \mathcal{L}(\mathcal{H}, H)$ as in Proposition 2.3. Let Z be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and take $C = \langle \cdot, c \rangle$ for some $0 \neq c \in \mathcal{D}(A)$. Clearly we can define the bounded linear operator $C^- : \mathbb{C} \to Z$ by $C^- y = y \frac{c}{\|c\|^2}$ for each $y \in \mathbb{C}$, then $CC^- y = y \frac{\langle c, c \rangle}{\|c\|^2} = y$ for each $y \in \mathbb{C}$ (i.e. C^- is a right inverse of C). Furthermore, $\mathrm{ran}(C^-) \subset \mathcal{D}(A)$.*

It remains to show that $C^- \delta_0 \in \mathcal{D}(\mathcal{T}_{A,S|_\mathcal{H}})$. By the results of [1, 59] we know that $\mathcal{T}_{A,S|_\mathcal{H}}$ generates the so-called implemented semigroup $\mathcal{G}(t)$ on $\mathcal{L}(\mathcal{H}, Z)$ given by $\mathcal{G}(t)\Pi = T_A(t)\Pi T_S(-t)|_\mathcal{H}$, $\Pi \in \mathcal{L}(\mathcal{H}, Z)$, where $T_A(t)$ is the C_0-semigroup generated by A on Z and $T_S(-t)|_\mathcal{H}$ is the C_0-group generated by $-S|_\mathcal{H}$ on $\mathcal{H}$. This implemented semigroup is strongly continuous in the strong operator topology, and by Theorem 1.17 in [59] it suffices to show that the limit $\lim_{h \to 0^+} \frac{\mathcal{G}(h)C^- \delta_0 - C^- \delta_0}{h}$ exists in the strong operator topology of $\mathcal{L}(\mathcal{H}, Z)$.

Let $f \in \mathcal{H}$ be arbitrary. Then $f \in \mathcal{D}(S|_\mathcal{H})$ because $S|_\mathcal{H} \in \mathcal{L}(\mathcal{H})$. Furthermore, since $S|_\mathcal{H}$ is the

differential operator, the limit $\lim_{h \to 0} \frac{f(t+h)-f(t)}{h}$ exists for all $t \in \mathbb{R}$. Then

$$\lim_{h \to 0^+} \frac{\mathcal{G}(h)C^- \delta_0 f - C^- \delta_0 f}{h} = \lim_{h \to 0^+} \frac{T_A(h)C^- \delta_0 T_S(-h)|_{\mathcal{H}} f - C^- \delta_0 f}{h} \tag{6.11}$$

$$= \lim_{h \to 0^+} \frac{T_A(h)C^- f(-h) - C^- f(0)}{h} \tag{6.12}$$

$$= \lim_{h \to 0^+} \frac{T_A(h)C^-[f(-h) - f(0)] + T_A(h)C^- f(0) - C^- f(0)}{h} \tag{6.13}$$

It is evident that $\lim_{h \to 0^+} \frac{T_A(h)C^-[f(-h)-f(0)]}{h} = \lim_{h \to 0^+} T_A(h)C^- \frac{[f(-h)-f(0)]}{h}$ exists, because $T_A(t)$ is a C_0-semigroup and C^- is continuous. Similarly $\lim_{h \to 0^+} \frac{T_A(h)C^- f(0) - C^- f(0)}{h}$ exists because $\mathrm{ran}(C^-) \in \mathcal{D}(A)$. Consequently the limit $\lim_{h \to 0^+} \frac{\mathcal{G}(h)C^- \delta_0 - C^- \delta_0}{h}$ exists in the strong operator topology of $\mathcal{L}(\mathcal{H}, Z)$, and so $C^- \delta_0 \in \mathcal{D}(\mathcal{T}_{A,S|_{\mathcal{H}}})$.

In Proposition 6.9 we only allowed the operator P in the exosystem vary. Next we shall also consider perturbations to the parameters A, B and G of the plant and the controller. The following elementary lemma will be important in the subsequent proofs.

Lemma 6.11. *Let U and V be Banach spaces, let $R \subset U$ be a closed subspace and let $M : \mathcal{D}(M) \subset U \to V$ be a closed linear operator. Then $\ker(M)$ is a closed subspace of U and the restriction $M|_R : \mathcal{D}(M|_R) = \mathcal{D}(M) \cap R \to V : M|_R u = Mu \, \forall u \in \mathcal{D}(M|_R)$ is a closed operator.*

Proof. Let $(u_n)_{n \in \mathbb{N}} \subset \ker(M)$ be such that $\lim_{n \to \infty} u_n = u \in U$. Then $(Mu_n)_{n \in \mathbb{N}} = (0)_{n \in \mathbb{N}} \subset V$ is a Cauchy sequence, hence convergent in V, i.e. $\lim_{n \to \infty} Mu_n = y$ for some $y \in V$. But it is easy to see that necessarily $y = 0$. Since M is a closed operator, $u \in \mathcal{D}(M)$ and $Mu = y = 0$, i.e. $u \in \ker(M)$. Consequently $\ker(M)$ is a closed subspace of U.

Let $(r_n)_{n \in \mathbb{N}} \subset \mathcal{D}(M|_R) = \mathcal{D}(M) \cap R$ be such that $\lim_{n \to \infty} r_n = r \in R$ and such that $\lim_{n \to \infty} M|_R r_n = y \in V$. Then $\lim_{n \to \infty} r_n = r \in U$ and $\lim_{n \to \infty} Mr_n = y \in V$. Since M is a closed operator $\mathcal{D}(M) \subset U \to V$, we must have that $r \in \mathcal{D}(M) \cap R = \mathcal{D}(M|_R)$ and $Mr = M|_R r = y$. $\qquad\square$

Theorem 6.12. *Assume that C has a right inverse $C^- \in \mathcal{L}(H, Z)$ such that $C^- Q \in \mathcal{D}(\mathcal{T}_{A,S})$. Let F, G and J in (4.1) be such that the closed loop operator $\mathcal{A}$ generates a strongly stable C_0-semigroup on $Z \times X$ and for every $P \in \mathcal{L}(W, Z)$ there exist a unique $\Pi \in \mathcal{L}(W, Z)$ and a unique $\Lambda \in \mathcal{L}(W, X)$ such that $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$ and $\Lambda(\mathcal{D}(S)) \subset \mathcal{D}(F)$, and the extended regulator equations (6.8) are satisfied. Then the controller (4.1) solves the EFRP for every $P \in \mathcal{L}(W, Z)$ in such a way that*

output regulation is conditionally robust with respect to all perturbations to G, and all small enough perturbations to A and B.

Proof. Our first observation is that the regulator equations (6.8) do not depend on G. Hence we may perturb G arbitrarily as long as closed loop stability is preserved, according to Proposition 6.9 and according to our assumption about the independence of the perturbations (Assumption 6.1).

Let $\mathcal{M} = \ker([\mathbf{C}\&\mathbf{D}]) \cap \ker([\&\mathcal{T}_{F,S}])$. Then by Lemma 6.11 $\mathcal{M}$ is a closed subspace of $\mathcal{L}(W, Z) \times \mathcal{L}(W, X)$, and the restriction $[\mathcal{T}_{A,S}\&\mathbf{B}]|_{\mathcal{M}}$ is a closed operator.

The crucial steps of the proof are the following.

(i) We show that the extended regulator equations (6.8) have a unique solution for every $P \in \mathcal{L}(W, Z)$ if and only if the the restriction $[\mathcal{T}_{A,S}\&\mathbf{B}]|_{\mathcal{M}}$ is a closed bijection $\mathcal{M} \to \mathcal{L}(W, Z)$.

(ii) Bounded additive perturbations to A and B result in a bounded additive perturbation to $[\mathcal{T}_{A,S}\&\mathbf{B}]$, whereas the space $\mathcal{M}$ is not affected by these perturbations. Thus such perturbations to A and B result in a bounded additive perturbation to $[\mathcal{T}_{A,S}\&\mathbf{B}]|_{\mathcal{M}}$.

(iii) If the perturbations to A and B are small enough, then the perturbed operator $[\mathcal{T}_{A,S}\&\mathbf{B}]|_{\mathcal{M}}^{p}$ is still a closed bijection $\mathcal{M} \to \mathcal{L}(W, Z)$.

The desired result then follows via another application of item (i) and Proposition 6.9. We now proceed to the proofs.

(i) Let $P \in \mathcal{L}(W, Z)$ be arbitrary. If the extended regulator equations (6.8) have a unique solution, then there exists a unique element $\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right) \in \mathcal{D}(\mathcal{T}_{A,S}) \times \mathcal{D}(\mathcal{T}_{F,S}) \subset \mathcal{D}([\mathcal{T}_{A,S}\&\mathbf{B}]) \cap \mathcal{D}([\&\mathcal{T}_{F,S}])$ such that

$$[\mathcal{T}_{A,S}\&\mathbf{B}]\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right) = \mathcal{T}_{A,S}\Pi + \mathbf{B}\Lambda = A\Pi - \Pi S + BJ\Lambda = -P \tag{6.14}$$

$$[\&\mathcal{T}_{F,S}]\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right) = F\Lambda - \Lambda S = 0 \tag{6.15}$$

$$[\mathbf{C}\&\mathbf{D}]\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right) = C\Pi + DJ\Lambda = 0 \tag{6.16}$$

Consequently $\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right) \in \mathcal{M}$, and $[\mathcal{T}_{A,S}\&\mathbf{B}]|_{\mathcal{M}}$ is a closed bijection $\mathcal{M} \to \mathcal{L}(W, Z)$.

On the other hand, if the restriction $[\mathcal{T}_{A,S}\&\mathbf{B}]|_{\mathcal{M}}$ is a closed bijection $\mathcal{M} \to \mathcal{L}(W, Z)$, then for every $P \in \mathcal{L}(W, Z)$ there exists a unique element $\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right) \in \mathcal{M} \cap \mathcal{D}([\mathcal{T}_{A,S}\&\mathbf{B}])$ such that

$[\mathcal{T}_{A,S}\&\mathbf{B}]\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right) = -P$. Equations (6.14)-(6.16) then show that the extended regulator equations (6.8) have a unique solution for every such P.

(ii) Let $A^p = A + \Delta_A$ and $B^p = B + \Delta_B$, where $\Delta_A \in \mathcal{L}(Z)$ and $\Delta_B \in \mathcal{L}(H,Z)$. Then since the perturbations are independent of each other and since the Banach space $\mathcal{M}$ does not depend on A or B, it is not affected by these perturbations. Moreover clearly $\mathcal{D}(\mathcal{T}_{A,S}) = \mathcal{D}(\mathcal{T}_{A^p,S})$. In fact, $\mathcal{T}_{A^p,S}\Pi = (A + \Delta_A)\Pi - \Pi S = \mathcal{T}_{A,S}\Pi + \Delta_A\Pi$ for each $\Pi \in \mathcal{D}(\mathcal{T}_{A,S})$, so that $\mathcal{T}_{A^p,S} = \mathcal{T}_{A,S} + \Delta_{\mathcal{T}}$. Here the bounded linear operator $\Delta_{\mathcal{T}} \in \mathcal{L}(\mathcal{L}(W,Z))$ is defined as $\Delta_{\mathcal{T}}\Pi = \Delta_A\Pi$ for each $\Pi \in \mathcal{L}(W,Z)$. Similarly, the perturbed operator $\mathbf{B}^p = \mathbf{B} + \Delta_{\mathbf{B}}$ where the bounded linear operator $\Delta_{\mathbf{B}}\Lambda = \Delta_B J\Lambda$ for each $\Lambda \in \mathcal{L}(W,X)$. Hence for all $\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right) \in \mathcal{D}([\mathcal{T}_{A,S}\&\mathbf{B}])$ we have that the perturbed operator is

$$[\mathcal{T}_{A,S}\&\mathbf{B}]^p\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right) = \mathcal{T}_{A^p,S}\Pi + \mathbf{B}^p\Lambda = \mathcal{T}_{A,S}\Pi + \mathbf{B}\Lambda + \Delta\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right) \tag{6.17}$$

where the bounded linear operator $\Delta : \mathcal{L}(W,Z) \times \mathcal{L}(W,X) \to \mathcal{L}(W,Z)$ is given by $\Delta\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right) = \Delta_{\mathcal{T}}\Pi + \Delta_{\mathbf{B}}\Lambda = \Delta_A\Pi + \Delta_B J\Lambda$ for each $\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right) \in \mathcal{L}(W,Z) \times \mathcal{L}(W,X)$. In conclusion $\mathcal{D}([\mathcal{T}_{A,S}\&\mathbf{B}]^p) = \mathcal{D}([\mathcal{T}_{A,S}\&\mathbf{B}])$ and $[\mathcal{T}_{A,S}\&\mathbf{B}]^p = [\mathcal{T}_{A,S}\&\mathbf{B}] + \Delta$.

(iii) From item (ii) we immediately see that as $\|\Delta_A\| \to 0$ and $\|\Delta_B\| \to 0$, so must $\|\Delta\| \to 0$. Since $\mathcal{M}$ and $\mathcal{L}(W,Z)$ are Banach spaces and $[\mathcal{T}_{A,S}\&\mathbf{B}]|_{\mathcal{M}}$ is a closed bijection $\mathcal{M} \to \mathcal{L}(W,Z)$, by the Open Mapping Theorem it must be boundedly invertible. By Theorem IV.1.16 in [57], if $\|\Delta\|$ is (i.e. if $\|\Delta_A\|$ and $\|\Delta_B\|$ are) sufficiently small, then the perturbed operator $[\mathcal{T}_{A,S}\&\mathbf{B}]|_{\mathcal{M}}^p$ is still boundedly invertible, that is, a closed bijection $\mathcal{M} \to \mathcal{L}(W,Z)$.

$\square$

Corollary 6.13. *Let the assumptions of Theorem 6.12 hold. If in addition the closed loop operator $\mathcal{A}$ generates an exponentially stable C_0-semigroup, then the controller (4.1) solves the EFRP for every $P \in \mathcal{L}(W,Z)$ in such a way that output regulation is robust with respect to all small enough perturbations to A, B and G.*

We now turn our attention to the case in which the operators C, D, G and Q undergo perturbations while the other operators in the closed loop system are held fixed.

Theorem 6.14. *Assume that the operators F, G and J in (4.1) are such that the conditions of Theorem 4.4 hold for some $Q = Q_0 \in \mathcal{L}(W,H)$. Assume, in addition, that for every $Q \in \mathcal{L}(W,H)$*

there exist unique operators $\Pi \in \mathcal{L}(W, Z)$ and $\Lambda \in \mathcal{L}(W, X)$ such that $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$ and $\Lambda(\mathcal{D}(S)) \subset \mathcal{D}(F)$, and the extended regulator equations

$$A\Pi + BJ\Lambda = \Pi S \quad in\ \mathcal{D}(S) \tag{6.18a}$$

$$F\Lambda = \Lambda S \quad in\ \mathcal{D}(S) \tag{6.18b}$$

$$C\Pi + DJ\Lambda = Q \quad in\ W \tag{6.18c}$$

are satisfied. Then the controller (4.1) solves the EFRP for every $Q \in \mathcal{L}(W, H)$ in such a way that output regulation is conditionally robust with respect to all perturbations to G, and all small enough perturbations to C and D.

Proof. The proof of this result closely parallels the proof of Theorem 6.12. Again, since the extended regulator equations (4.3) do not depend on G and since all perturbations are assumed to be independent of each other (Assumption 6.1), we may perturb G arbitrarily as long as closed loop stability is preserved, according to Theorem 4.4.

Let $\mathcal{N} = \ker([\mathcal{T}_{A,S} \& \mathbf{B}]) \cap \ker([\& \mathcal{T}_{F,S}])$. Then, by Lemma 6.11, $\mathcal{N}$ is a closed subspace of $\mathcal{L}(W, Z) \times \mathcal{L}(W, X)$, and the restriction $[\mathbf{C} \& \mathbf{D}]|_{\mathcal{N}}$ is a bounded linear operator.

The crucial steps of the proof are the following.

(i) We show that the extended regulator equations (6.18) have a unique solution for each $Q \in \mathcal{L}(W, H)$ if and only if the the restriction $[\mathbf{C} \& \mathbf{D}]|_{\mathcal{N}}$ is a bijection $\mathcal{N} \to \mathcal{L}(W, H)$.

(ii) Bounded additive perturbations to C and D result in a bounded additive perturbation to $[\mathbf{C} \& \mathbf{D}]$, whereas the space $\mathcal{N}$ is not affected by these perturbations. Thus, these perturbations result in a bounded and additive perturbation to $[\mathbf{C} \& \mathbf{D}]|_{\mathcal{N}}$.

(iii) If the perturbations to C and D are small enough, then the perturbed operator $[\mathbf{C} \& \mathbf{D}]|_{\mathcal{N}}^p$ is still a bounded bijection $\mathcal{N} \to \mathcal{L}(W, H)$.

(iv) The extended regulator equations (4.3), with C and D replaced by C^p and D^p, have a solution for each Q whenever the perturbations to C and D are small enough.

The desired result then follows from Theorem 4.4. We now proceed to the proofs.

(i) Let $Q \in \mathcal{L}(W, H)$ be arbitrary. If the extended regulator equations (6.18) have a unique solution, then there exists a unique element $\left(\begin{smallmatrix} \Pi \\ \Lambda \end{smallmatrix} \right) \in \mathcal{D}(\mathcal{T}_{A,S}) \times \mathcal{D}(\mathcal{T}_{F,S}) \subset \mathcal{D}([\mathcal{T}_{A,S} \& \mathbf{B}]) \cap$

$\mathcal{D}([\&\mathcal{T}_{F,S}])$ such that

$$[\mathcal{T}_{A,S}\&\mathbf{B}]\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right) = \mathcal{T}_{A,S}\Pi + \mathbf{B}\Lambda = A\Pi - \Pi S + BJ\Lambda = 0 \tag{6.19}$$

$$[\&\mathcal{T}_{F,S}]\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right) = F\Lambda - \Lambda S = 0 \tag{6.20}$$

$$[\mathbf{C}\&\mathbf{D}]\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right) = C\Pi + DJ\Lambda = Q \tag{6.21}$$

Consequently $\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right) \in \mathcal{N}$, and $[\mathbf{C}\&\mathbf{D}]|_{\mathcal{N}}$ is a bounded bijection $\mathcal{N} \to \mathcal{L}(W,H)$.

On the other hand, if the restriction $[\mathbf{C}\&\mathbf{D}]|_{\mathcal{N}}$ is a bounded bijection $\mathcal{N} \to \mathcal{L}(W,H)$, then for every $Q \in \mathcal{L}(W,H)$ there exists a unique element $\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right) \in \mathcal{N}$ such that $[\mathbf{C}\&\mathbf{D}]|_{\mathcal{N}}\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right) = Q$. Equations (6.19)-(6.21) then show that the extended regulator equations (6.18) have a unique solution for every $Q \in \mathcal{L}(W,H)$.

(ii) Let $C^p = C + \Delta_C$ and $D^p = D + \Delta_D$, where $\Delta_C \in \mathcal{L}(Z,H)$ and $\Delta_D \in \mathcal{L}(H)$. Then since perturbations are independent of each other and since the Banach space $\mathcal{N}$ does not depend on C or D, $\mathcal{N}$ is not affected by these perturbations. Moreover clearly $[\mathbf{C}\&\mathbf{D}]^p\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right) = C^p\Pi + D^p\Lambda = (C+\Delta_C)\Pi + (D+\Delta_D)J\Lambda = [\mathbf{C}\&\mathbf{D}]\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right) + \Delta_{\mathbf{CD}}\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right)$ for each $\Pi \in \mathcal{L}(W,Z)$ and $\Lambda \in \mathcal{L}(W,X)$. Here the bounded linear operator $\Delta_{\mathbf{CD}} \in \mathcal{L}(\mathcal{L}(W,Z) \times \mathcal{L}(W,X), \mathcal{L}(W,H))$ is defined by $\Delta_{\mathbf{CD}}\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right) = \Delta_C\Pi + \Delta_D J\Lambda$ for each $\Pi \in \mathcal{L}(W,Z)$ and $\Lambda \in \mathcal{L}(W,X)$.

(iii) From item (ii) we immediately see that as $\|\Delta_C\| \to 0$ and $\|\Delta_D\| \to 0$, so must $\|\Delta_{\mathbf{CD}}\| \to 0$. Since $\mathcal{N}$ and $\mathcal{L}(W,H)$ are Banach spaces and $[\mathbf{C}\&\mathbf{D}]|_{\mathcal{N}}$ is a bounded bijection $\mathcal{N} \to \mathcal{L}(W,H)$, by the Open Mapping Theorem it must be boundedly invertible. By Theorem IV.1.16 in [57], if $\|\Delta_{\mathbf{CD}}\|$ is (i.e. if $\|\Delta_C\|$ and $\|\Delta_D\|$ are) sufficiently small, then the perturbed operator $[\mathbf{C}\&\mathbf{D}]|_{\mathcal{N}}^p$ is still boundedly invertible, that is, a bounded bijection $\mathcal{N} \to \mathcal{L}(W,H)$.

(iv) Let $\left(\begin{smallmatrix}\Pi_0\\\Lambda_0\end{smallmatrix}\right) \in \mathcal{D}(\mathcal{T}_{A,S}) \times \mathcal{D}(\mathcal{T}_{F,S})$ denote a solution of the extended regulator equations (4.3) for the nominal value $Q = Q_0$. Let $Q \in \mathcal{L}(W,H)$ be arbitrary and set $Q^p = Q - C^p\Pi_0 - D^p J\Lambda_0$. Assuming that the perturbations to C and D are small enough, applying items (iii) and (i) again shows that for this $Q^p \in \mathcal{L}(W,H)$ the extended regulator equations (6.18), with C and

D replaced by C^p and D^p respectively, have a unique solution $\left(\begin{smallmatrix}\Pi^p\\\Lambda^p\end{smallmatrix}\right)$. Moreover

$$[\mathcal{T}_{A,S}\&\mathbf{B}]\left(\begin{smallmatrix}\Pi_0+\Pi^p\\\Lambda_0+\Lambda^p\end{smallmatrix}\right) = \mathcal{T}_{A,S}(\Pi_0 + \Pi^p) + \mathbf{B}(\Lambda_0 + \Lambda^p) = \mathcal{T}_{A,S}\Pi_0 + \mathbf{B}\Lambda_0 = -P \qquad (6.22)$$

$$[\&\mathcal{T}_{F,S}]\left(\begin{smallmatrix}\Pi_0+\Pi^p\\\Lambda_0+\Lambda^p\end{smallmatrix}\right) = F(\Lambda_0 + \Lambda^p) - (\Lambda_0 + \Lambda^p)S = 0 \qquad (6.23)$$

$$[\mathbf{C}\&\mathbf{D}]^p\left(\begin{smallmatrix}\Pi_0+\Pi^p\\\Lambda_0+\Lambda^p\end{smallmatrix}\right) = C^p(\Pi_0 + \Pi^p) + D^p J(\Lambda_0 + \Lambda^p) \qquad (6.24)$$

$$= C^p\Pi_0 + D^p J\Lambda_0 + Q - C^p\Pi_0 - D^p J\Lambda_0 = Q \qquad (6.25)$$

Consequently the operators $\Pi_0 + \Pi^p$ and $\Lambda_0 + \Lambda^p$ solve the extended regulator equations (4.3), with C and D replaced by C^p and D^p respectively.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Corollary 6.15. *Let the assumptions of Theorem 6.14 hold. If in addition the closed loop operator $\mathcal{A}$ generates an exponentially stable C_0-semigroup, then the controller (4.1) solves the EFRP for every $Q \in \mathcal{L}(W, H)$ in such a way that output regulation is robust with respect to all small enough perturbations to C, D and G.*

The following theorem is the main result of this section. It provides a sufficient condition that small perturbations to the operators A, B, C, D, G, J, P and Q do not destroy EFRP type output regulation. A remarkable feature in the result is that, in contrast to Theorem 6.12, we obtain conditional robustness — even with respect to more general perturbations — without having to assume the existence of a right inverse C^- for C such that $C^-Q \in \mathcal{D}(\mathcal{T}_{A,S})$. However, we do have to assume a more restricted kind of unique solvability of the extended regulator equations (4.3) than in either Theorem 6.12 or Theorem 6.14.

Theorem 6.16. *Let F, G and J in the controller (4.1) be such that the closed loop operator $\mathcal{A}$ generates a strongly stable C_0-semigroup $T_\mathcal{A}(t)$ on $Z \times X$. Assume that, in addition, for every $P \in \mathcal{L}(W, Z)$ and every $Q \in \mathcal{L}(W, H)$ there exist unique operators $\Pi \in \mathcal{L}(W, Z)$ and $\Lambda \in \mathcal{L}(W, X)$ such that $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$ and $\Lambda(\mathcal{D}(S)) \subset \mathcal{D}(F)$, and the extended regulator equations (4.3) are satisfied. Then the controller (4.1) solves the EFRP for every $P \in \mathcal{L}(W, Z)$ and every $Q \in \mathcal{L}(W, H)$ in such a way that output regulation is conditionally robust with respect to all perturbations to G, and all small enough perturbations to A, B, C, D and J.*

Proof. As before, since the extended regulator equations (4.3) do not depend on G, by the independece of the perturbations we may perturb G arbitrarily as long as closed loop stability is preserved,

according to Theorem 4.4. Moreover, by Lemma 6.11 the restriction $[\mathcal{T}_{A,S}\&\mathbf{B}]|_{\ker([\&\mathcal{T}_{F,S}])}$ of the closed operator $[\mathcal{T}_{A,S}\&\mathbf{B}]$ to the closed subspace $\ker([\&\mathcal{T}_{F,S}])$ of $\mathcal{L}(W,Z) \times \mathcal{L}(W,X)$ is again a closed operator.

We can now define a linear operator matrix $\mathbf{K} : \mathcal{D}(\mathbf{K}) = \mathcal{D}([\mathcal{T}_{A,S}\&\mathbf{B}]) \cap \ker([\&\mathcal{T}_{F,S}]) \subset \mathcal{L}(W,Z) \times \mathcal{L}(W,X) \to \mathcal{L}(W,Z) \times \mathcal{L}(W,H)$ by

$$\mathbf{K}\begin{pmatrix}\Pi \\ \Lambda\end{pmatrix} = \begin{pmatrix}\mathcal{T}_{A,S} & \mathbf{B} \\ \mathbf{C} & \mathbf{D}\end{pmatrix}\begin{pmatrix}\Pi \\ \Lambda\end{pmatrix} = \begin{pmatrix}\mathcal{T}_{A,S}\Pi + \mathbf{B}\Lambda \\ \mathbf{C}\Pi + \mathbf{D}\Lambda\end{pmatrix} = \begin{pmatrix}[\mathcal{T}_{A,S}\&\mathbf{B}]\begin{pmatrix}\Pi \\ \Lambda\end{pmatrix} \\ [\mathbf{C}\&\mathbf{D}]\begin{pmatrix}\Pi \\ \Lambda\end{pmatrix}\end{pmatrix} \tag{6.26}$$

for each $\begin{pmatrix}\Pi \\ \Lambda\end{pmatrix} \in \mathcal{D}(\mathbf{K})$.

In analogy with the previous proofs, we proceed by showing the following:

(i) $\mathbf{K}$ is a closed operator $\mathcal{D}(\mathbf{K}) = \mathcal{D}([\mathcal{T}_{A,S}\&\mathbf{B}]) \cap \ker([\&\mathcal{T}_{F,S}]) \to \mathcal{L}(W,Z) \times \mathcal{L}(W,H)$.

(ii) The extended regulator equations (4.3) have a unique solution for each $P \in \mathcal{L}(W,Z)$ and every $Q \in \mathcal{L}(W,H)$ if and only if $\mathbf{K}$ is a bijection $\mathcal{D}(\mathbf{K}) \to \mathcal{L}(W,Z) \times \mathcal{L}(W,H)$.

(iii) Bounded linear additive perturbations to A, B, C, D and J result in a bounded linear additive perturbation $\Delta_{\mathbf{K}}$ to $\mathbf{K}$, which is independent of the perturbations to G, such that the perturbed operator is $\mathbf{K}^p = \mathbf{K} + \Delta_{\mathbf{K}}$ with $\mathcal{D}(\mathbf{K}^p) = \mathcal{D}(\mathbf{K})$.

(iv) If the perturbations to A, B, C, D and J are small enough, then the perturbed operator $\mathbf{K}^p$ is still a bounded bijection $\mathcal{D}(\mathbf{K}) \to \mathcal{L}(W,Z) \times \mathcal{L}(W,H)$.

The desired result then follows via another application of item (ii) above and Theorem 4.4. It remains to prove the above properties.

(i) Let $\left(\begin{pmatrix}\Pi_n \\ \Lambda_n\end{pmatrix}\right)_{n\in\mathbb{N}} \subset \mathcal{D}(\mathbf{K})$ be such that $\lim_{n\to\infty}\begin{pmatrix}\Pi_n \\ \Lambda_n\end{pmatrix} = \begin{pmatrix}\Pi \\ \Lambda\end{pmatrix} \in \mathcal{L}(W,Z) \times \mathcal{L}(W,X)$ and $\lim_{n\to\infty}\mathbf{K}\begin{pmatrix}\Pi_n \\ \Lambda_n\end{pmatrix} = \begin{pmatrix}P \\ Q\end{pmatrix} \in \mathcal{L}(W,Z) \times \mathcal{L}(W,H)$. Then for each $n \in \mathbb{N}$ we have $\begin{pmatrix}\Pi_n \\ \Lambda_n\end{pmatrix} \in \mathcal{D}([\mathcal{T}_{A,S}\&\mathbf{B}]) \cap \ker([\&\mathcal{T}_{F,S}]) = \mathcal{D}([\mathcal{T}_{A,S}\&\mathbf{B}]|_{\ker([\&\mathcal{T}_{F,S}])})$. Moreover, $\lim_{n\to\infty}[\mathcal{T}_{A,S}\&\mathbf{B}]\begin{pmatrix}\Pi_n \\ \Lambda_n\end{pmatrix} = \lim_{n\to\infty}[\mathcal{T}_{A,S}\&\mathbf{B}]|_{\ker([\&\mathcal{T}_{F,S}])}\begin{pmatrix}\Pi_n \\ \Lambda_n\end{pmatrix} = P$. Since the operator $[\mathcal{T}_{A,S}\&\mathbf{B}]|_{\ker([\&\mathcal{T}_{F,S}])}$ is closed (cf. Lemma 6.11), $\begin{pmatrix}\Pi \\ \Lambda\end{pmatrix} \in \mathcal{D}([\mathcal{T}_{A,S}\&\mathbf{B}]) \cap \ker([\&\mathcal{T}_{F,S}])$ and we also have $[\mathcal{T}_{A,S}\&\mathbf{B}]\begin{pmatrix}\Pi \\ \Lambda\end{pmatrix} = [\mathcal{T}_{A,S}\&\mathbf{B}]|_{\ker([\&\mathcal{T}_{F,S}])}\begin{pmatrix}\Pi \\ \Lambda\end{pmatrix} = P$. Furthermore, since $[\mathbf{C}\&\mathbf{D}]$ is bounded, $[\mathbf{C}\&\mathbf{D}]\begin{pmatrix}\Pi \\ \Lambda\end{pmatrix} = Q$. Putting things back together we obtain $\begin{pmatrix}\Pi \\ \Lambda\end{pmatrix} \in \mathcal{D}(\mathbf{K})$ and $\mathbf{K}\begin{pmatrix}\Pi \\ \Lambda\end{pmatrix} = \begin{pmatrix}P \\ Q\end{pmatrix}$, which shows that $\mathbf{K}$ is closed.

(ii) By a direct calculation it can be shown that for all $P \in \mathcal{L}(W, Z)$ and all $Q \in \mathcal{L}(W, H)$ the extended regulator equations (4.3) have a unique solution Π, Λ if and only if

$$[\mathcal{T}_{A,S} \& \mathbf{B}]\begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} = -P \tag{6.27}$$

$$[\mathbf{C} \& \mathbf{D}]\begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} = Q \tag{6.28}$$

for a unique $\begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} \in \mathcal{D}([\mathcal{T}_{A,S} \& \mathbf{B}]) \cap \ker([\& \mathcal{T}_{F,S}])$. Hence the extended regulator equations (4.3) have a unique solution for each $P \in \mathcal{L}(W, Z)$ and every $Q \in \mathcal{L}(W, H)$ if and only if $\mathbf{K}$ is a bijection $\mathcal{D}(\mathbf{K}) \to \mathcal{L}(W, Z) \times \mathcal{L}(W, H)$.

(iii) Let $A^p = A + \Delta_A, B^p = B + \Delta_B, C^p = C + \Delta_C, D^p = D + \Delta_D, J^p = J + \Delta_J$, where the perturbations are bounded and linear on suitable spaces. These perturbations induce a perturbation to $\mathbf{K}$. The resulting perturbed operator $\mathbf{K}^p$ is given by

$$\mathbf{K}^p \begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} = \begin{pmatrix} [\mathcal{T}_{A,S} \& \mathbf{B}]^p \begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} \\ [\mathbf{C} \& \mathbf{D}]^p \begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \mathcal{T}_{A+\Delta_A, S} \Pi + (B + \Delta_B)(J + \Delta_J)\Lambda \\ (C + \Delta_C)\Pi + (D + \Delta_D)(J + \Delta_J)\Lambda \end{pmatrix} \tag{6.29}$$

$$= \begin{pmatrix} \mathcal{T}_{A,S} \Pi + BJ\Lambda \\ C\Pi + DJ\Lambda \end{pmatrix} + \begin{pmatrix} \Delta_A \Pi + B\Delta_J \Lambda + \Delta_B(J + \Delta_J)\Lambda \\ \Delta_C \Pi + D\Delta_J \Lambda + \Delta_D(J + \Delta_J)\Lambda \end{pmatrix} \tag{6.30}$$

$$= \mathbf{K} \begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} + \begin{pmatrix} \Delta_A & B\Delta_J + \Delta_B(J + \Delta_J) \\ \Delta_C & D\Delta_J + \Delta_D(J + \Delta_J) \end{pmatrix} \begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} \tag{6.31}$$

for every $\begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} \in \mathcal{D}(\mathbf{K})$. Hence bounded linear additive perturbations to A, B, C, D and J result in a bounded linear additive perturbation $\Delta_{\mathbf{K}} = \begin{pmatrix} \Delta_A & B\Delta_J + \Delta_B(J+\Delta_J) \\ \Delta_C & D\Delta_J + \Delta_D(J+\Delta_J) \end{pmatrix}$ to $\mathbf{K}$, such that for the perturbed operator we have $\mathcal{D}(\mathbf{K}^p) = \mathcal{D}(\mathbf{K} + \Delta_{\mathbf{K}}) = \mathcal{D}(\mathbf{K})$. Moreover, by the independence of the perturbations $\Delta_{\mathbf{K}}$ does not depend on the perturbations to G.

(iv) From item (iii) we immediately see that as $\|\Delta_A\| \to 0, \|\Delta_B\| \to 0, \|\Delta_C\| \to 0, \|\Delta_D\| \to 0$ and $\|\Delta_J\| \to 0$, so must $\|\Delta_{\mathbf{K}}\| \to 0$. Since $\mathbf{K}$ with domain $\mathcal{D}(\mathbf{K})$ is a closed bijection between two Banach spaces, by the Open Mapping Theorem it must be boundedly invertible. By Theorem IV.1.16 in [57], if $\|\Delta_{\mathbf{K}}\|$ is (i.e. if $\|\Delta_A\|, \|\Delta_B\|, \|\Delta_C\|, \|\Delta_D\|$ and $\|\Delta_J\|$ are) sufficiently small, then the perturbed operator $\mathbf{K}^p$ is still boundedly invertible, that is, a closed bijection.

<div align="right">□</div>

The following corollary is particularly useful in the case that the exosystem is finite-dimensional.

Corollary 6.17. *Let the assumptions of Theorem 6.16 hold. If in addition the closed loop operator $\mathcal{A}$ generates an exponentially stable C_0-semigroup, then the controller (4.1) solves the EFRP for each $P \in \mathcal{L}(W, Z)$ and $Q \in \mathcal{L}(W, H)$ in such a way that output regulation is robust with respect to all small enough perturbations to A, B, C, D, G and J.*

Remark 6.18. A converse for Corollary 6.17 will be proved in Corollary 6.23. In particular, it turns out that in this case the unique solvability of the extended regulator equations (4.3) is actually necessary for robust output regulation.

We point out that Corollary 6.17 is new even for finite-dimensional systems, although its assumptions resemble those found in the finite-dimensional control literature. In particular, Corollary 6.17 generalizes the work of Francis [29] and Francis and Wonham [32] by allowing for perturbations also in C, D and Q. Moreover, it generalizes the results of Davison [24] by allowing for the parameters F, G and J of the controller to be free modulo closed loop stability and unique solvability of the extended regulator equations (4.3) (in Davison's work the matrix F contains a suitable reduplication of the maximal cyclic component of the matrix S, for example).

We emphasize that the results of this section are general perturbation results in the sense that we have not made any specific choice of the parameters F, G and J in the error feedback controller (4.1). Unfortunately, at this stage it is not at all clear *how* the operators F, G and J should be chosen in order to meet the assumptions of these results. Of course, we can always reformulate the extended regulator equations (4.3) as

$$\begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} S = \begin{pmatrix} A & BJ \\ GC & F+GDJ \end{pmatrix} \begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} + \begin{pmatrix} P \\ -GQ \end{pmatrix} \quad \text{in } \mathcal{D}(S) \tag{6.32a}$$

$$Q = \begin{pmatrix} C & DJ \end{pmatrix} \begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} \quad \text{in } W \tag{6.32b}$$

and then attempt to directly apply the results of Chapter 8 to obtain unique solutions; this is possible because the equations (6.32) are of the form (3.10). However, as we shall see in Section 6.2 below, a better approach is to observe that the equation (6.32b) is embedded in the equation (6.32a). Hence, all we really need is the solvability of (6.32a) and some additional condition which ensures that (6.32b) is also satisfied. This additional requirement turns out to be an internal model condition, as suggested by the finite-dimensional theory (see Section 6.2 below).

6.2   The internal model structure

In this section we shall study the necessary and sufficient structure of error feedback controllers
(4.1) which achieve conditionally robust output regulation in the sense of the EFRP. A crucial
observation that we make (see the proof of Theorem 6.20) is that *every* sufficiently stable closed
loop EFRP control system in a sense contains the error zeroing dynamics even without assuming
the solvability of the extended regulator equations (4.3). The main results of this section show
that the desired dynamical behaviour of conditionally robust output regulation is then realized if
(and sometimes only if) the dynamic controller (4.1) has the following structure:

Definition 6.19 (Internal model structure). A controller (4.1) has the internal model structure if
the following implication holds:

$$\forall \Lambda \in \mathcal{D}(\mathcal{T}_{F,S}),\ \forall \Delta \in \mathcal{L}(W,H):\ \Lambda S = F\Lambda + G\Delta \ \text{in}\ \mathcal{D}(S) \implies \Delta = 0 \qquad (6.33)$$

Here $\mathcal{D}(\mathcal{T}_{F,S})$ is as in Definition 6.7.

We shall see shortly that sufficient closed loop stability and the internal model structure together
imply the solvability of the extended regulator equations (4.3) — and thus also output regulation
by Theorem 4.4.

6.2.1   A generalization of the Internal Model Principle

Theorem 6.20 below generalizes the Internal Model Principle of Francis and Wonham [32] in such a
way that no purely finite-dimensional concepts are utilized. We have to assume exponential closed
loop stability — which may be unreasonable if $\dim(W) = \infty$ — in order to achieve the desired
equivalence, but we emphasize that this assumption will be weakened in Subsection 6.2.2 where
we shall present more sufficient conditions for conditionally robust output regulation. Moreover,
the following result is new even for finite-dimensional exosystems (2.1).

Theorem 6.20. *Let F, G and J in (4.1) be chosen such that the closed loop operator $\mathcal{A} = \left(\begin{smallmatrix} A & BJ \\ GC & F+GDJ \end{smallmatrix}\right)$ generates an exponentially stable C_0-semigroup on $Z \times X$. Then the controller
(4.1) solves the EFRP for every $P \in \mathcal{L}(W,Z)$ and every $Q \in \mathcal{L}(W,H)$ in the exosystem (2.2) in
such a way that*

1. *$\|e(t)\| \leq Me^{-\omega t}[\|z(0)\| + \|x(0)\| + \|w(0)\|]$ for all $t \geq 0$ and some $M, \omega > 0$ which do not
   depend on the initial conditions $z(0) \in Z$, $x(0) \in X$, and $w(0) \in W$,*

    2. output regulation is robust with respect all such perturbations to A, B, C, D, G and J which

       preserve the exponential closed loop stability,

if and only if the controller has the internal model structure.

Proof. (*Necessity.*) Assume that for every $P \in \mathcal{L}(W, Z)$ and every $Q \in \mathcal{L}(W, H)$ in the exosystem (2.2) we achieve robust output regulation with respect all such independent perturbations to the operators A, B, C, D, G and J which preserve exponential closed loop stability. Let $\Delta \in \mathcal{L}(W, H)$ and $\Lambda \in \mathcal{D}(\mathcal{T}_{F,S})$ be arbitrary operators such that $\Lambda S = F\Lambda + G\Delta$ in $\mathcal{D}(S)$. We must show that this implies $\Delta = 0$.

    Observe that by our assumptions, output regulation in the sense of the EFRP occurs for the particular operators $Q = \Delta - DJ\Lambda \in \mathcal{L}(W, H)$ and for $P = BJ\Lambda \in \mathcal{L}(W, Z)$ in the exosystem (2.2). Let $\Theta(t) = \left(\begin{smallmatrix} z(t) \\ x(t) \end{smallmatrix} \right) \in Z \times X$ and define the operators

$$\mathcal{A} = \begin{pmatrix} A & BJ \\ GC & F + GDJ \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} BJ\Lambda \\ -G(\Delta - DJ\Lambda) \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} C & DJ \end{pmatrix} \qquad (6.34)$$

with suitable domains of definition. Then we may write the closed loop system (4.2) (with $P = BJ\Lambda$ and $Q = \Delta - DJ\Lambda$) as

$$\begin{pmatrix} \dot{\Theta}(t) \\ \dot{w}(t) \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{P} \\ 0 & S \end{pmatrix} \begin{pmatrix} \Theta(t) \\ w(t) \end{pmatrix}, \quad \Theta(0) \in Z \times X, \quad w(0) \in W, \quad t \geq 0 \qquad (6.35a)$$

$$e(t) = \mathcal{C}\Theta(t) - (\Delta - DJ\Lambda)w(t), \quad t \geq 0 \qquad (6.35b)$$

Since $\mathcal{A}$ generates an exponentially stable C_0−semigroup and since S generates an isometric C_0−group, by the results in [88, 90] (see also Section A.2) there exists a unique operator $\left(\begin{smallmatrix} \Pi_0 \\ \Lambda_0 \end{smallmatrix} \right) \in \mathcal{L}(W, Z \times X)$ such that $\left(\begin{smallmatrix} \Pi_0 \\ \Lambda_0 \end{smallmatrix} \right) S = \mathcal{A} \left(\begin{smallmatrix} \Pi_0 \\ \Lambda_0 \end{smallmatrix} \right) + \mathcal{P}$ in $\mathcal{D}(S)$, i.e.

$$\begin{pmatrix} \Pi_0 \\ \Lambda_0 \end{pmatrix} S = \begin{pmatrix} A & BJ \\ GC & F + GDJ \end{pmatrix} \begin{pmatrix} \Pi_0 \\ \Lambda_0 \end{pmatrix} + \begin{pmatrix} BJ\Lambda \\ -G\Delta + GDJ\Lambda \end{pmatrix} \quad \text{in } \mathcal{D}(S) \qquad (6.36)$$

Let $w(0) \in W$ be arbitrary and let $\Theta(0) = \left(\begin{smallmatrix} z(0) \\ x(0) \end{smallmatrix} \right) = \left(\begin{smallmatrix} \Pi_0 \\ \Lambda_0 \end{smallmatrix} \right) w(0)$. Using Lemma 3.5 it is then straight forward to show that for every every $t \geq 0$

$$\int_0^t T_\mathcal{A}(t - \tau)\mathcal{P}T_S(\tau)w(0)d\tau = \left(\begin{smallmatrix} \Pi_0 \\ \Lambda_0 \end{smallmatrix} \right) T_S(t)w(0) - T_\mathcal{A}(t) \left(\begin{smallmatrix} \Pi_0 \\ \Lambda_0 \end{smallmatrix} \right) w(0) \qquad (6.37)$$

and that the corresponding tracking error $e(t)$ is as follows:

$$e(t) = \mathcal{C}\Theta(t) - (\Delta - DJ\Lambda)w(t) \tag{6.38}$$

$$= \mathcal{C}T_{\mathcal{A}}(t)\Theta(0) + \mathcal{C}\int_0^t T_{\mathcal{A}}(t-\tau)\mathcal{P}T_S(\tau)w(0)d\tau - (\Delta - DJ\Lambda)T_S(t)w(0) \tag{6.39}$$

$$= \mathcal{C}T_{\mathcal{A}}(t)\left[\Theta(0) - \left(\begin{smallmatrix}\Pi_0\\\Lambda_0\end{smallmatrix}\right)w(0)\right] + \left[\mathcal{C}\left(\begin{smallmatrix}\Pi_0\\\Lambda_0\end{smallmatrix}\right) - (\Delta - DJ\Lambda)\right]T_S(t)w(0) \tag{6.40}$$

$$= (C\Pi_0 + DJ\Lambda_0 - \Delta + DJ\Lambda)T_S(t)w(0) \tag{6.41}$$

Precisely as in the proof of Theorem 3.20 we deduce using the exponential decay of $\|e(t)\|$ that necessarily $C\Pi_0 + DJ\Lambda_0 - \Delta + DJ\Lambda = 0$ in W. However, clearly $\Pi_0 = 0$ and $\Lambda_0 = -\Lambda$ satisfy the equation (6.36). By uniqueness we must then have that $\Delta = C0 + DJ\Lambda - DJ\Lambda = 0$. This shows that the controller necessarily has the internal model structure.

(*Sufficiency.*) Let $P \in \mathcal{L}(W, Z)$ and $Q \in \mathcal{L}(W, H)$ be arbitrary and set $\mathcal{P} = \left(\begin{smallmatrix}P\\-GQ\end{smallmatrix}\right)$. Let $A^p = A + \Delta_A, B^p = B + \Delta_B, C^p = C + \Delta_C, D^p = D + \Delta_D, J^p = J + \Delta_J$, where the independent perturbations are bounded and linear on suitable spaces. Assume that the exponential stability of the closed loop system is not affected by these perturbations. Let the perturbed closed loop operator be denoted by $\mathcal{A}^p$. Since $\mathcal{A}^p$ generates an exponentially stable C_0−semigroup and since S generates an isometric C_0−group, by the results in [88, 90] (see also Section A.2) there exists a unique operator $\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right) \in \mathcal{L}(W, Z \times X)$ such that $\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right)S = \mathcal{A}^p\left(\begin{smallmatrix}\Pi\\\Lambda\end{smallmatrix}\right) + \mathcal{P}$ in $\mathcal{D}(S)$, i.e.

$$\begin{pmatrix}\Pi\\\Lambda\end{pmatrix}S = \begin{pmatrix}A^p & B^pJ^p\\GC^p & F + GD^pJ^p\end{pmatrix}\begin{pmatrix}\Pi\\\Lambda\end{pmatrix} + \begin{pmatrix}P\\-GQ\end{pmatrix} \quad \text{in } \mathcal{D}(S) \tag{6.42}$$

But this shows that $\Lambda S = F\Lambda + G(C^p\Pi + D^pJ^p\Lambda - Q)$ in $\mathcal{D}(S)$, and since the controller has the internal model structure, ultimately $C^p\Pi + D^pJ^p\Lambda = Q$ in W and the extended regulator equations (4.3) for the perturbed operators are satisfied. The result now follows by Theorem 4.10. We remark that since the extended regulator equations (4.3) do not depend on G, this operator may be subject to arbitrary independent perturbations as long as closed loop exponential stability holds. $\qquad\square$

Remark 6.21. Taking a glance at the proof of Theorem 4 in [32][7] immediately reveals that if $\dim(W) < \infty$, if $\dim(Z) < \infty$ and if $\dim(H) < \infty$, then every dynamic controller (4.1) solving the

---

[7]This is the classical finite-dimensional result showing that a degree of robustness in output regulation can be obtained by incorporating a suitably reduplicated (and suitably controllable and observable) copy of the maximal cyclic component of the exosystem matrix S in F.

EFRP by utilizing the classical Internal Model Principle (i.e. Theorem 4 in [32]) has the internal model structure. However, it should be pointed out that in this case Theorem 6.20 above also guarantees robustness with respect to C, D and Q which is not the case in [32]. On the other hand, Francis and Wonham [32] also consider the case in which the tracking error $e(t)$ is only *readable* from some other signal; we shall not do that here.

Remark 6.22. In Theorem 6.20 the operators S and F cannot be perturbed if robust output regulation is to be maintained. This is in accordance with the the finite-dimensional robust output regulation theory where it is well-known that the system matrix S of the exosystem and the suitably reduplicated copy of its maximal cyclic component in the matrix F cannot be subject to perturbations if robust output regulation is to be maintained [24, 29, 32]. Although some parts of F may in practice tolerate perturbations, by convention (and for the sake of simplicity) nowhere in this chapter do we allow for perturbations to the operators S and F.

Recall from Section 6.1 that the unique solvability of the extended regulator equations (4.3) is a precursor for conditionally robust output regulation. Now Theorem 6.20 immediately yields the following result which illustrates a degree of *necessity* of the unique solvability of the extended regulator equations (4.3) for robust output regulation. In particular, Corollary 6.23 below is a partial converse for Corollary 6.17.

Corollary 6.23. *Let F, G and J in (4.1) be chosen such that the closed loop operator $\mathcal{A}$ generates an exponentially stable C_0-semigroup on $Z \times X$ and such that the controller (4.1) solves the EFRP for every $P \in \mathcal{L}(W, Z)$ and every $Q \in \mathcal{L}(W, H)$ robustly as described in items 1 and 2 of Theorem 6.20. Then the extended regulator equations (4.3) have a unique solution $\Pi \in \mathcal{L}(W, Z)$, $\Lambda \in \mathcal{L}(W, X)$ for each $P \in \mathcal{L}(W, Z)$ and every $Q \in \mathcal{L}(W, H)$.*

Proof. Under the assumptions the controller (4.1) must have the internal model structure, according to Theorem 6.20. The sufficiency part of the proof of Theorem 6.20 then readily shows that the extended regulator equations (4.3) have a solution $\Pi \in \mathcal{L}(W, Z), \Lambda \in \mathcal{L}(W, X)$. On the other hand, *any* such solution operators $\widehat{\Pi}, \widehat{\Lambda}$ also satisfy the equation

$$\begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} S = \begin{pmatrix} A & BJ \\ GC & F + GDJ \end{pmatrix} \begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} + \begin{pmatrix} P \\ -GQ \end{pmatrix} \quad \text{in } \mathcal{D}(S) \tag{6.43}$$

But the solution of this equation is unique because the closed loop system is exponentially stable and S generates an isometric C_0-group (see Section A.2). $\qquad \square$

Remark 6.24. Let $D = 0$. If $\dim(W) < \infty$, if $\dim(Z) < \infty$ and if $\dim(H) < \infty$ then under the assumptions of Corollary 6.23 for every $P \in \mathcal{L}(W, Z)$ there exist operators $\Pi \in \mathcal{L}(W, Z)$ and $\Gamma \in \mathcal{L}(W, H)$ such that

$$P = A\Pi - \Pi S + B\Gamma \tag{6.44a}$$

$$0 = C\Pi \tag{6.44b}$$

This condition has been shown to be necessary for robust output regulation with respect to P in Theorem 2a of [29]. Francis [29] has also demonstrated that this condition is equivalent to

$$\mathrm{ran} \begin{pmatrix} A - \lambda I & B \\ C & 0 \end{pmatrix} = Z \times H, \quad \forall \lambda \in \sigma(S) \tag{6.45}$$

which is the one arising in the work of Davison and Goldenberg [19] and Wonham [93].

6.2.2 Conditional robustness results based on the internal model structure

We shall next present some more sufficient conditions for conditionally robust output regulation. In contrast to the previous subsection, here we shall only require that the closed loop system operator $\mathcal{A}$ generates a strongly stable C_0−semigroup $T_{\mathcal{A}}(t)$ on $Z \times X$. Our results again utilize the concept of internal model structure, but we shall also need an assumption about the unique solvability of a Sylvester type operator equation. This assumption is always satisfied for an exponentially stable $T_{\mathcal{A}}(t)$, but need not be so for a strongly stable $T_{\mathcal{A}}(t)$.

Theorem 6.25. *Let F, G and J in the controller (4.1) be chosen such that*

1. *the closed loop system operator $\mathcal{A} = \begin{pmatrix} A & BJ \\ GC & F+GDJ \end{pmatrix}$ generates a strongly stable C_0-semigroup on $Z \times X$,*

2. *the controller (4.1) has the internal model structure,*

3. *for every $\mathcal{P} \in \mathcal{L}(W, Z \times X)$ there exists a unique $Y \in \mathcal{L}(W, Z \times X)$ such that $Y(\mathcal{D}(S)) \subset \mathcal{D}(\mathcal{A})$ and $YS = \mathcal{A}Y + \mathcal{P}$ in $\mathcal{D}(S)$.*

Then the controller (4.1) solves the EFRP for all $P \in \mathcal{L}(W, Z)$ and all $Q \in \mathcal{L}(W, H)$ in such a way that output regulation is conditionally robust with respect to small enough perturbations to A, B, C, D, G and J.

Proof. Let $P \in \mathcal{L}(W, Z)$ and $Q \in \mathcal{L}(W, H)$ be arbitrary and set $\mathcal{P} = \left(\begin{smallmatrix} P \\ -GQ \end{smallmatrix} \right)$. Let $A^p = A + \Delta_A, B^p = B + \Delta_B, C^p = C + \Delta_C, D^p = D + \Delta_D, J^p = J + \Delta_J$, where the perturbations are bounded and linear on suitable spaces. For conditional robustness, we assume that the perturbed closed loop operator $\mathcal{A}^p$ is still the generator of a strongly stable C_0-semigroup on $Z \times X$. By our assumptions the linear Sylvester operator $\mathcal{T}_{\mathcal{A},S}$ defined by

$$\mathcal{D}(\mathcal{T}_{\mathcal{A},S}) = \{ Y \in \mathcal{L}(W, Z \times X) \mid Y(\mathcal{D}(S)) \subset \mathcal{D}(\mathcal{A}), \exists \mathcal{P} \in \mathcal{L}(W, Z \times X) : $$

$$\mathcal{P}w = \mathcal{A}Yw - YSw \, \forall w \in \mathcal{D}(S) \} \tag{6.46a}$$

$$\mathcal{T}_{\mathcal{A},S}Y = \mathcal{P} \tag{6.46b}$$

is a boundedly invertible closed operator $\mathcal{D}(\mathcal{T}_{\mathcal{A},S}) \subset \mathcal{L}(W, Z \times X) \to \mathcal{L}(W, Z \times X)$ [3]; consequently so is $\mathcal{T}_{\mathcal{A}^p,S}$ as long as the above perturbations to A, B, C, D and J are small enough ([57] p. 196). Hence there exists a unique operator $\left(\begin{smallmatrix} \Pi \\ \Lambda \end{smallmatrix} \right) \in \mathcal{L}(W, Z \times X)$ such that $\left(\begin{smallmatrix} \Pi \\ \Lambda \end{smallmatrix} \right)S = \mathcal{A}^p \left(\begin{smallmatrix} \Pi_1 \\ \Pi_2 \end{smallmatrix} \right) + \mathcal{P}$ in $\mathcal{D}(S)$. As in the proof of the sufficiency part of Theorem 6.20 this and the internal model structure show that the extended regulator equations (4.3) for the perturbed parameters have a solution. Observe that G can then also be perturbed because these equations do not depend on it. The result follows by Theorem 4.4. $\qquad \square$

We emphasize that in practice the unique solvability of the Sylvester type operator equation $YS = \mathcal{A}Y + \mathcal{P}$ in $\mathcal{D}(S)$ for all $\mathcal{P} \in \mathcal{L}(W, Z \times X)$, which is required in Theorem 6.25, can often be established without any knowledge of the actual solution operator Y (see Section A.2 and [3, 8, 88, 90]). One such case is that in which $\mathcal{A}$ also generates an exponentially stable C_0-semigroup; another such case is presented below.

Corollary 6.26. *Let $S \in \mathcal{L}(W)$. Let F, G and J in the controller (4.1) be chosen such that*

    1. the closed loop operator $\mathcal{A}$ generates a strongly stable C_0-semigroup on $Z \times X$,

    2. the controller (4.1) has the internal model structure,

    3. $\sigma(\mathcal{A}) \cap \sigma(S) = \emptyset$.

Then the controller (4.1) solves the EFRP for all $P \in \mathcal{L}(W, Z)$ and all $Q \in \mathcal{L}(W, H)$ in such a way that output regulation is conditionally robust with respect to those perturbations to A, B, C, D, G and J for which the corresponding perturbation $\Delta_{\mathcal{A}}$ to the closed loop operator $\mathcal{A}$ satisfies $\|\Delta_{\mathcal{A}}\| < \min_{\lambda \in \sigma(S)} \|R(\lambda, \mathcal{A})\|^{-1}$.

Proof. By [90] (see also Section A.2) for every $\mathcal{P} \in \mathcal{L}(W, Z \times X)$ there exists a unique $Y \in$ $\mathcal{L}(W, Z \times X)$ such that $Y(\mathcal{D}(S)) \subset \mathcal{D}(\mathcal{A})$ and $YS = \mathcal{A}Y + \mathcal{P}$ in $\mathcal{D}(S)$. The result follows by Theorem 6.25 and Remark IV.3.2 in [57] which guarantees that the assumption in item 3 is not altered by the perturbations (observe that $\sigma(S) \subset i\mathbb{R} \cap \rho(\mathcal{A})$ is a compact set). We point out that by the independence of the perturbations, G can be treated separately of the other operators, as in Theorem 6.25. $\qquad \square$

Remark 6.27. It is not assumed in Corollary 6.26 that $\dim(W) < \infty$.

If the spectra of the closed loop operator $\mathcal{A}$ and the exosystem operator S are not disjoint, i.e. $\sigma(\mathcal{A}) \cap \sigma(S) \neq \emptyset$, then the Sylvester operator equation $YS = \mathcal{A}Y + \mathcal{P}$ in $\mathcal{D}(S)$ does *not* have a unique solution for every $\mathcal{P} \in \mathcal{L}(W, Z \times X)$ [90]. On the other hand, even if it is true that $\sigma(\mathcal{A}) \cap \sigma(S) = \emptyset$ then this Sylvester equation does not necessarily have a unique solution unless $S \in \mathcal{L}(W)$ [90]. Hence Theorem 6.25 and its Corollary 6.26 are not always applicable in practice (they *are* applicable and *new* for finite-dimensional exosystems (2.1)). However, it is still possible that the above operator equation *does* have a unique solution for every $\mathcal{P} \in \mathcal{L}(W, Z \times X)$ which is also in some *smaller* subspace $\mathcal{L}(W_0, Z \times X)$ of $\mathcal{L}(W, Z \times X)$. This situation occurs e.g. in certain repetitive control applications (see Section 6.7). We next show how Theorem 6.25 can be extended to this case. We need some elementary preliminary lemmata.

Lemma 6.28. *Let W_0 be a Banach space such that $W \hookrightarrow W_0$. Let U be a Banach space. Then $\mathcal{L}(W_0, U) \hookrightarrow \mathcal{L}(W, U)$.*

Proof. Let $Y \in \mathcal{L}(W_0, U)$ and let $w \in W \hookrightarrow W_0$. Then

$$\|Yw\|_U \leq \|Y\|_{\mathcal{L}(W_0, U)} \|w\|_{W_0} \leq c\|Y\|_{\mathcal{L}(W_0, U)} \|w\|_W \tag{6.47}$$

for some constant $c > 0$. Hence $Y \in \mathcal{L}(W, U)$. Moreover, we have

$$\|Y\|_{\mathcal{L}(W, U)} = \sup_{w \in W, \|w\|_W \leq 1} \|Yw\|_U \leq \sup_{w \in W, \|w\|_W \leq 1} c\|Y\|_{\mathcal{L}(W_0, U)} \|w\|_W \tag{6.48}$$

$$= c\|Y\|_{\mathcal{L}(W_0, U)} \tag{6.49}$$

so that the identity operator $\mathcal{L}(W_0, U) \to \mathcal{L}(W, U)$ is indeed continuous. $\qquad \square$

Now assume that W_0 is some fixed Banach space such that $W \hookrightarrow W_0$. Then the Sylvester operator $\mathcal{T}_{\mathcal{A},S} : \mathcal{D}(\mathcal{T}_{\mathcal{A},S}) \subset \mathcal{L}(W, Z \times X) \to \mathcal{L}(W, Z \times X)$ defined in (6.46) can be restricted to an

operator $\mathcal{T}_{\mathcal{A},S}|$ whose image is in $\mathcal{L}(W_0, Z \times X)$ as follows:

$$\mathcal{D}(\mathcal{T}_{\mathcal{A},S}|) = \{ Y \in \mathcal{D}(\mathcal{T}_{\mathcal{A},S}) \mid \mathcal{T}_{\mathcal{A},S}Y \in \mathcal{L}(W_0, Z \times X) \} \tag{6.50a}$$

$$\mathcal{T}_{\mathcal{A},S}|Y = \mathcal{T}_{\mathcal{A},S}Y, \quad \forall Y \in \mathcal{D}(\mathcal{T}_{\mathcal{A},S}|) \tag{6.50b}$$

Lemma 6.29. *The above operator $\mathcal{T}_{\mathcal{A},S}|$ is a closed operator $\mathcal{W} \to \mathcal{L}(W_0, Z \times X)$, where $\mathcal{W} = \overline{\mathcal{D}(\mathcal{T}_{\mathcal{A},S}|)}^{\|\cdot\|_{\mathcal{L}(W,Z \times X)}}$.*

Proof. Let $(Y_n)_{n \in \mathbb{N}} \subset \mathcal{D}(\mathcal{T}_{\mathcal{A},S}|) \subset \mathcal{D}(\mathcal{T}_{\mathcal{A},S})$ be such that $\lim_{n \to \infty} Y_n = Y$ (convergence in $\mathcal{L}(W, Z \times X)$) and $\lim_{n \to \infty} \mathcal{T}_{\mathcal{A},S}|Y_n = P$ (convergence in $\mathcal{L}(W_0, Z \times X)$). Then $Y \in \mathcal{L}(W, Z \times X)$ and $P \in \mathcal{L}(W_0, Z \times X)$ because these spaces are Banach spaces. Moreover,

$$\lim_{n \to \infty} \|\mathcal{T}_{\mathcal{A},S}Y_n - P\|_{\mathcal{L}(W,Z \times X)} \le c \lim_{n \to \infty} \|\mathcal{T}_{\mathcal{A},S}|Y_n - P\|_{\mathcal{L}(W_0,Z \times X)} = 0 \tag{6.51}$$

for some constant $c > 0$ because $\mathcal{L}(W_0, Z \times X) \hookrightarrow \mathcal{L}(W, Z \times X)$, according to Lemma 6.28. Hence by the closedness of $\mathcal{T}_{\mathcal{A},S}$ in $\mathcal{L}(W, Z \times X)$ (cf. [3]) we have $Y \in \mathcal{D}(\mathcal{T}_{\mathcal{A},S})$ and $\mathcal{T}_{\mathcal{A},S}Y = P$. But $P \in \mathcal{L}(W_0, Z \times X)$, so that by definition $Y \in \mathcal{D}(\mathcal{T}_{\mathcal{A},S}|)$ and $\mathcal{T}_{\mathcal{A},S}|Y = P$, i.e. $\mathcal{T}_{\mathcal{A},S}|$ is a closed operator $\mathcal{W} \to \mathcal{L}(W_0, Z \times X)$. $\qquad\square$

By the Open Mapping Theorem and by above it is clear that $\mathcal{T}_{\mathcal{A},S}|$ is a boundedly invertible operator $\mathcal{D}(\mathcal{T}_{\mathcal{A},S}|) \to \mathcal{L}(W_0, Z \times X)$ if and only if for all $\mathcal{P} \in \mathcal{L}(W_0, Z \times X)$ the operator equation $YS = \mathcal{A}Y + \mathcal{P}$ in $\mathcal{D}(S)$ has a unique solution $Y \in \mathcal{L}(W, Z \times X)$, with $Y(\mathcal{D}(S)) \subset \mathcal{D}(\mathcal{A})$. This motivates us to define $(W, W_0)-$*admissible perturbations* to the closed loop system as such operators $\Delta_{\mathcal{A}} \in \mathcal{L}(Z \times X)$ which do not destroy the unique solvability of this operator equation:

Definition 6.30. Let W and W_0 be as in the above. Let $0 \in \rho(\mathcal{T}_{\mathcal{A},S}|)$. We say that an operator $\Delta_{\mathcal{A}} \in \mathcal{L}(Z \times X)$ is $(W, W_0)-$admissible if $0 \in \rho(\mathcal{T}_{\mathcal{A}+\Delta_{\mathcal{A}},S}|)$. Here the perturbed operator $\mathcal{T}_{\mathcal{A}+\Delta_{\mathcal{A}},S}| : \mathcal{D}(\mathcal{T}_{\mathcal{A}+\Delta_{\mathcal{A}},S}|) \subset \mathcal{L}(W, Z \times X) \to \mathcal{L}(W_0, Z \times X)$ is defined by

$$\mathcal{D}(\mathcal{T}_{\mathcal{A}+\Delta_{\mathcal{A}},S}|) = \{ Y \in \mathcal{D}(\mathcal{T}_{\mathcal{A}+\Delta_{\mathcal{A}},S}) \mid \mathcal{T}_{\mathcal{A}+\Delta_{\mathcal{A}},S}Y \in \mathcal{L}(W_0, Z \times X) \} \tag{6.52a}$$

$$\mathcal{T}_{\mathcal{A}+\Delta_{\mathcal{A}},S}|Y = \mathcal{T}_{\mathcal{A}+\Delta_{\mathcal{A}},S}Y, \quad \forall Y \in \mathcal{D}(\mathcal{T}_{\mathcal{A}+\Delta_{\mathcal{A}},S}|) \tag{6.52b}$$

and the perturbed Sylvester operator $\mathcal{T}_{\mathcal{A}+\Delta_{\mathcal{A}},S}$ is defined in the obvious analogy with (6.46).

Remark 6.31. If $W = W_0$ then every sufficiently small operator $\Delta_{\mathcal{A}} \in \mathcal{L}(Z \times X)$ is (W, W_0)-admissible by [57] p. 196 and by the fact that $\mathcal{D}(\mathcal{T}_{\mathcal{A}+\Delta_{\mathcal{A}},S}|) = \mathcal{D}(\mathcal{T}_{\mathcal{A}+\Delta_{\mathcal{A}},S}) = \mathcal{D}(\mathcal{T}_{\mathcal{A},S}) = \mathcal{D}(\mathcal{T}_{\mathcal{A},S}|)$

and $\mathcal{T}_{\mathcal{A}+\Delta_{\mathcal{A}},S}| = \mathcal{T}_{\mathcal{A}+\Delta_{\mathcal{A}},S} = \mathcal{T}_{\mathcal{A},S} + \Delta^{\mathcal{T}}$ where $\Delta^{\mathcal{T}} Y = \Delta_{\mathcal{A}} Y$ for all $Y \in \mathcal{L}(W, Z \times X)$. However, in the general case that $W \hookrightarrow W_0$ we shall need some additional structure for the perturbations $\Delta_{\mathcal{A}}$ to ensure that $0 \in \rho(\mathcal{T}_{\mathcal{A}+\Delta_{\mathcal{A}},S}|)$.

The following result generalizes Theorem 6.25 for $(W, W_0)-$admissible perturbations.

Theorem 6.32. *Let W and W_0 be as in the above. Let F, G and J in the controller (4.1) be chosen such that*

1. *the closed loop operator $\mathcal{A}$ generates a strongly stable C_0-semigroup on $Z \times X$,*

2. *the controller (4.1) has the internal model structure,*

3. *for every $\mathcal{P} \in \mathcal{L}(W_0, Z \times X)$ there exists a unique $Y \in \mathcal{L}(W, Z \times X)$ such that $Y(\mathcal{D}(S)) \subset \mathcal{D}(\mathcal{A})$ and $YS = \mathcal{A}Y + \mathcal{P}$ in $\mathcal{D}(S)$.*

Then the controller (4.1) solves the EFRP for all $P \in \mathcal{L}(W_0, Z)$ and all $Q \in \mathcal{L}(W_0, H)$ in the following way: Output regulation is conditionally robust with respect to such perturbations to A, B, C, D, G and J for which the corresponding perturbation $\Delta_{\mathcal{A}}$ to the closed loop operator $\mathcal{A}$ is $(W, W_0)-$admissible.

Proof. Let $P \in \mathcal{L}(W_0, Z) \hookrightarrow \mathcal{L}(W, Z)$ and $Q \in \mathcal{L}(W_0, H) \hookrightarrow \mathcal{L}(W, H)$ be arbitrary and set $\mathcal{P} = \begin{pmatrix} P \\ -GQ \end{pmatrix} \in \mathcal{L}(W_0, Z \times X)$. Let $A^p = A + \Delta_A, B^p = B + \Delta_B, C^p = C + \Delta_C, D^p = D + \Delta_D, J^p = J + \Delta_J$, where the perturbations are bounded and linear on suitable spaces. We assume for conditional robustness that the corresponding perturbed closed loop operator $\mathcal{A}^p = \mathcal{A} + \Delta_{\mathcal{A}}$ is still the generator of a strongly stable C_0-semigroup on $Z \times X$. By our assumptions there exists a unique operator $\begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} \in \mathcal{D}(\mathcal{T}_{\mathcal{A}+\Delta_{\mathcal{A}},S}|) \subset \mathcal{L}(W, Z \times X)$ such that $\begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} S = \mathcal{A}^p \begin{pmatrix} \Pi \\ \Lambda \end{pmatrix} + \mathcal{P}$ in $\mathcal{D}(S)$. As in the proof of the sufficiency part of Theorem 6.20 this and the internal model structure show that the extended regulator equations (4.3) for the perturbed parameters have a solution. The result now follows, as G can also be perturbed; the extended regulator equations (4.3) do not depend on G. $\qquad\square$

The following corollary demonstrates the usefulness of the concept of $(W, W_0)-$admissible perturbations in such output regulation problems where the exosystem (2.2) is infinite-dimensional.

Corollary 6.33. *Let E be a Banach space and for a given sequence $(\omega_n)_{n \in I} \subset \mathbb{R}$ of distinct frequencies consider the function spaces $W = \mathcal{E} = H_{AP}(E, f_n, \omega_n)$, $W_0 = H_{AP}(E, g_n, \omega_n)$, with*

$f_n \geq g_n$ for all $n \in I$. Let $S = S|_{\mathcal{E}}$ generate the left translation C_0-group on $\mathcal{E}$. Let the sequences $(f_n)_{n \in I}$ and $(g_n)_{n \in I}$, and the operators F, G and J in (4.1) be such that

1. the closed loop operator $\mathcal{A}$ generates a strongly stable C_0-semigroup on $Z \times X$,

2. the controller (4.1) has the internal model structure,

3. $\sigma(\mathcal{A}) \cap \{\, i\omega_n \mid n \in I \,\} = \emptyset$ and $\sum_{n \in I} \frac{\|R(i\omega_n, \mathcal{A})\|^2 g_n^2}{f_n^2} < \infty$.

Then the controller (4.1) solves the EFRP for all $P \in \mathcal{L}(W_0, Z)$ and all $Q \in \mathcal{L}(W_0, H)$ in the following way: Output regulation is conditionally robust with respect to such perturbations to A, B, C, D, G and J for which the corresponding perturbation $\Delta_{\mathcal{A}}$ to the closed loop operator $\mathcal{A}$ satisfies $\sup_{n \in I} \|\Delta_{\mathcal{A}} R(i\omega_n, \mathcal{A})\| < 1$.

Proof. By Proposition 2.16 it is evident that $W \hookrightarrow W_0$. For every $n \in I$ define the bounded linear operator $P_n : \mathcal{E} \to \mathcal{E}$ such that $P_n h = \widehat{h}(n) e^{i\omega_n \cdot}$ for all $h = \sum_{n \in I} \widehat{h}(n) e^{i\omega_n \cdot} \in \mathcal{E}$ (the operators P_n can be explicitly defined using e.g. the Fourier-Bohr transform [63]; observe that the coefficients $\widehat{h}(n)$ are unique by construction). Let $\mathcal{P} \in \mathcal{L}(W_0, Z \times X)$ be arbitrary. Then the linear operator $Y : \mathcal{E} \to Z \times X$ defined by $Yh = \sum_{n \in I} R(i\omega_n, \mathcal{A}) \mathcal{P} P_n h$ for all $h \in \mathcal{E}$ is bounded:

$$\|Yh\|_{Z \times X} \leq \sum_{n \in I} \|R(i\omega_n, \mathcal{A}) \mathcal{P} P_n h\|_{Z \times X} \tag{6.53}$$

$$\leq \sum_{n \in I} \|R(i\omega_n, \mathcal{A})\|_{\mathcal{L}(Z \times X)} \|\mathcal{P} \widehat{h}(n) e^{i\omega_n \cdot}\|_{Z \times X} \tag{6.54}$$

$$\leq \sum_{n \in I} \|R(i\omega_n, \mathcal{A})\|_{\mathcal{L}(Z \times X)} \|\mathcal{P}\|_{\mathcal{L}(W_0, Z \times X)} \|\widehat{h}(n)\|_E g_n \tag{6.55}$$

$$= \|\mathcal{P}\|_{\mathcal{L}(W_0, Z \times X)} \sum_{n \in I} \|R(i\omega_n, \mathcal{A})\|_{\mathcal{L}(Z \times X)} \frac{g_n}{f_n} \|\widehat{h}(n)\|_E f_n \tag{6.56}$$

$$\leq \|\mathcal{P}\|_{\mathcal{L}(W_0, Z \times X)} \sqrt{\sum_{n \in I} \frac{\|R(i\omega_n, \mathcal{A})\|^2_{\mathcal{L}(Z \times X)} g_n^2}{f_n^2}} \sqrt{\sum_{n \in I} \|\widehat{h}(n)\|^2_E f_n^2} \tag{6.57}$$

$$\leq \|\mathcal{P}\|_{\mathcal{L}(W_0, Z \times X)} \sqrt{\sum_{n \in I} \frac{\|R(i\omega_n, \mathcal{A})\|^2_{\mathcal{L}(Z \times X)} g_n^2}{f_n^2}} \|h\|_{\mathcal{E}} \tag{6.58}$$

by the Schwartz inequality, because clearly $P_n h = \widehat{h}(n) e^{i\omega_n \cdot} \in W_0$ for all $n \in I$.

Since $S|_{\mathcal{E}}$ is just the differential operator, it is clear that $Y \in \mathcal{L}(\mathcal{E}, Z \times X)$ is the unique operator satisfying $Y S|_{\mathcal{E}} h = \mathcal{A} Y h + \mathcal{P} h$ for each $h \in \text{span}\{\, a_n e^{i\omega_n \cdot} \mid a_n \in E, n \in I_k \,\}$, where the index set I_k is finite. Using the boundedness of Y, the closedness of $\mathcal{A}$ and the fact that

$P_n S|_{\mathcal{E}} h = i\omega_n P_n h = S|_{\mathcal{E}} P_n h$ for all $h \in \mathcal{D}(S|_{\mathcal{E}})$ it is not difficult to show that $Y(\mathcal{D}(S|_{\mathcal{E}})) \subset \mathcal{D}(\mathcal{A})$ and that Y is the unique solution of the operator equation $YS|_{\mathcal{E}} = \mathcal{A}Y + \mathcal{P}$ in $\mathcal{D}(S|_{\mathcal{E}})$ (see also Theorem 8.11). Consequently $0 \in \rho(\mathcal{T}_{\mathcal{A},S}|)$ in the notation (6.50).

In order to be able to apply Theorem 6.32 it remains to show that the perturbation $\Delta_{\mathcal{A}}$ is $(W, W_0)-$admissible under the assumption $\sup_{n \in I} \|\Delta_{\mathcal{A}} R(i\omega_n, \mathcal{A})\| < 1$. Since for all $n \in I$ we have $i\omega_n I - \mathcal{A} - \Delta_{\mathcal{A}} = [I - \Delta_{\mathcal{A}} R(i\omega_n, \mathcal{A})](i\omega_n I - \mathcal{A})$, under our assumption $i\omega_n \in \rho(\mathcal{A} + \Delta_{\mathcal{A}})$ and we can expand $R(i\omega_n, \mathcal{A} + \Delta_{\mathcal{A}})$ using a Neumann series as

$$R(i\omega_n, \mathcal{A} + \Delta_{\mathcal{A}}) = R(i\omega_n, \mathcal{A}) \sum_{k=0}^{\infty} \left[\Delta_{\mathcal{A}} R(i\omega_n, \mathcal{A})\right]^k \tag{6.59}$$

Hence there exists $m > 0$ which is independent of $n \in I$ such that $\|R(i\omega_n, \mathcal{A} + \Delta_{\mathcal{A}})\| \leq m\|R(i\omega_n, \mathcal{A})\|$ for all $n \in I$. A reasoning similar to the one given above then immediately shows that the operator equation $YS|_{\mathcal{E}} = [\mathcal{A} + \Delta_{\mathcal{A}}]Y + \mathcal{P}$ in $\mathcal{D}(S|_{\mathcal{E}})$ also has a unique solution $Y \in \mathcal{L}(\mathcal{E}, Z \times X)$ for all $\mathcal{P} \in \mathcal{L}(W_0, Z \times X)$. Thus the perturbation $\Delta_{\mathcal{A}}$ is $(W, W_0)-$admissible and the desired result follows from Theorem 6.32. $\qquad\square$

Remark 6.34. In Corollary 6.33 $\|R(i\omega_n, \mathcal{A})\|$ need not be uniformly bounded in n unless $\mathcal{A}$ generates an exponentially stable C_0-semigroup or the index set I is finite. Thus, in general it is possible that $\sigma(S)$ and $\sigma(\mathcal{A})$ *overlap* at $\pm i\infty$. To the author's knowledge this has not been possible in any earlier robustness result in the existing output regulation theory.

It is probably appropriate to conclude this section with a rough-and-ready summary and practical interpretation of the above (conditional) robustness results. So, assuming that a given error feedback controller (4.1) has the internal model structure and that it achieves strong closed loop stability, we have the following:

- Output regulation is exponentially fast and robust with respect to certain stability-preserving perturbations if the closed loop system is also exponentially stable (cf. Theorem 6.20).

- Output regulation is conditionally robust with respect to certain small perturbations, if the exogenous signals do not contain arbitrarily high frequencies and if the (complex) spectrum of these signals does not mix with that of the closed loop system operator (cf. Corollary 6.26).

- If the exogenous signals do contain arbitrarily high frequencies, but none of these (complex) frequencies $i\omega_n$ is part of the spectrum of the closed loop operator $\mathcal{A}$, if P, Q and the

exogenous signals are sufficiently continuous, and if the order of growth of $\|R(i\omega_n, \mathcal{A})\|$ as $|\omega_n| \to \infty$ is in a sense controlled, then output regulation is conditionally robust with respect to certain perturbations which do not increase this order of growth (cf. Corollary 6.33).

In the above, of course, only certain parts of the closed loop system are allowed to be subject to perturbations.

6.3 Characterizations for the internal model structure

Having (hopefully) convinced the reader that such dynamic controllers (4.1) which have the internal model structure play a central role in linear robust output regulation problems, in this section we shall establish general characterizations for those controllers which have this structure. Our approach is mostly based on the recent theory of implemented semigroups due to Alber and Kühnemund [1, 59]; a brief review of this theory can be found in Appendix A.3. However, we shall begin with a simple but useful geometric characterization for the internal model structure. In a subsequent example this geometric characterization will reveal the relation of our robustness results to the (finite-dimensional) structurally stable synthesis algorithm of Francis [29].

Theorem 6.35. *Let $\mathcal{G} \in \mathcal{L}(\mathcal{L}(W, H), \mathcal{L}(W, X))$ be defined such that $\mathcal{G}\Delta = G\Delta$ for each $\Delta \in \mathcal{L}(W, H)$. A controller (4.1) has the internal model structure if and only if $\ker(\mathcal{G}) = \{0\}$ and $\operatorname{ran}(\mathcal{T}_{F,S}) \cap \operatorname{ran}(\mathcal{G}) = \{0\}$.*

Proof. Assume first that the controller has the internal model structure. If $\mathcal{G}\Delta = 0$ for some $\Delta \in \mathcal{L}(W, H)$, then $\Lambda S = F\Lambda + G\Delta$ in $\mathcal{D}(S)$ for $\Lambda = 0$, which implies $\Delta = 0$, and hence $\mathcal{G}$ is injective. On the other hand, for arbitrary $Y \in \operatorname{ran}(\mathcal{T}_{F,S}) \cap \operatorname{ran}(\mathcal{G})$ we have $Y = F\Lambda - \Lambda S = G\Delta$ for some $\Lambda \in \mathcal{D}(\mathcal{T}_{F,S})$ and some $\Delta \in \mathcal{L}(W, H)$. Whence $Y = 0$ because $\Delta = 0$ by the internal model structure.

Assume then that $\ker(\mathcal{G}) = \{0\}$ and $\operatorname{ran}(\mathcal{T}_{F,S}) \cap \operatorname{ran}(\mathcal{G}) = \{0\}$. Let $\Lambda S = F\Lambda + G\Delta$ in $\mathcal{D}(S)$ for some $\Lambda \in \mathcal{D}(\mathcal{T}_{F,S})$ and some $\Delta \in \mathcal{L}(W, H)$. Then $\Lambda S - F\Lambda = G\Delta \in \operatorname{ran}(\mathcal{T}_{F,S}) \cap \operatorname{ran}(\mathcal{G}) = \{0\}$. However, $\mathcal{G}$ is injective, and so $\Delta = 0$. This implies that the controller has the internal model structure. $\square$

Example 6.36. *Let the spaces Z, H and $W = \mathcal{H} \subset BUC(\mathbb{R}, H)$ be finite-dimensional and let us take $S = S|_{\mathcal{H}}$ in accordance with Proposition 2.3. In this example we show, using Theorem 6.35,*

that the structurally stable synthesis algorithm (SSSA) of Francis (see Theorem 2b in [29]), when applied to any EFRP where $S = S|_\mathcal{H}$, results in a controller having the internal model structure. Recall from [29] that the SSSA essentially amounts to (i) designing an observer for an extended system containing the plant and a $\dim(H)-$fold direct sum of the maximal cyclic component S_0 of S, and then (ii) applying a control which is obtained from a related pure gain synthesis.

Let us assume that the SSSA can be used to solve a given EFRP where $S = S|_\mathcal{H}$. Observe that $S|_\mathcal{H}$ generates the isometric left shift group on the finite-dimensional space $\mathcal{H} \subset BUC(\mathbb{R}, H)$. Consequently there exists a finite set $\{\, i\omega_n \mid n \in I \,\}$ of complex frequencies such that $i\omega_n \neq i\omega_m$ for $n \neq m$ and such that the exponentials $(e^{i\omega_n \cdot})_{n \in I}$ form a basis in $\mathcal{H}$. This implies that $T_S(t)|_\mathcal{H} f = \sum_{n \in I} f_n e^{i\omega_n(\cdot + t)} = \sum_{n \in I} f_n e^{i\omega_n \cdot} e^{i\omega_n t}$ for every $t \in \mathbb{R}$ and every $f \in \mathcal{H}$ (for some unique sequence $(f_n)_{n \in I} \subset H$). Thus the differential operator $S = S|_\mathcal{H}$ can be represented by a diagonal matrix having a $\dim(H)-$fold reduplication of each of the frequencies $i\omega_n$, $n \in I$, on the diagonal. Then according to the Corollary on p. 306 of [9] the minimal polynomial $p(\lambda)$ of S is just the product $\Pi_{n \in I}(\lambda - i\omega_n)$ of distinct linear factors. Since the minimal polynomial of S_0 is the same as that of S [29], the matrix representation of $S = S|_\mathcal{H}$ is similar[8] to a block diagonal matrix S_D having a $\dim(H)-$fold reduplication of the maximal cyclic component S_0 of S on the block diagonal [29]. Also observe that the space $W = \mathcal{H}$ that we use here is just the space $\mathcal{H}_{2e} = \mathcal{H}_{21} \oplus \cdots \oplus \mathcal{H}_{21}$ ($\dim(H)-$fold copy) employed in [29].

Using the notation of Theorem 6.35, from the relations (48) and (49) in [29] we readily see that $\ker(\mathcal{G}) = \{0\}$ and that $\operatorname{ran}(\mathcal{T}_{F,S_D}) \cap \operatorname{ran}(\mathcal{G}) = \{0\}$ (in the notation of Francis we have $\mathcal{G} = \underline{B}_{cze}$ and $\mathcal{T}_{F,S_D} = \underline{A}_{ce}$). It is then evident that also $\operatorname{ran}(\mathcal{T}_{F,S}) \cap \operatorname{ran}(\mathcal{G}) = \{0\}$. By Theorem 6.35 the resulting controller has the internal model structure.

Remark 6.37. In [29] Francis proved that the SSSA results in a controller which is robust (structurally stable) with respect to small perturbations to the parameters A, B, P, G and J of the closed loop system. However, by Theorem 6.20, in the setting of Example 6.36 we also achieve robustness with respect to small perturbations to C and arbitrary perturbations to Q. To the author's knowledge, this has not been explicitly pointed out before in the finite-dimensional output regulation literature (although the result is probably known in the folklore).

Clearly the injectivity of the operator $\mathcal{G}$ in Theorem 6.35 is easy to verify; in particular, if G is

---

[8]The similarity transform is given in the Corollary on p. 306 of [9].

injective, then so is $\mathcal{G}$. However, the range intersection property in Theorem 6.35 may be far from trivial to verify in practice. Consequently, it is appropriate to present also other characterizations for the internal model structure. The next one is of spectral nature:

Theorem 6.38. *Let F generate a uniformly bounded C_0-semigroup $T_F(t)$ on X. If*

$$\sup_{\lambda>0}\|R(\lambda,\mathcal{T}_{F,S})G\Delta\|=\infty \quad \forall\Delta\in\mathcal{L}(W,H)\setminus\{0\} \tag{6.60}$$

then the controller (4.1) *has the internal model structure. Conversely, if X is reflexive and if the controller* (4.1) *has the internal model structure, then condition* (6.60) *holds.*

Proof. Let $\Lambda S=F\Lambda+G\Delta$ in $\mathcal{D}(S)$ for some $\Lambda\in\mathcal{D}(\mathcal{T}_{F,S})$ and some $\Delta\in\mathcal{L}(W,H)$. Then according to Theorem 12 in [88] the family Δ_λ, $\lambda>0$, of bounded linear operators defined by

$$\Delta_\lambda w=\int_0^\infty e^{-\lambda t}T_F(t)G\Delta T_S(-t)wdt \quad \forall\lambda>0\,\forall w\in W \tag{6.61}$$

is uniformly bounded for $\lambda>0$. However, by the Alber-Kühnemund theory [1, 59] (see in particular p. 372 of [1] and Remark 2.15 in [1]), the Sylvester operator $\mathcal{T}_{F,S}$ generates the uniformly bounded implemented semigroup $\mathcal{F}(t)=T_F(t)\cdot T_S(-t)$ on the operator space $\mathcal{L}(W,X)$. Hence for each $\lambda>0$ the resolvent $R(\lambda,\mathcal{T}_{F,S})$ can be given in terms of a Laplace integral:

$$\left[R(\lambda,\mathcal{T}_{F,S})G\Delta\right]w=\int_0^\infty e^{-\lambda t}[\mathcal{F}(t)G\Delta]wdt \tag{6.62}$$

$$=\int_0^\infty e^{-\lambda t}T_F(t)G\Delta T_S(-t)wdt \tag{6.63}$$

$$=\Delta_\lambda w \quad \forall\lambda>0\,\forall w\in W \tag{6.64}$$

Consequently if

$$\sup_{\lambda>0}\|R(\lambda,\mathcal{T}_{F,S})G\Delta\|=\infty \quad \forall\Delta\in\mathcal{L}(W,H)\setminus\{0\} \tag{6.65}$$

we must have that $\Delta=0$, i.e. the controller has the internal model structure.

Conversely, if X is reflexive, if the controller has the internal model structure and if we also have $\sup_{\lambda>0}\|R(\lambda,\mathcal{T}_{F,S})G\Delta\|<\infty$ for some nonzero $\Delta\in\mathcal{L}(W,H)$, then the family of operators Δ_λ defined above is uniformly bounded for $\lambda>0$. According to Theorem 12 in [88] this implies that there exists $\Lambda\in\mathcal{L}(W,X)$ such that $\Lambda S=F\Lambda+G\Delta$ in $\mathcal{D}(S)$. However, since the controller has the internal model structure, necessarily $\Delta=0$, which is a contradiction. This shows that the condition (6.60) holds. $\qquad\square$

Remark 6.39. In the dynamic state feedback controller of Subsection 4.5.2 we chose $F = S$ which always generates a uniformly bounded C_0-semigroup on $X = W$. Consequently, the assumption in Theorem 6.38 about the uniform boundedness of $T_F(t)$ is realistic in many applications.

Assuming that F generates a uniformly bounded C_0-semigroup $T_F(t)$ on X, the condition (6.60) clearly holds if $G\Delta$ is an eigenvector, corresponding to the eigenvalue $\lambda = 0$, of the linear Sylvester operator $\mathcal{T}_{F,S}$ for all nonzero $\Delta \in \mathcal{L}(W, H)$. In this case $R(\lambda, \mathcal{T}_{F,S})G\Delta = \frac{1}{\lambda}G\Delta$ for all $\Delta \in \mathcal{L}(W, H)$ and all $\lambda > 0$. However, in general the condition (6.60) and the uniform boundedness of $T_F(t)$ only imply that $0 \in \sigma_A(\mathcal{T}_{F,S})$ (the approximate point spectrum). In fact, for all nonzero $\Delta \in \mathcal{L}(W, H)$ the operators $G\Delta$ can be used to construct a corresponding approximate eigenvector for $\mathcal{T}_{F,S}$ (see Proposition IV.1.10 in [28]). In the remainder of this section we shall further investigate the interesting case in which the internal model structure property of a controller (4.1) can be reduced to (part of) $G\Delta$ being an eigenvector of $\mathcal{T}_{F,S}$ for all operators Δ. It turns out to be possible to drop the above explicit assumption about the uniform boundedness of $T_F(t)$. We shall employ the following assumption throughout the remainder of this section.

Assumption 6.40. *There exists $\lambda_0 > 0$ such that $(0, \lambda_0) \in \rho(\mathcal{T}_{F,S})$ and*

$$\sup_{0<\lambda<\lambda_0} \|\lambda R(\lambda, \mathcal{T}_{F,S})\| < \infty \tag{6.66}$$

Assumption 6.40 covers the case of a uniformly bounded $T_F(t)$:

Proposition 6.41. *If F generates a uniformly bounded C_0-semigroup $T_F(t)$, then Assumption 6.40 holds.*

Proof. By the Alber-Kühnemund theory [1, 59], the Sylvester operator $\mathcal{T}_{F,S}$ generates the implemented semigroup $\mathcal{F}(t) = T_F(t) \cdot T_S(-t)$ on $\mathcal{L}(W, X)$ which is strongly continuous in the strong operator topology. Furthermore, by Lemma 3.14 in [59] $\sup_{t\geq 0}\|\mathcal{F}(t)\| = \sup_{t\geq 0}\|T_F(t)\|\|T_S(-t)\| < \infty$, so that $\mathcal{F}(t)$ is uniformly bounded. By Theorem 1.28 in [59], $\mathcal{T}_{F,S}$ is a Hille-Yosida operator such that

$$\|R(\lambda, \mathcal{T}_{F,S})^k\| \leq \frac{M}{\lambda^k} \quad \forall k \in \mathbb{N}, \forall \lambda > 0 \tag{6.67}$$

for some $M \geq 1$. Consequently Assumption 6.40 holds. $\square$

The next result provides additional conditions under which the internal model structure property of a controller (4.1) follows from $G\Delta$ being an eigenvector (corresponding to the eigenvalue 0) of $\mathcal{T}_{F,S}$ for all nonzero operators $\Delta \in \mathcal{L}(W, H)$. For such controllers, $\text{ran}(G) \subset X$ is a

$T_F(t)-$invariant subspace on which the dynamical behaviour of the semigroup $T_F(t)$ is in a sense described by that of $T_S(t)$. Observe that below $T_F(t)$ need not be uniformly bounded.

Proposition 6.42. *Let the operator $\mathcal{G}$ (cf. Theorem 6.35) be injective. If, in addition, for every $\Delta \in \mathcal{L}(W, H)$ and every $t \geq 0$, it is true that $T_F(t)G\Delta = G\Delta T_S(t)$, or equivalently $G\Delta \in \ker(\mathcal{T}_{F,S})$, then the controller (4.1) has the internal model structure.*

Proof. Since Assumption 6.40 holds, for every $0 < \lambda < \lambda_0$ and for each $\Lambda \in \mathcal{L}(W, X)$ we have that $\mathcal{T}_{F,S} R(\lambda, \mathcal{T}_{F,S})\Lambda = \lambda R(\lambda, \mathcal{T}_{F,S})\Lambda - \Lambda$. Thus $\Lambda \in \overline{\mathrm{ran}}(\mathcal{T}_{F,S})$ if and only if $\lim_{\lambda \searrow 0} \lambda R(\lambda, \mathcal{T}_{F,S})\Lambda = 0$. Moreover, $\lambda R(\lambda, \mathcal{T}_{F,S})\Lambda = \Lambda$ for all $0 < \lambda < \lambda_0$ if and only if $\Lambda \in \ker(\mathcal{T}_{F,S})$ (see p. 263 of [2] for more details). In particular we have $\ker(\mathcal{T}_{F,S}) \cap \overline{\mathrm{ran}}(\mathcal{T}_{F,S}) = \{0\}$. Hence whenever $\mathrm{ran}(\mathcal{G}) \subset \ker(\mathcal{T}_{F,S})$ is true, we also have $\mathrm{ran}(\mathcal{G}) \cap \mathrm{ran}(\mathcal{T}_{F,S}) \subset \ker(\mathcal{T}_{F,S}) \cap \overline{\mathrm{ran}}(\mathcal{T}_{F,S}) = \{0\}$, from which the result immediately follows by Theorem 6.35. Thus it only remains to show that $G\Delta \in \ker(\mathcal{T}_{F,S})$ if and only if $T_F(t)G\Delta = G\Delta T_S(t)$ for all $t \geq 0$.

Let $\Delta \in \mathcal{L}(W, H)$ be such that $T_F(t)G\Delta = G\Delta T_S(t)$ for every $t \geq 0$. Then we also have that $\mathcal{F}(t)G\Delta = T_F(t)G\Delta T_S(-t) = G\Delta$ for all $t \geq 0$ and all $\Delta \in \mathcal{L}(W, H)$. Here $\mathcal{F}(t)$ is again the semigroup implemented by $T_F(t)$ and $T_S(-t)$ on $\mathcal{L}(W, X)$ [59]. Consequently,

$$[\mathcal{T}_{F,S} G\Delta]w = \lim_{t \searrow 0} \frac{[\mathcal{F}(t)G\Delta]w - G\Delta w}{t} = 0 \quad \forall w \in W \tag{6.68}$$

so that indeed $G\Delta \in \ker(\mathcal{T}_{F,S})$. Conversely, if $G\Delta \in \ker(\mathcal{T}_{F,S})$, then $\mathcal{F}(t)G\Delta = G\Delta$ by Proposition 1.16 (b) in [59]. The result now follows. $\square$

Unfortunately, in applications the assumption that $G\Delta$ is an eigenvector of $\mathcal{T}_{F,S}$ for *every* $0 \neq \Delta \in \mathcal{L}(W, H)$ may be a severe restriction; we do not know if it can be satisfied in any nontrivial situation. However, assuming the injectivity of $\mathcal{G}$ and a considerable amount of additional structure for $\mathcal{T}_{F,S}$ we can prove that it is in fact necessary and sufficient for the internal model structure that for every nonzero $\Delta \in \mathcal{L}(W, H)$ some *part* of $G\Delta$ is an eigenvector of $\mathcal{T}_{F,S}$ (corresponding to the eigenvalue 0). This condition can be quite easily verified e.g. in the finite-dimensional case, as will be demonstrated shortly in an example. We then obtain our last complete characterization for those controllers which have the internal model structure:

Theorem 6.43. *Assume that $\mathcal{T}_{F,S}$ is uniformly Abel-ergodic, i.e. $\lim_{\lambda \searrow 0} \lambda R(\lambda, \mathcal{T}_{F,S})$ converges in $\mathcal{L}(\mathcal{L}(W, X))$. Then the controller (4.1) has the internal model structure if and only if for all nonzero $\Delta \in \mathcal{L}(W, H)$ $\lim_{\lambda \searrow 0} \lambda R(\lambda, \mathcal{T}_{F,S})G\Delta \neq 0$, and the operator $\mathcal{G}$ is injective.*

Proof. If the controller (4.1) has the internal model structure, then $\mathcal{G}$ is injective by Theorem 6.35. By Proposition 4.3.15 in [2], $\mathrm{ran}(\mathcal{T}_{F,S})$ is closed in $\mathcal{L}(W, X)$. The uniform Abel-ergodicity of $\mathcal{T}_{F,S}$ and Proposition 4.3.2 in [2] immediately give the decomposition $\mathcal{L}(W, X) = \ker(\mathcal{T}_{F,S}) \oplus \mathrm{ran}(\mathcal{T}_{F,S})$. Hence for every nonzero $\Delta \in \mathcal{L}(W, H)$ we have $G\Delta = F\Lambda - \Lambda S + M$ for some $\Lambda \in \mathcal{D}(\mathcal{T}_{F,S})$ and some $M \in \ker(\mathcal{T}_{F,S})$. By the internal model structure, $M \neq 0$ (for otherwise $\Delta = 0$). On the other hand, by Corollary 4.3.2 in [2] the operator $P_{ker} = \lim_{\lambda \searrow 0} \lambda R(\lambda, \mathcal{T}_{F,S})$ is the projection onto $\ker(\mathcal{T}_{F,S})$ along $\mathrm{ran}(\mathcal{T}_{F,S})$. Hence $\lim_{\lambda \searrow 0} \lambda R(\lambda, \mathcal{T}_{F,S})G\Delta = M \neq 0$.

Conversely, assume that for all nonzero $\Delta \in \mathcal{L}(W, H)$ we have $\lim_{\lambda \searrow 0} \lambda R(\lambda, \mathcal{T}_{F,S})G\Delta \neq 0$ and that $\mathcal{G}$ is injective. Let $\Lambda \in \mathcal{D}(\mathcal{T}_{F,S})$ and $\Delta \in \mathcal{L}(W, H)$ be such that $\Lambda S = F\Lambda + G\Delta$. Then by the above direct sum decomposition of $\mathcal{L}(W, X)$, it is true that $\lim_{\lambda \searrow 0} \lambda R(\lambda, \mathcal{T}_{F,S})G\Delta = 0$. But this is possible only if $\Delta = 0$. Hence the controller has the internal model structure. □

Corollary 6.44. *Let W, H and X be finite-dimensional spaces. Let F generate a uniformly bounded C_0-semigroup on X. Then the controller (4.1) has the internal model structure if and only if for all nonzero $\Delta \in \mathcal{L}(W, H)$ $\lim_{\lambda \searrow 0} \lambda R(\lambda, \mathcal{T}_{F,S})G\Delta \neq 0$, and $\mathcal{G}$ is injective.*

Proof. Under our assumptions $\mathcal{L}(W, X)$ is also finite-dimensional. Moreover, the semigroup $\mathcal{F}(t)$ implemented by $T_F(t)$ and $T_S(-t)$ on $\mathcal{L}(W, X)$ is C_0 and uniformly bounded [1, 59]. Since every generator of a bounded C_0-semigroup on a reflexive Banach space is (strongly) Abel-ergodic (by Corollary 4.3.5 in [2] and Proposition 4.3.4 in [2]), and since in finite-dimensional spaces strong convergence implies uniform convergence, the operator $\mathcal{T}_{F,S}$ is uniformly Abel-ergodic. The result follows by Theorem 6.43. □

Example 6.45. *Let us consider the Davison-type controller introduced in Subsection 4.5.2 in the special finite-dimensional case $W = \mathbb{C}^2$, $H = \mathbb{C}$ and*

$$S = \begin{pmatrix} i\omega_1 & 0 \\ 0 & i\omega_2 \end{pmatrix}, \quad \omega_1 \neq \omega_2, \quad \omega_1, \omega_2 \in \mathbb{R} \tag{6.69}$$

We shall find sufficient conditions for the internal model structure of the controller (4.32) using Theorem 6.43. In this setup the dynamic state feedback controller (4.32) utilizes $F = S$ and $G = G_0$ for some $G_0 \in \mathcal{L}(H, W)$. Let $G_0 = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ and $\Delta = \begin{pmatrix} d_1 & d_2 \end{pmatrix}$ so that $G_0\Delta = \begin{pmatrix} g_1 d_1 & g_1 d_2 \\ g_2 d_1 & g_2 d_2 \end{pmatrix}$. According to

Proposition 4.3.4 in [2] we have that

$$\lim_{\lambda \searrow 0} \lambda R(\lambda, \mathcal{T}_{S,S}) G_0 \Delta = \lim_{t \to \infty} \frac{1}{t} \int_0^t T_S(t) G_0 \Delta T_S(-t) dt \tag{6.70}$$

$$= \lim_{t \to \infty} \frac{1}{t} \int_0^t \begin{pmatrix} e^{i\omega_1 t} & 0 \\ 0 & e^{i\omega_2 t} \end{pmatrix} \begin{pmatrix} g_1 d_1 & g_1 d_2 \\ g_2 d_1 & g_2 d_2 \end{pmatrix} \begin{pmatrix} e^{-i\omega_1 t} & 0 \\ 0 & e^{-i\omega_2 t} \end{pmatrix} dt \tag{6.71}$$

$$= \begin{pmatrix} g_1 d_1 & 0 \\ 0 & g_2 d_2 \end{pmatrix} \tag{6.72}$$

because $\mathcal{T}_{S,S}$ generates the implemented (uniformly bounded and strongly continuous) semigroup $\mathcal{F}(t) = T_S(t) \cdot T_S(-t)$ on $\mathcal{L}(W)$. Clearly whenever we choose G_0 such that $g_1 \neq 0$ and $g_2 \neq 0$, we have that $\lim_{\lambda \searrow 0} \lambda R(\lambda, \mathcal{T}_{S,S}) G_0 \Delta \neq 0$ unless $\Delta = 0$, i.e. $d_1 = d_2 = 0$. For this choice of G_0 the controller (4.32) has the internal model structure.

6.4 The internal model structure in the case $\sigma(F) = \sigma(S)$

In Section 6.3 we presented some abstract characterizations for the internal model structure for general operators F and G in the controller (4.1). However, in practice the particular case $\sigma(F) = \sigma(S)$ turns out to be very important. For example, the generalizations of Davison's dynamic state feedback controller presented in Subsection 4.5.2 employ the choices $X = W$ and $F = S$. On the other hand, later on in this chapter we shall also prove that the verification of the internal model structure property for the Francis-type controllers of Subsection 4.5.1 can often be *reduced* to the case $\sigma(F) = \sigma(S)$. Hence this special case deserves additional attention.

Recall that we assume throughout this thesis that S generates an isometric C_0-group on some Banach space W. Suppose that S is also bounded, that X and W are actually Hilbert spaces and that $F \in \mathcal{L}(X)$ also generates an isometric C_0-group on X. Then since the operators S and F are also normal, there exists unique spectral measures E^S and E^F associated with S and F [86]. Thus, for example, if ξ is any Borel measurable set in the plane, then $E^S(\xi)$ is a projection that reduces S, and the spectrum of the restriction of S to the range of $E^S(\xi)$ is contained in the closure of ξ [81, 86]. In the following we let ξ_λ^k denote the open disk with center $i\lambda \in i\mathbb{R}$ and radius $\frac{1}{k}$, $k \in \mathbb{N}$.

Theorem 6.46. *Assume that X and W are Hilbert spaces, that $S \in \mathcal{L}(W)$ and that $F \in \mathcal{L}(X)$ generates an isometric C_0-group on X. Let E^S and E^F be the spectral measures as in the above*

and assume that $\sigma(F) = \sigma(S)$. If $G \in \mathcal{L}(H, X)$ is such that for all $\Delta \in \mathcal{L}(W, H)$ the implication

$$\left[\lim_{k \to \infty} \|E^F(\xi_\lambda^k) G \Delta E^S(\xi_\lambda^k)\| = 0 \; \forall i\lambda \in \sigma(S) = \sigma(F) \right] \implies \Delta = 0 \qquad (6.73)$$

holds true, then also the implication (6.33) holds true, i.e. the controller (4.1) with F and G as in the above has the internal model structure.

Proof. Let $\Lambda \in \mathcal{D}(\mathcal{T}_{F,S}) = \mathcal{L}(W, X)$ and $\Delta \in \mathcal{L}(W, H)$ be such that $\Lambda S = F\Lambda + G\Delta$ in $\mathcal{D}(S) = W$. Then $G\Delta \in \operatorname{ran}(\mathcal{T}_{F,S})$ and according to Theorem 2.2 in [81] this implies that for every $i\lambda \in \sigma(S) = \sigma(F)$ we have $\lim_{k \to \infty} \|E^F(\xi_\lambda^k) G \Delta E^S(\xi_\lambda^k)\| = 0$. By our assumptions this implies $\Delta = 0$. $\qquad \square$

It is a well-known (but nontrivial[9]) fact that the generator of an isometric C_0-group on a Banach space Y is *decomposable*, i.e. for every compact set $\gamma \subset i\mathbb{R}$ there exists a maximal spectral subspace $M_Y(\gamma)$ (a maximal closed invariant subspace of Y on which the generator is bounded and has spectrum contained in γ). Since $\cup_{n \in \mathbb{N}} M_Y([-in, in])$ is also dense in Y [90] we suspect that the boundedness assumption for S and F in Theorem 6.46 can be generally removed without much effort. However, since in many interesting output regulation applications (e.g. in repetitive control) the spectrum $\sigma(S)$ is known to be a discrete set, the following result suffices for our purposes; observe that neither S nor F need be bounded in it.

Theorem 6.47. *Let F generate an isometric C_0-group on a Banach space X such that $\sigma(F) = \sigma(S)$ are discrete. Let $P_{i\lambda}^F$ denote the spectral projection on X corresponding to any (isolated) point $i\lambda \in \sigma(F)$. If $G \in \mathcal{L}(H, X)$ is such that for all isolated points $i\lambda$ of $\sigma(F)$ the operators $P_{i\lambda}^F G : H \to \operatorname{ran}(P_{i\lambda}^F)$ are injective, then the implication (6.33) holds true, i.e. the controller (4.1) with these parameters F and G has the internal model structure.*

Proof. Let $P_{i\lambda}^S$ denote the spectral projection on W corresponding to any isolated point $i\lambda \in \sigma(S)$. Then by Gelfand's Theorem ([2] Corollary 4.4.8) and our assumptions the spectra $\sigma_p(S) = \sigma(S) = \sigma(F) = \sigma_p(F)$ consist of isolated points only, such that $P_{i\lambda}^S S = i\lambda P_{i\lambda}^S = S P_{i\lambda}^S$ and $P_{i\lambda}^F F = i\lambda P_{i\lambda}^F = F P_{i\lambda}^F$ for all $i\lambda \in \sigma(S) = \sigma(F)$. If $\Lambda \in \mathcal{D}(\mathcal{T}_{F,S})$ and $\Delta \in \mathcal{L}(W, H)$ are such that $\Lambda S = F\Lambda + G\Delta$ in $\mathcal{D}(S)$, then for all $i\lambda \in \sigma(S) = \sigma(F)$ we have $0 = P_{i\lambda}^F[\Lambda S - F\Lambda - G\Delta]P_{i\lambda}^S w = P_{i\lambda}^F G \Delta P_{i\lambda}^S w$ for all $w \in W$. But by our assumption the operators $P_{i\lambda}^F G$ are injective, so that $\Delta P_{i\lambda}^S w = 0$ for all $w \in W$ and all $i\lambda \in \sigma(S)$. Since the spectra $\sigma(S) = \sigma(F)$ are discrete, now the maximal spectral subspace $M_W([-in, in]) = \operatorname{span}\{\operatorname{ran} P_{i\lambda_k}^S \mid k \in I_n\}$ for some finite set I_n of indices, and hence

---

[9] see Appendix A and [90] p. 400.

$\Delta = 0$ in $M_W([-in, in])$ for all $n \in \mathbb{N}$. An extension by continuity then implies $\Delta = 0$ in W because $\cup_{n \in \mathbb{N}} M_W([-in, in])$ is dense in W (see Appendix A). $\qquad\square$

Example 6.48. *Let $H = \mathbb{C}^N$ for some $N \in \mathbb{N}$ and take $X = \mathcal{H} \overset{s}{\hookrightarrow} BUC(\mathbb{R}, \mathbb{C}^N)$, with $F = S|_{\mathcal{H}}$, in accordance with Proposition 2.3. Assume that $\sigma(F)$ is discrete and $G \in \mathcal{L}(\mathbb{C}^N, X)$. Then for each $i\lambda \in \sigma(F)$ the operator $P_{i\lambda}^F$ maps $\mathcal{H}$ to $\{ ae^{i\lambda} \mid a \in \mathbb{C}^N \}$, which is an $N-$dimensional linear space. Since $G : \mathbb{C}^N \to \mathcal{H}$, the operator $P_{i\lambda}^F G$ can be represented by an $N \times N$ matrix. In fact, each $y \in \mathbb{C}^N$ has the representation $y = \sum_{n=1}^N \langle y, e_n \rangle_{\mathbb{C}^N} e_n$ in terms of the natural orthornormal basis for $\mathbb{C}^N$. Upon defining $\psi_n = Ge_n \in \mathcal{H}$ for each $1 \le n \le N$ we have*

$$P_{i\lambda}^F Gy = \sum_{n=1}^N \langle y, e_n \rangle P_{i\lambda}^F \psi_n = \begin{pmatrix} P_{i\lambda}^F \psi_1 & \cdots & P_{i\lambda}^F \psi_N \end{pmatrix} \begin{pmatrix} \langle y, e_1 \rangle \\ \vdots \\ \langle y, e_N \rangle \end{pmatrix} \tag{6.74}$$

$$= e^{i\lambda \cdot} \begin{pmatrix} \hat{\psi}_{11}(i\lambda) & \cdots & \hat{\psi}_{1N}(i\lambda) \\ \vdots & \vdots & \vdots \\ \hat{\psi}_{N1}(i\lambda) & \cdots & \hat{\psi}_{NN}(i\lambda) \end{pmatrix} \begin{pmatrix} \langle y, e_1 \rangle \\ \vdots \\ \langle y, e_N \rangle \end{pmatrix} \tag{6.75}$$

where for each $1 \le n \le N$ we have defined

$$P_{i\lambda}^F \psi_n = \begin{pmatrix} \hat{\psi}_{1n}(i\lambda) \\ \vdots \\ \hat{\psi}_{Nn}(i\lambda) \end{pmatrix} e^{i\lambda \cdot} \in \{ ae^{i\lambda \cdot} \mid a \in \mathbb{C}^N \} = \operatorname{ran}(P_{i\lambda}^F) \tag{6.76}$$

Consequently, in order that $P_{i\lambda}^F G$ is injective for all $i\lambda \in \sigma(F)$, the $N \times N$ matrix in (6.75) — consisting of the "combined λth Fourier coefficients" of the functions $\psi_n = Ge_n \in \mathcal{H}$ for $1 \le n \le N$ — must be nonsingular for all $i\lambda \in \sigma(F)$. This is of course reflected in an appropriate choice of the operator G.

Example 6.49. *Let $\alpha > \frac{1}{2}$, $p > 0$, $H = \mathbb{C}$ and let $X = H_{per}^\alpha(0, p)$ (the Sobolev space of $\alpha-$differentiable $p-$periodic scalar functions, see Chapter 2). Let $F = S|_{H_{per}^\alpha(0,p)}$ in accordance with Proposition 2.3. If we take any $G \in \mathcal{L}(\mathbb{C}, X)$, with $Gu = gu$ for some $g \in X$ and all $u \in \mathbb{C}$, such that the pair (F, G) is approximately controllable, then the operator $P_{i\lambda}^F G$ is injective for all $i\lambda \in \sigma(F) = \{ i\frac{2\pi n}{p} \mid n \in \mathbb{Z} \}$. In fact, X is a Hilbert space and F is a Riesz spectral operator (cf. [17]) whose eigenvectors constitute an orthonormal basis $(\phi_n)_{n \in \mathbb{Z}}$ of (weighted) exponentials on X. By Theorem 4.2.3 in [17] the approximate controllability of the pair (F, G) is equivalent*

to $\langle g, \phi_n \rangle \neq 0$ for all $n \in \mathbb{Z}$. But a direct calculation shows that $P_{i\lambda}^F Gu = \langle g, \phi_n \rangle \phi_n u$ for each $i\lambda = i\frac{2\pi n}{p} \in \sigma(F)$ and all $u \in \mathbb{C}$; hence $P_{i\lambda}^F G$ is indeed injective for all $i\lambda \in \sigma(F)$.

6.5 Conditional robustness results for the controllers of Section 4.5

We are finally in a position to present verifiable sufficient conditions under which conditionally robust output regulation occurs for the two dynamic controllers introduced in Section 4.5. Our key idea is to replace the copy of the operator S in (4.26) and (4.32) by an *auxiliary operator* S_a defined on an *auxiliary Banach space* W_a, such that in a sense S_a "resembles[10]" S. This approach makes it possible, for example, to regard the operators P and Q which are embedded in the operator F in (4.26) as *design parameters*. This feature is very convenient in the stabilization of the closed loop system; observe that it is not present e.g. in Theorem 4.15 where P and Q must coincide with the corresponding operators of the exosystem (2.2), as $W_a = W$.

In the present section conditionally robust output regulation with respect to certain perturbations is obtained by stabilizing the closed loop system (containing S_a) strongly and by relying on the internal model structure. We shall provide conditions under which the verification of the internal model structure can be traced to Theorem 6.47 for $F = S_a$.

6.5.1 Conditional robustness results for the Francis-type controllers

We begin our study with the controller (4.26) which generalizes those of Francis [29] and Byrnes et al. [12]. In Assumption 6.50 below the items 1-5 are natural infinite-dimensional generalizations of the requirements in the structurally stable synthesis algorithm (SSSA) of Francis [29]. On the other hand, the item 6 in Assumption 6.50 below arises naturally from the results of Section 6.4.

Assumption 6.50. *There exists an operator S_a which generates an isometric C_0-group on a Banach space W_a, and the following are true:*

1. *There is no feedthrough, i.e. $D = 0$,*

2. *The spectra $\sigma(S_a) = \sigma(S)$ are discrete,*

---

[10]Of course, we do not exclude the possibility that $W_a = W$ and $S_a = S$.

3. *There exist $P_a \in \mathcal{L}(W_a, Z)$ and $Q_a \in \mathcal{L}(W_a, H)$ and $G = \left(\begin{smallmatrix} G_1 \\ G_2 \end{smallmatrix} \right) \in \mathcal{L}(H, Z \times W_a)$ such that $\left(\begin{smallmatrix} A & P_a \\ 0 & S_a \end{smallmatrix} \right) - \left(\begin{smallmatrix} G_1 \\ G_2 \end{smallmatrix} \right) \left(\begin{smallmatrix} C & -Q_a \end{smallmatrix} \right)$ generates a strongly stable C_0-semigroup on $Z \times W_a$,*

4. *There exists $K \in \mathcal{L}(Z, H)$ such that $A + BK$ generates an exponentially stable C_0-semigroup on Z,*

5. *There exist $\Pi \in \mathcal{L}(W_a, Z)$, such that $\Pi(\mathcal{D}(S_a)) \subset \mathcal{D}(A)$, and $\Gamma \in \mathcal{L}(W_a, H)$ satisfying the following regulator equations:*

$$\Pi S_a = A\Pi + B\Gamma + P_a \quad in \ \mathcal{D}(S_a) \tag{6.77a}$$

$$Q_a = C\Pi \quad in \ W_a \tag{6.77b}$$

6. *For every $i\omega \in \sigma(S_a)$ the operator $P_{i\omega}^{S_a} G_2 : H \to \operatorname{ran}(P_{i\omega}^{S_a})$ is injective. Here $P_{i\omega}^{S_a}$ is the spectral projection corresponding to the (isolated) point $i\omega \in \sigma(S_a)$.*

Remark 6.51. The methods of Subsection 4.6.3 are directly applicable in verifying item 3 of Assumption 6.50; observe that here P_a and Q_a can be freely chosen (modulo item 5). The methods of Chapter 8 then apply to item 5 of this assumption.

Lemma 6.52. *Under Assumption 6.50 the dynamic controller (4.1) with*

$$F = \begin{pmatrix} A + BK - G_1 C & P_a + B(\Gamma - K\Pi) + G_1 Q_a \\ -G_2 C & S_a + G_2 Q_a \end{pmatrix}, \quad G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}, \quad J = \begin{pmatrix} K & \Gamma - K\Pi \end{pmatrix} \tag{6.78}$$

on the state space $X = Z \times W_a$ has the internal model structure. Moreover, the closed loop operator $\mathcal{A} = \left(\begin{smallmatrix} A & BJ \\ GC & F \end{smallmatrix} \right)$ generates a strongly stable C_0-semigroup on $Z \times X$.

Remark 6.53. In (6.78) the operators $P_a \in \mathcal{L}(W_a, Z)$ and $Q_a \in \mathcal{L}(W_a, H)$ need not coincide with the operators $P \in \mathcal{L}(W, Z)$ and $Q \in \mathcal{L}(W, H)$ of the exosystem (2.2).

Proof of Lemma 6.52. Assume that $\Delta \in \mathcal{L}(W, H)$ and $\Lambda = \left(\begin{smallmatrix} \Lambda_1 \\ \Lambda_2 \end{smallmatrix} \right) \in \mathcal{D}(\mathcal{T}_{F,S}) \subset \mathcal{L}(W, X)$ are such that $\Lambda S = F\Lambda + G\Delta$ in $\mathcal{D}(S)$. Then

$$\begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix} S = \begin{pmatrix} A + BK - G_1 C & P_a + B(\Gamma - K\Pi) + G_1 Q_a \\ -G_2 C & S_a + G_2 Q_a \end{pmatrix} \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix} + \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} \Delta \quad in \ \mathcal{D}(S) \tag{6.79}$$

whence $\Lambda_2 S = S_a \Lambda_2 + G_2(Q_a \Lambda_2 - C\Lambda_1 + \Delta)$ in $\mathcal{D}(S)$. It follows from Theorem 6.47 that $Q_a \Lambda_2 - C\Lambda_1 + \Delta = 0$ in W. Hence also $\Lambda_2 S = S_a \Lambda_2$ in $\mathcal{D}(S)$.

On the other hand, by (6.79) we also have

$$\Lambda_1 S = (A + BK - G_1 C)\Lambda_1 + (P_a + B(\Gamma - K\Pi) + G_1 Q_a)\Lambda_2 + G_1 \Delta \tag{6.80}$$

$$= (A + BK)\Lambda_1 + (P_a + B(\Gamma - K\Pi))\Lambda_2 + G_1(Q_a \Lambda_2 - C\Lambda_1 + \Delta) \tag{6.81}$$

$$= (A + BK)\Lambda_1 + (P_a + B(\Gamma - K\Pi))\Lambda_2 \quad \text{in } \mathcal{D}(S) \tag{6.82}$$

Consequently $\binom{\Lambda_1}{\Lambda_2} S = \left(\begin{smallmatrix} A+BK & B(\Gamma-K\Pi)+P_a \\ 0 & S_a \end{smallmatrix} \right) \binom{\Lambda_1}{\Lambda_2}$ in $\mathcal{D}(S)$. But since by our assumptions the regulator equations (6.77) are satisfied, we have

$$\begin{pmatrix} I & \Pi \\ 0 & I \end{pmatrix} \begin{pmatrix} A + BK & 0 \\ 0 & S_a \end{pmatrix} \begin{pmatrix} I & -\Pi \\ 0 & I \end{pmatrix} = \begin{pmatrix} A + BK & B(\Gamma - K\Pi) + P_a \\ 0 & S_a \end{pmatrix} \tag{6.83}$$

so that $\left(\begin{smallmatrix} I & -\Pi \\ 0 & I \end{smallmatrix} \right) \binom{\Lambda_1}{\Lambda_2} S = \left(\begin{smallmatrix} A+BK & 0 \\ 0 & S_a \end{smallmatrix} \right) \left(\begin{smallmatrix} I & -\Pi \\ 0 & I \end{smallmatrix} \right) \binom{\Lambda_1}{\Lambda_2}$ in $\mathcal{D}(S)$. Therefore $(\Lambda_1 - \Pi\Lambda_2)S = (A+BK)(\Lambda_1 - \Pi\Lambda_2)$ in $\mathcal{D}(S)$. But $A + BK$ generates an exponentially stable C_0−semigroup, so that by the uniqueness the only operator $M \in \mathcal{L}(W, Z)$ satisfying $MS = (A + BK)M$ in $\mathcal{D}(S)$ is $M = 0$ (see [90] and Section A.2). We thus have $\Lambda_1 = \Pi\Lambda_2$ in W.

Finally recall from the above that $Q_a \Lambda_2 - C\Lambda_1 + \Delta = 0$ in W, so that by the regulator equations (6.77) we have $Q_a \Lambda_2 - C\Pi\Lambda_2 + \Delta = Q_a \Lambda_2 - Q_a \Lambda_2 + \Delta = \Delta = 0$. This shows that the controller (4.1) with the above parameters has the internal model structure. The strong stability of the closed loop semigroup $T_{\mathcal{A}}(t)$ follows as in the proof of Theorem 4.15, upon replacing S by S_a, P by P_a and Q by Q_a. □

The next result follows immediately from Theorem 6.20, Theorem 6.25 and Theorem 6.32 using Lemma 6.52.

Corollary 6.54. *Let Assumption 6.50 hold such that $W \hookrightarrow W_a$. Let $\mathcal{A} = \left(\begin{smallmatrix} A & BJ \\ GC & F \end{smallmatrix} \right)$, with the operators F, G and J of (4.1) as in (6.78). Then the following hold.*

- *If for every $\mathcal{P} \in \mathcal{L}(W_a, Z \times X)$ there exists a unique $Y \in \mathcal{L}(W, Z \times X)$ such that $Y(\mathcal{D}(S)) \subset \mathcal{D}(\mathcal{A})$ and $YS = \mathcal{A}Y + \mathcal{P}$ in $\mathcal{D}(S)$, then the controller (4.1) solves the EFRP for every $P \in \mathcal{L}(W_a, Z) \subset \mathcal{L}(W, Z)$ and $Q \in \mathcal{L}(W_a, H) \subset \mathcal{L}(W, H)$ in the exosystem, and the following holds: Output regulation is conditionally robust with respect to such perturbations to A, B, C, G and J for which the corresponding perturbation $\Delta_{\mathcal{A}}$ to the closed loop operator $\mathcal{A}$ is (W, W_a)−admissible.*

- *If for every $\mathcal{P} \in \mathcal{L}(W, Z \times X)$ there exists a unique $Y \in \mathcal{L}(W, Z \times X)$ such that $Y(\mathcal{D}(S)) \subset \mathcal{D}(\mathcal{A})$ and $YS = \mathcal{A}Y + \mathcal{P}$ in $\mathcal{D}(S)$, then the controller (4.1) solves the EFRP for every $P \in \mathcal{L}(W, Z)$ and $Q \in \mathcal{L}(W, H)$ in the exosystem, in such a way that output regulation is conditionally robust with respect to all small enough perturbations to A, B, C, G and J.*

- *If $\mathcal{A}$ generates an exponentially stable C_0-semigroup on $Z \times X$, then the controller (4.1) solves the EFRP for every $P \in \mathcal{L}(W, Z)$ and $Q \in \mathcal{L}(W, H)$ in the exosystem, in such a way that output regulation is robust with respect to small enough perturbations to A, B, C, G and J. Moreover, asymptotic tracking of the reference signals in the presence of disturbances is exponentially fast.*

The special case $W_a = W$ and $S_a = S$ is of course possible in Corollary 6.54. However, if $\dim(W) = \infty$, then it is in general wise to only have $W \hookrightarrow W_a$. In the case of $p-$periodic scalar reference signals, for example, we should consider choosing $W = H_{per}^{\beta}(0, p)$ and $W_a = H_{per}^{\alpha}(0, p)$, with $S = S|_{H_{per}^{\beta}(0,p)}$ and $S_a = S|_{H_{per}^{\alpha}(0,p)}$, in accordance with Proposition 2.3. Here $\beta > \alpha > \frac{1}{2}$ guarantees that $W \hookrightarrow W_a$, and the idea is that if β is *considerably larger* than α, then the reference signals are much smoother than what the error feedback controller (6.78) is prepared to asymptotically track. Corollary 6.33 then reveals that in some cases *sufficient smoothness of the reference signals implies a degree of conditional robustness*. We refer the reader to Section 6.7 for an application of this principle, and we point out that this feature is of course only interesting in the case of an infinite-dimensional exosystem (2.2).

Remark 6.55. In [12] (Theorem IV.2) Byrnes et al. have proved a complete characterization for the existence and construction of error feedback controllers achieving (possibly nonrobust) output regulation for a given finite-dimensional exogenous system (2.1) with fixed operators P and Q. They employed controllers (4.1) with parameters as in (4.26). In their work the closed loop system operator $\mathcal{A}$ generates an exponentially stable C_0-semigroup; hence Corollary 6.54 above reveals that if G_2 is chosen appropriately in their controller, then the controller does not have to be changed if the matrices P and Q in the exosystem are changed. Moreover, in this case output regulation is in fact robust with respect to sufficiently small perturbations to certain parameters of the closed loop system.

Remark 6.56. If Z, X, H and $W = W_a = \mathcal{H} \subset BUC(\mathbb{R}, H)$ are finite-dimensional spaces and if $S = S_a = S|_{\mathcal{H}}$ in accordance with Proposition 2.3, then by Example 6.36 the matrix F in (6.78)

contains the matrix S_a which is similar to a block diagonal matrix S_D utilizing a $\dim(H)-$fold reduplication of the maximal cyclic component of S. Hence the above robustness results are in accordance with the corresponding finite-dimensional theory [29].

6.5.2 Conditional robustness results for the Davison-type controllers

We shall now establish conditions under which the generalization (4.32) of Davison's dynamic state feedback controller (cf. [39]) achieves conditionally robust output regulation. We shall also show how this state feedback controller can be used in the design of a dynamic controller (4.1) which does not employ direct feedback from the state of the plant. As in Subsection 6.5.1 also here it will be useful to prove the results in the more general case that the operator S in (4.32) is replaced by an auxiliary operator S_a which "resembles" S (of course $S_a = S$ is again possible).

The items 1 and 2 in Assumption 6.57 below are natural generalizations of the corresponding finite-dimensional assumptions in [39]. On the other hand, the third item in Assumption 6.57 below arises naturally from the results of Section 6.4.

Assumption 6.57. *There exists an operator S_a which generates an isometric C_0-group on a Banach space W_a, and the following are true:*

1. *The spectra $\sigma(S) = \sigma(S_a)$ are discrete,*

2. *There exist $K_1 \in \mathcal{L}(Z, H)$, $K_2 \in \mathcal{L}(W_a, H)$ and $G_0 \in \mathcal{L}(H, W_a)$ such that the operator $\left(\begin{smallmatrix} A+BK_1 & BK_2 \\ G_0(C+DK_1) & S_a+G_0DK_2 \end{smallmatrix} \right)$ generates a strongly stable C_0-semigroup on $Z \times W_a$,*

3. *For every isolated point $i\omega \in \sigma(S_a)$ the operator $P_{i\omega}^{S_a} G_0 : H \to \operatorname{ran}(P_{i\omega}^{S_a})$ is injective. Here $P_{i\omega}^{S_a}$ is the spectral projection corresponding to the (isolated) point $i\omega \in \sigma(S_a)$.*

Remark 6.58. The methods of Subsection 4.6.4 are readily applicable in verifying item 2 above.

Under Assumption 6.57 we shall consider the dynamic state feedback controller

$$\dot{x}(t) = S_a x(t) + G_0 e(t), \quad x(0) \in W_a, \quad t \geq 0 \tag{6.84a}$$

$$u(t) = K_1 z(t) + K_2 x(t) \tag{6.84b}$$

which for $W = W_a$ and $S = S_a$ reduces to that in (4.32). It is evident that an error feedback output regulation problem for the controller (6.84) can be studied as an EFRP for a plant (1.1) where A is replaced by $A + BK_1$ and C is replaced by $C + DK_1$. We obtain:

Lemma 6.59. *Let Assumption 6.57 hold and consider a plant (1.1) where A is replaced by $A+BK_1$ and C is replaced by $C + DK_1$. Set $X = W_a$ with $F = S_a$, $G = G_0$, $J = K_2$. Then the resulting dynamic controller (4.1) has the internal model structure, and the closed loop operator $\mathcal{A} = \left(\begin{smallmatrix} A+BK_1 & BJ \\ G(C+DK_1) & F+GDJ \end{smallmatrix} \right)$ generates a strongly stable C_0-semigroup on $Z \times W_a$.*

Proof. This result follows directly from Theorem 6.47 and Assumption 6.57. $\qquad\square$

The next result follows immediately from Theorem 6.20, Theorem 6.25 and Theorem 6.32 using Lemma 6.59.

Corollary 6.60. *Let Assumption 6.57 hold such that $W \hookrightarrow W_a$. Consider a plant where A is replaced by $A + BK_1$ and C is replaced by $C + DK_1$. Consider the controller (4.1) whose parameters F, G and J are as in Lemma 6.59.*

- *If for every $\mathcal{P} \in \mathcal{L}(W_a, Z \times X)$ there exists a unique $Y \in \mathcal{L}(W, Z \times X)$ such that $Y(\mathcal{D}(S)) \subset \mathcal{D}(\mathcal{A})$ and $YS = \mathcal{A}Y + \mathcal{P}$ in $\mathcal{D}(S)$, then the controller (4.1) solves the EFRP for every $P \in \mathcal{L}(W_a, Z) \subset \mathcal{L}(W, Z)$ and $Q \in \mathcal{L}(W_a, H) \subset \mathcal{L}(W, H)$ in the exosystem, in the following way: Output regulation is conditionally robust with respect to such perturbations to A, B, C, D, K_1, K_2 and G_0 for which the corresponding perturbation $\Delta_{\mathcal{A}}$ to the closed loop operator $\mathcal{A}$ is $(W, W_a)-$admissible.*

- *If for every $\mathcal{P} \in \mathcal{L}(W, Z \times X)$ there exists a unique $Y \in \mathcal{L}(W, Z \times X)$ such that $Y(\mathcal{D}(S)) \subset \mathcal{D}(\mathcal{A})$ and $YS = \mathcal{A}Y + \mathcal{P}$ in $\mathcal{D}(S)$, then the controller (4.1) solves the EFRP for every $P \in \mathcal{L}(W, Z)$ and $Q \in \mathcal{L}(W, H)$ in the exosystem, in such a way that output regulation is conditionally robust with respect to all small enough perturbations to A, B, C, D, K_1, K_2 and G_0.*

- *If $\mathcal{A}$ generates an exponentially stable C_0-semigroup on $Z \times X$, then the controller (4.1) solves the EFRP for every $P \in \mathcal{L}(W, Z)$ and $Q \in \mathcal{L}(W, H)$ in the exosystem, in such a way that output regulation is robust with respect to all small enough perturbations to A, B, C, D, K_1, K_2 and G_0. Moreover, asymptotic tracking of the reference signals in the presence of disturbances is exponentially fast.*

Now that we have studied conditionally robust output regulation for the dynamic state feedback controller (6.84), we turn to the construction of a dynamic controller (4.1) which does not involve

direct feedback from the state of the plant. In addition to Assumption 6.57 we shall need the exponential detectability of the pair (A, C).

Lemma 6.61. *Let Assumption 6.57 hold such that $W \hookrightarrow W_a$. Assume, in addition, that there exists $L \in \mathcal{L}(H, Z)$ such that $A - LC$ generates an exponentially stable C_0-semigroup on Z. Let $X = Z \times W_a$ and choose the parameters of the controller (4.1) as follows:*

$$F = \begin{pmatrix} A + BK_1 - L(C + DK_1) & (B - LD)K_2 \\ 0 & S_a \end{pmatrix}, \quad G = \begin{pmatrix} L \\ G_0 \end{pmatrix}, \quad J = \begin{pmatrix} K_1 & K_2 \end{pmatrix} \qquad (6.85)$$

Then the controller (4.1) has the internal model structure and the closed loop operator $\mathcal{A} = \begin{pmatrix} A & BJ \\ GC & F+GDJ \end{pmatrix}$ generates a strongly stable C_0-semigroup on $Z \times X$.

Proof. If $\Lambda = \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix} \in \mathcal{D}(\mathcal{T}_{F,S})$ and $\Delta \in \mathcal{L}(W, H)$ are such that $\Lambda S = F\Lambda + G\Delta$ in $\mathcal{D}(S)$, then

$$\begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix} S = \begin{pmatrix} A + BK_1 - L(C + DK_1) & (B - LD)K_2 \\ 0 & S_a \end{pmatrix} \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix} + \begin{pmatrix} L \\ G_0 \end{pmatrix} \Delta \quad \text{in } \mathcal{D}(S) \qquad (6.86)$$

Hence $\Lambda_2 S = S_a \Lambda_2 + G_0 \Delta$ in $\mathcal{D}(S)$. The internal model structure of the controller (4.1) can now be easily verified as in Theorem 6.47. We prove that the closed loop system operator $\mathcal{A}$ generates a strongly stable C_0-semigroup on $Z \times X$. With F, G and J as in (6.85), the closed loop system operator becomes

$$\mathcal{A} = \begin{pmatrix} A & BJ \\ GC & F + GDJ \end{pmatrix} = \begin{pmatrix} A & BK_1 & BK_2 \\ LC & A + BK_1 - LC & BK_2 \\ G_0 C & G_0 D K_1 & S_a + G_0 D K_2 \end{pmatrix} \qquad (6.87)$$

Applying the similarity transform U given as

$$U = \begin{pmatrix} I & -I & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \qquad (6.88)$$

we see that $\mathcal{A}$ is similar to the operator $\widetilde{\mathcal{A}} = U \mathcal{A} U^{-1}$ having the expression

$$\widetilde{\mathcal{A}} = \begin{pmatrix} A - LC & 0 & 0 \\ LC & A + BK_1 & BK_2 \\ G_0 C & G_0(C + DK_1) & S_a + G_0 D K_2 \end{pmatrix} \qquad (6.89)$$

which — by Assumption 6.57 and the assumption that $A - LC$ generates an exponentially stable C_0-semigroup on Z — generates a strongly stable C_0-semigroup on $Z \times W_a$. Consequently, the similar closed loop semigroup $T_{\mathcal{A}}(t)$ is also strongly stable. □

The next result follows immediately from Theorem 6.20, Theorem 6.25 and Theorem 6.32 using Lemma 6.61.

Corollary 6.62. *Let Assumption 6.57 hold such that $W \hookrightarrow W_a$. Assume, in addition, that there exists $L \in \mathcal{L}(H, Z)$ such that $A - LC$ generates an exponentially stable C_0-semigroup on Z. Consider the controller (4.1) whose parameters F, G and J are as in (6.85).*

- *If for every $\mathcal{P} \in \mathcal{L}(W_a, Z \times X)$ there exists a unique $Y \in \mathcal{L}(W, Z \times X)$ such that $Y(\mathcal{D}(S)) \subset \mathcal{D}(\mathcal{A})$ and $YS = \mathcal{A}Y + \mathcal{P}$ in $\mathcal{D}(S)$, then the controller (4.1) solves the EFRP for every $P \in \mathcal{L}(W_a, Z) \subset \mathcal{L}(W, Z)$ and $Q \in \mathcal{L}(W_a, H) \subset \mathcal{L}(W, H)$ in the exosystem, in the following way: Output regulation is conditionally robust with respect to such perturbations to A, B, C, D, G and J for which the corresponding perturbation $\Delta_{\mathcal{A}}$ to the closed loop operator $\mathcal{A}$ is $(W, W_a)-$admissible.*

- *If for every $\mathcal{P} \in \mathcal{L}(W, Z \times X)$ there exists a unique $Y \in \mathcal{L}(W, Z \times X)$ such that $Y(\mathcal{D}(S)) \subset \mathcal{D}(\mathcal{A})$ and $YS = \mathcal{A}Y + \mathcal{P}$ in $\mathcal{D}(S)$, then the controller (4.1) solves the EFRP for every $P \in \mathcal{L}(W, Z)$ and $Q \in \mathcal{L}(W, H)$ in the exosystem, in such a way that output regulation is conditionally robust with respect to all small enough perturbations to A, B, C, D, G and J.*

- *If $\mathcal{A}$ generates an exponentially stable C_0-semigroup on $Z \times X$, then the controller (4.1) solves the EFRP for every $P \in \mathcal{L}(W, Z)$ and $Q \in \mathcal{L}(W, H)$ in the exosystem, in such a way that output regulation is robust with respect to all small enough perturbations to A, B, C, D, G and J. Moreover, asymptotic tracking of the reference signals in the presence of disturbances is exponentially fast.*

6.6 A case study: Robustification of output regulation

In some applications it may be sensible to *trade* perfect output regulation without guaranteed robustness to almost perfect output regulation with guaranteed robustness. The purpose of the present case study section is to illustrate how this can be done in the case that $H = \mathbb{C}^M$ for some

$M \in \mathbb{N}$ and in the case that the true reference signals — for which perfect regulation is *not* required — are in some generalized Sobolev space $\mathcal{G} = H_{AP}(H, f_n, \omega_n)$ with fixed sequences $(\omega_n)_{n \in I}$ and $(f_n)_{n \in I}$ (see Chapter 2). In order to avoid trivialities, we assume in this section that I is an infinite set of indices, so that $\mathcal{G}$ is an infinite-dimensional space.

Let $\epsilon > 0$ be the desired tracking accuracy in the sense that $\limsup_{t \to \infty} \|e(t)\| < \epsilon \|y_{ref}\|_{\mathcal{G}}$ is required for all reference signals $y_{ref} \in \mathcal{G}$ and all initial states $z(0) \in Z$ and $x(0) \in X$ of the plant (1.1) and the error feedback controller (4.1). By our construction there exists $N \in \mathbb{N}$ such that $\sum_{|n|>N} f_n^{-2} < \epsilon^2$. Then for all $y_{ref} = \sum_{n \in I} \widehat{y_{ref}}(n) e^{i \omega_n \cdot} \in \mathcal{G}$ we have by the Schwartz inequality that

$$\Big\| y_{ref} - \sum_{|n| \leq N} \widehat{y_{ref}}(n) e^{i \omega_n \cdot} \Big\|_\infty \leq \sum_{|n|>N} \Big\| \widehat{y_{ref}}(n) e^{i \omega_n \cdot} \Big\|_\infty \leq \sum_{|n|>N} \| \widehat{y_{ref}}(n) \|_H \qquad (6.90)$$

$$\leq \sqrt{\sum_{|n|>N} f_n^{-2}} \sqrt{\sum_{n \in I} \| \widehat{y_{ref}}(n) \|_H f_n^2} \leq \epsilon \| y_{ref} \|_{\mathcal{G}} \qquad (6.91)$$

Consequently, whenever it is possible to asymptotically track the *approximation* signal $y_{ref}^N = \sum_{|n| \leq N} \widehat{y_{ref}}(n) e^{i \omega_n \cdot}$ of $y_{ref} = \sum_{n \in I} \widehat{y_{ref}}(n) e^{i \omega_n \cdot}$ in the sense of the EFRP, it is also possible to asymptotically track y_{ref} with accuracy $\epsilon > 0$ in the above sense. In fact, in this case

$$\limsup_{t \to \infty} \|y(t) - y_{ref}(t)\| \leq \limsup_{t \to \infty} \|y(t) - y_{ref}^N(t)\| + \limsup_{t \to \infty} \|y_{ref}^N(t) - y_{ref}(t)\| < \epsilon \|y_{ref}\|_{\mathcal{G}} \quad (6.92)$$

for all $y_{ref} \in \mathcal{G}$ and for all initial states $z(0) \in Z$ and $x(0) \in X$ of the plant (1.1) and the error feedback controller (4.1) achieving the asymptotic tracking of the approximations $y_{ref}^N(t)$.

Let $N \in \mathbb{N}$ be fixed as in the above, and define $g_n = f_n$ for all $|n| \leq N$ and $g_n = 0$ for $|n| > N$. We shall next provide sufficient conditions that the above approximations y_{ref}^N can be asymptotically tracked robustly with respect to perturbations to some of the control system's parameters. In accordance with Proposition 2.3 this amounts to solving the EFRP robustly for $W = \mathcal{H} = H_{AP}(H, g_n, \omega_n)$ and $S = S|_{\mathcal{H}}$ (observe that $\dim(\mathcal{H}) < \infty$). In order to accomplish this we make the following standing assumption.

Assumption 6.63. *There is no feedthrough, i.e. $D = 0$, and there exist $K \in \mathcal{L}(Z, H)$ and $L \in \mathcal{L}(H, Z)$ such that $A + BK$ and $A - LC$ generate exponentially stable C_0-semigroups on Z. Moreover, the set $\{ i\omega_n \mid |n| \leq N \} \subset \rho(A)$ and the matrix $H(i\omega_n) = CR(i\omega_n, A)B$ is nonsingular for all $|n| \leq N$ (recall that we have assumed that $H = \mathbb{C}^M$ for some $M \in \mathbb{N}$).*

Let us define the bounded linear operators $P_n : \mathcal{H} \to \mathcal{H}$, $|n| \leq N$, by $P_n f = \widehat{f}(n)e^{i\omega_n \cdot}$ for all $f = \sum_{|n| \leq N} \widehat{f}(n)e^{i\omega_n \cdot} \in \mathcal{H}$. In the main result of this section we shall utilize the following operators (which are well-defined under Assumption 6.63):

$$\Gamma_0 = \sum_{|n| \leq N} H(i\omega_n)^{-1}\delta_0 P_n \in \mathcal{L}(\mathcal{H}, H) \tag{6.93}$$

$$\Pi_0 = \sum_{|n| \leq N} R(i\omega_n, A)B\Gamma_0 P_n \in \mathcal{L}(\mathcal{H}, Z) \tag{6.94}$$

$$\hat{P} = B\Gamma_0 - L\delta_0 \in \mathcal{L}(\mathcal{H}, Z) \tag{6.95}$$

$$\hat{Q} = 2\delta_0 \in \mathcal{L}(\mathcal{H}, H) \tag{6.96}$$

$$\Gamma = \sum_{|n| \leq N} H(i\omega_n)^{-1}[\hat{Q} - CR(i\omega_n, A)\hat{P}]P_n \in \mathcal{L}(\mathcal{H}, H) \tag{6.97}$$

$$\Pi = \sum_{|n| \leq N} R(i\omega_n, A)[B\Gamma + \hat{P}]P_n \in \mathcal{L}(\mathcal{H}, Z) \tag{6.98}$$

$$G_2 = -\delta_0^* \in \mathcal{L}(H, \mathcal{H}) \tag{6.99}$$

$$G_1 = L - \Pi_0 \delta_0^* \in \mathcal{L}(H, Z) \tag{6.100}$$

where $\delta_0 f = f(0)$ for all $f \in \mathcal{H}$ and δ_0^* is the adjoint operator of δ_0 with respect to the inner product on $\mathcal{H}$ given in Proposition 2.16.

Remark 6.64. It is an elementary calculation to verify that

$$\Pi_0 S|_{\mathcal{H}} = A\Pi_0 + B\Gamma_0 \quad \text{in } \mathcal{D}(S|_{\mathcal{H}}) = \mathcal{H} \tag{6.101a}$$

$$\delta_0 = C\Pi_0 \quad \text{in } \mathcal{H} \tag{6.101b}$$

and that

$$\Pi S|_{\mathcal{H}} = A\Pi + B\Gamma + \hat{P} \quad \text{in } \mathcal{D}(S|_{\mathcal{H}}) = \mathcal{H} \tag{6.102a}$$

$$\hat{Q} = C\Pi \quad \text{in } \mathcal{H} \tag{6.102b}$$

Theorem 6.65 below is our main result in this section. It presents one possible choice for the dynamic controller (4.1) for robust asymptotic tracking of the above approximations y_{ref}^N in the presence of certain disturbances.

Theorem 6.65. *Under Assumption 6.63 the dynamic controller (4.1) with $X = Z \times \mathcal{H}$ and*

$$F = \begin{pmatrix} A + BK - G_1 C & \hat{P} + B(\Gamma - K\Pi) + G_1 \hat{Q} \\ -G_2 C & S|_{\mathcal{H}} + G_2 \hat{Q} \end{pmatrix}, \quad G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}, \quad J = \begin{pmatrix} K & \Gamma - K\Pi \end{pmatrix}$$

(6.103)

where the related operators are defined in (6.93)-(6.100), solves the EFRP (with $W = \mathcal{H}$) for all $P \in \mathcal{L}(\mathcal{H}, Z)$ and $Q \in \mathcal{L}(\mathcal{H}, H)$ in the exosystem (2.2). Moreover, output regulation is exponentially fast and robust with respect to small enough perturbations to the operators A, B, C, G and J.

Proof. We show that all requirements of Assumption 6.50 are satisfied for $W_a = W = \mathcal{H}$, $S_a = S|_{\mathcal{H}}$, $P_a = \hat{P}$ and $Q_a = \hat{Q}$, and that the closed loop operator $\mathcal{A} = \begin{pmatrix} A & BJ \\ GC & F \end{pmatrix}$ generates an exponentially stable C_0−semigroup on $Z \times X$. That robust and exponentially fast output regulation indeed occurs follows by Corollary 6.54.

The items 1,2,4 and 5 of Assumption 6.50 follow easily from Assumption 6.63 and the fact that $\dim(\mathcal{H}) < \infty$. As in the proof of Proposition 4.28 we see that $\delta_0^* : H \to \mathcal{H} : u \to \sum_{|n| \leq N} f_n^{-1} u e^{i\omega_n \cdot}$ for all $u \in H$. Consequently for every $i\omega_n \in \sigma(S|_{\mathcal{H}})$ the operator $P_{i\omega_n}^{S|_{\mathcal{H}}} G_2 : H \to \operatorname{ran}(P_{i\omega_n}^{S|_{\mathcal{H}}})$ is injective (i.e. item 6 of Assumption 6.50 is true) because $P_{i\omega_n}^{S|_{\mathcal{H}}} G_2 u = -P_{i\omega_n}^{S|_{\mathcal{H}}} \sum_{|k| \leq N} f_k^{-1} u e^{i\omega_k \cdot} = -f_n^{-1} u e^{i\omega_n \cdot}$. Here $P_{i\omega_n}^{S|_{\mathcal{H}}}$ is of course the spectral projection corresponding to the (isolated) point $i\omega_n \in \sigma(S|_{\mathcal{H}})$, for $|n| \leq N$.

As in the proof of Theorem 4.15 we can show that the closed loop operator $\mathcal{A}$ is similar to the operator

$$\widetilde{\mathcal{A}} = \begin{pmatrix} A + BK & -BK & -B(\Gamma - K\Pi) \\ 0 & A - G_1 C & \hat{P} + G_1 \hat{Q} \\ 0 & -G_2 C & S|_{\mathcal{H}} + G_2 \hat{Q} \end{pmatrix} = \begin{pmatrix} A + BK & \Delta \\ 0 & \mathcal{A}_F \end{pmatrix}$$

(6.104)

where the operator $\mathcal{A}_F = \begin{pmatrix} A - G_1 C & \hat{P} + G_1 \hat{Q} \\ -G_2 C & S|_{\mathcal{H}} + G_2 \hat{Q} \end{pmatrix}$ satisfies

$$\begin{pmatrix} I & -\Pi_0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A - G_1 C & \hat{P} + G_1 \hat{Q} \\ -G_2 C & S|_{\mathcal{H}} + G_2 \hat{Q} \end{pmatrix} \begin{pmatrix} I & \Pi_0 \\ 0 & I \end{pmatrix} =$$

(6.105)

$$\begin{pmatrix} A - G_1 C + \Pi_0 G_2 C & A\Pi_0 - G_1 C\Pi_0 + \Pi_0 G_2 C\Pi_0 + \hat{P} + G_1 \hat{Q} - \Pi_0 S|_{\mathcal{H}} - \Pi_0 G_2 \hat{Q} \\ -G_2 C & S|_{\mathcal{H}} + G_2 \hat{Q} - G_2 C\Pi_0 \end{pmatrix} =$$

(6.106)

$$\begin{pmatrix} A - LC & 0 \\ \delta_0^* C & S|_{\mathcal{H}} - \delta_0^* \delta_0 \end{pmatrix}$$

(6.107)

so that by the finite-dimensionality of $\mathcal{H}$ and by Corollary 4.26 $\mathcal{A}_F$ generates an exponentially stable C_0−semigroup on $Z \times \mathcal{H}$. Hence also item 3 of Assumption 6.50 is true. Finally, by similarity and the assumption that $A + BK$ generates an exponentially stable C_0−semigroup on Z the closed loop operator $\mathcal{A}$ also generates an exponentially stable C_0−semigroup on $Z \times X$.

<div align="right">□</div>

Since in Theorem 6.65 we regulate signals generated by a particular *finite-dimensional* neutrally stable exosystem (2.1), also the error feedback output regulation theory of Byrnes et al. [12] (Theorem IV.2 in [12] in particular) is applicable in this situation. Thus, in order to justify Theorem 6.65 as being new and useful, it is important to accentuate the following differences between Theorem 6.65 and the results in [12]:

- Exponential detectability of the pair $\left(\left(\begin{smallmatrix} A & P \\ 0 & S \end{smallmatrix} \right), \left(\begin{smallmatrix} C & -Q \end{smallmatrix} \right) \right)$, i.e. hypothesis H3 in [12], does not have to be postulated here; it is part of the conclusion. In particular, here the operators P and Q correspond to the above *design parameters* $\hat{P}$ and $\hat{Q}$, while in [12] they correspond to the *particular* operators of the exosystem for which output regulation is to be achieved.

- The regulator equations (3.10) have been *a priori* solved here, whereas the use of Theorem IV.2 in [12] explicitly requires the solution of these regulator equations. Moreover, in [12] the operators P and Q in these equations must correspond to the particular operators utilized in the exosystem (2.1).

- Output regulation is guaranteed to be robust here, while the issue of robustness is not addressed in [12].

- We achieve approximate (with a desired accuracy $\epsilon > 0$) asymptotic tracking of all reference signals in the infinite-dimensional Sobolev space $H_{AP}(H, f_n, \omega_n)$, while no approximation results are presented in [12].

On the other hand, it has been proved in [12] that the nonsingularity of $H(i\omega)$ for all $i\omega \in \sigma(S)$ (which is assumed in the above) already implies the solvability of the regulator equations (3.10) regardless of the operators P and Q, whenever $\dim(W) < \infty$. Moreover, we point out that in [12] it is not explicitly assumed that the pair (A, C) is exponentially detectable, as is done in the above. Nonetheless, Theorem 6.65 above may sometimes be easier to apply in practice than Theorem

IV.2 in [12] because of the additional design parameters $\hat{P}$ and $\hat{Q}$ which need not coincide with the operators P and Q utilized in the exosystem.

6.7 A case study: Conditionally robust repetitive control for SISO systems

We now turn our attention to certain repetitive control applications. In this section we make the following standing assumption, which shows that our aim is to study the asymptotic tracking of all $p-$periodic scalar-valued reference signals in certain Sobolev spaces $H_{per}^\beta(0, p)$, $p > 0$ and $\beta > \frac{1}{2}$, using error feedback control.

Assumption 6.66. *The plant is a SISO system, i.e. $H = \mathbb{C}$, the operator A generates an exponentially stable C_0-semigroup on Z, $D = 0$, $W = \mathcal{H} = H_{per}^\beta(0, p)$, $W_a = \mathcal{G} = H_{per}^\alpha(0, p)$ for some $\beta > \alpha > \frac{1}{2}$ (where α is fixed at the outset and β is to be determined), $p > 0$ and $S = S|_{\mathcal{H}}$, $S_a = S|_{\mathcal{G}}$, $Q = \delta_0 \in \mathcal{L}(\mathcal{G}, \mathbb{C}) \cap \mathcal{L}(\mathcal{H}, \mathbb{C})$, in accordance with Proposition 2.3.*

Remark 6.67. The assumption that there is no feedthrough in the plant, i.e. $D = 0$, is deliberate. We aim to show that, contrary to the conventional repetitive control scheme [36, 96], in our framework conditionally robust output regulation of $p-$periodic signals with an infinite number of distinct frequency components is possible even if $D = 0$. This is a consequence of the fact that we do not require exponential closed loop stability. In the conventional repetitive control scheme internal (i.e. exponential) closed loop stability — which implies output regulation — can only be attained if the *finite-dimensional* plant is *not strictly proper* (cf. Chapter 1 or Section V of [96] and Proposition 2 of [36]).

In order to achieve output regulation with a degree of conditional robustness, we shall construct a dynamic controller (4.1) with parameters as in (6.78). We begin the controller design process by stabilizing the pair $(S|_{\mathcal{G}}, \delta_0)$ strongly using the pole-placement techniques of Chapter 4. Below, we denote $i\omega_n = i\frac{2\pi n}{p}$ for all $n \in \mathbb{Z}$. The following result is just a rephrasement of Proposition 4.28:

Lemma 6.68. *Let $\gamma > \alpha + \frac{1}{2}$. Then there exists $L \in \mathcal{L}(\mathbb{C}, \mathcal{G})$ such that*

1. *$S|_{\mathcal{G}} + L\delta_0$ generates a strongly stable C_0-semigroup on $\mathcal{G}$,*

2. *The resolvent satisfies $\|R(i\omega_n, S|_{\mathcal{G}} + L\delta_0)\| \leq C'\sqrt{1 + \omega_n^2}^\gamma$ for some $C' > 0$ and every $n \in \mathbb{Z}$,*

3. *There exists a unique $l \in \mathcal{G}$ such that $Lu = lu$ for every $u \in \mathbb{C}$ and $\langle l, \phi_n \rangle \neq 0$ for every $n \in \mathbb{Z}$.*

Here $(\phi_n)_{n \in \mathbb{Z}}$ denotes the orthonormal basis of (weighted) exponentials $c_n e^{i\omega_n \cdot} = \frac{e^{i\omega_n \cdot}}{\sqrt{1+\omega_n^2}^{\alpha}}$, $n \in \mathbb{Z}$, for $\mathcal{G}$, which are also the eigenvectors of $S|_{\mathcal{G}}$ corresponding to the eigenvalues $i\omega_n$.

Remark 6.69. The above strongly stabilizing feedback L for the pair $(S|_{\mathcal{G}}, \delta_0)$ is constructed in the proof of Proposition 4.28; see in particular (4.65).

The following is the main result of this section.

Theorem 6.70. *Let $\mathcal{G}$, γ and L be as in Lemma 6.68. Let $\beta > \gamma + \alpha + \epsilon$ for arbitrary $\epsilon > \frac{1}{2}$. Assume that there exist $\Pi \in \mathcal{L}(\mathcal{G}, Z)$ and $\Gamma \in \mathcal{L}(\mathcal{G}, \mathbb{C})$ such that $\Pi(\mathcal{D}(S|_{\mathcal{G}})) \subset \mathcal{D}(A)$ and the following regulator equations are satisfied:*

$$A\Pi + B\Gamma = \Pi S|_{\mathcal{G}} \quad in \ \mathcal{D}(S|_{\mathcal{G}}) \tag{6.108a}$$

$$C\Pi = \delta_0 \quad in \ \mathcal{G} \tag{6.108b}$$

Let $X = Z \times \mathcal{G}$ and set $F = \left(\begin{smallmatrix} A & B\Gamma \\ -LC & S|_{\mathcal{G}}+L\delta_0 \end{smallmatrix} \right)$, $J = \left(\begin{smallmatrix} 0 & \Gamma \end{smallmatrix} \right)$ and $G = \left(\begin{smallmatrix} 0 \\ L \end{smallmatrix} \right)$. Then the controller (4.1) with these parameters solves the EFRP for every $P \in \mathcal{L}(\mathcal{G}, Z)$ and each $Q \in \mathcal{L}(\mathcal{G}, \mathbb{C})$ in the exosystem, in the following sense: Output regulation is conditionally robust with respect to such perturbations to A, B, C, G and J for which the corresponding perturbation $\Delta_{\mathcal{A}}$ to the closed loop operator $\mathcal{A} = \left(\begin{smallmatrix} A & BJ \\ GC & F \end{smallmatrix} \right)$ satisfies $\sup_{n \in \mathbb{Z}} \|\Delta_{\mathcal{A}} R(i\omega_n, \mathcal{A})\| < 1$.

Proof. By Assumption 6.66 and Lemma 6.68 all conditions of Assumption 6.50 are satisfied for $P_a = 0$ and $Q_a = \delta_0 \in \mathcal{L}(\mathcal{G}, \mathbb{C})$. In particular, $\left(\begin{smallmatrix} A & 0 \\ 0 & S|_{\mathcal{G}} \end{smallmatrix} \right) - \left(\begin{smallmatrix} 0 \\ L \end{smallmatrix} \right) \left(\begin{smallmatrix} C & -\delta_0 \end{smallmatrix} \right)$ generates a strongly stable C_0−semigroup on $Z \times \mathcal{G}$, and in the notation of Lemma 6.68 and Assumption 6.50 we have that $P_{i\omega_n}^{S|_{\mathcal{G}}} Lu = \langle l, \phi_n \rangle \phi_n u$ for all $u \in \mathbb{C}$ and $n \in \mathbb{Z}$, so that $P_{i\omega_n}^{S|_{\mathcal{G}}} L$ is injective because $\langle l, \phi_n \rangle \neq 0$ for all $n \in \mathbb{Z}$. Thus by Lemma 6.52 the closed loop operator $\mathcal{A}$ generates a strongly stable C_0−semigroup on $Z \times X$. Moreover, a controller (4.1) with the above parameters has the internal model structure.

Applying the procedure in the proof of Theorem 4.15 we immediately see that the closed loop operator

$$\mathcal{A} = \begin{pmatrix} A & BJ \\ GC & F \end{pmatrix} = \begin{pmatrix} A & 0 & B\Gamma \\ 0 & A & B\Gamma \\ LC & -LC & S|_{\mathcal{G}}+L\delta_0 \end{pmatrix} \tag{6.109}$$

is similar to the operator $\widetilde{\mathcal{A}}$ given by

$$\widetilde{\mathcal{A}} = \begin{pmatrix} A & 0 & -B\Gamma \\ 0 & A & 0 \\ 0 & -LC & S|_{\mathcal{G}} + L\delta_0 \end{pmatrix} = \begin{pmatrix} A & -BJ \\ 0 & \mathcal{A}_s \end{pmatrix} \tag{6.110}$$

where we have defined $\mathcal{A}_s = \begin{pmatrix} A & 0 \\ -LC & S|_{\mathcal{G}}+L\delta_0 \end{pmatrix}$. By the triangular structure of the operators $\mathcal{A}_s$ and $\widetilde{\mathcal{A}}$, by the exponential stability of $T_A(t)$ and by Lemma 6.68, $\{\, i\omega_n \mid n \in \mathbb{Z} \,\} \subset \rho(\mathcal{A})$. Moreover, a direct calculation utilizing the triangular structure of $\mathcal{A}_s$ and the second item of Lemma 6.68 shows that $\|R(i\omega_n, \mathcal{A}_s)\| \leq M\sqrt{1+\omega_n^2}^{\gamma}$ for some $M > 0$ and all $n \in \mathbb{Z}$. Similarly, $\|R(i\omega_n, \widetilde{\mathcal{A}})\| \leq M_a\sqrt{1+\omega_n^2}^{\gamma}$ for some $M_a > 0$ and all $n \in \mathbb{Z}$. Finally, by similarity also $\|R(i\omega_n, \mathcal{A})\| \leq M'\sqrt{1+\omega_n^2}^{\gamma}$ for some $M' > 0$ and all $n \in \mathbb{Z}$. Then

$$\sum_{n\in\mathbb{Z}} \frac{\|R(i\omega_n, \mathcal{A})\|^2 (1+\omega_n^2)^{\alpha}}{(1+\omega_n^2)^{\beta}} \leq M'^2 \sum_{n\in\mathbb{Z}} \frac{(1+\omega_n^2)^{\alpha+\gamma}}{(1+\omega_n^2)^{\beta}} < \infty \tag{6.111}$$

The result now follows by Corollary 6.33 with $f_n = \sqrt{1+\omega_n^2}^{\beta}$ and $g_n = \sqrt{1+\omega_n^2}^{\alpha}$. $\square$

Remark 6.71. The logic and order behind the choice of the scalar parameters α, β and γ in Theorem 6.70 is this:

1. $\alpha > \frac{1}{2}$ is chosen to be sufficiently large to obtain bounded solutions to the regulator equations (6.108) (see e.g. condition (3.55) in Section 3.5); this choice of α also fixes $\mathcal{G}$.

2. γ is any real number larger than $\alpha + \frac{1}{2}$ (the smaller the better).

3. β is any real number larger than $\alpha + \gamma + \epsilon$ for $\epsilon > \frac{1}{2}$; this parameter β fixes $\mathcal{H}$ and hence the degree of smoothness required for the reference signals.

Remark 6.72. It is crucial in Theorem 6.70 that the closed loop system contains a stabilized copy of the differential operator $S|_{\mathcal{G}}$ on a *larger* space $\mathcal{G} = H^{\alpha}_{per}(0,p)$ than the space $\mathcal{H} = H^{\beta}_{per}(0,p)$ on which we require output regulation. Thus, here sufficient smoothness of the exogenous signals (with respect to the pivot space $\mathcal{G}$) implies a degree of conditional robustness in output regulation.

Remark 6.73. The regulator equations (6.108) can be solved using the methods of Chapter 8 or Section 3.5. In particular, the condition (3.55) completely characterizes the solvability of these equations if there are no transmission zeros of the plant on $\{\, i\omega_n \mid n \in \mathbb{Z} \,\}$.

6.8 Some concrete examples

The purpose of the present section is to provide some simple but concrete examples of (conditionally) robust output regulation for infinite-dimensional systems.

Example 6.74. *Consider the same disturbance-free controlled one-dimensional heat equation on the interval $[0,1]$, with Neumann boundary conditions, as in Example 1.1. In [12] (Example VI.2) Byrnes et al. designed a dynamic error feedback controller (4.1) such that the output of this heat plant asymptotically tracks the reference signal $y_{ref}(t) = \sin(2t)$ for all initial states of the plant. This controller is explicitly reproduced in the item (iv) of Example 1.1.*

However, Byrnes et al. [12] did not study the robustness of their controller, so we shall here provide an addendum to Example 1.1 and Example VI.2 in [12] by proving its robustness. Observe that the items 1-5 in Assumption 6.50 are satisfied for $W = W_a$, $S_a = S$ and $P = 0$, because evidently S generates an isometric C_0-group on W, $\begin{pmatrix} A & 0 \\ 0 & S \end{pmatrix} - \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}\begin{pmatrix} C & -Q \end{pmatrix}$ generates an exponentially stable C_0-semigroup and

$$\Pi S = A\Pi + B\Gamma \quad in \ W \tag{6.112a}$$

$$C\Pi = Q \quad in \ W \tag{6.112b}$$

(see [12] for more details). In order to be able to apply Corollary 6.54 we must show that $P_{i\gamma}^S G_2 : \mathbb{R} \to \mathrm{ran}(P_{i\gamma}^S)$ is injective for $\gamma = \pm 2$. But for $\gamma = 2$ and for all $u \in \mathbb{R}$ we have

$$P_{i\gamma}^S G_2 u = \frac{1}{2\pi i} \oint_\gamma R(\lambda, S) G_2 d\lambda u \tag{6.113}$$

$$= \frac{1}{2\pi i} \oint_\gamma \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda+i2} & 0 \\ 0 & \frac{1}{\lambda-i2} \end{pmatrix} \begin{pmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -3 \\ -3 \end{pmatrix} d\lambda u \tag{6.114}$$

$$= \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{3}{2} - \frac{3}{2}i \end{pmatrix} u \tag{6.115}$$

$$= \begin{pmatrix} -\frac{3}{2} + \frac{3}{2}i \\ -\frac{3}{2} - \frac{3}{2}i \end{pmatrix} u \tag{6.116}$$

and similarly for $\gamma = 2$ and all $u \in \mathbb{R}$

$$P_{i\gamma}^S G_2 u = \frac{1}{2\pi i} \oint_\gamma R(\lambda, S) G_2 d\lambda u = \begin{pmatrix} -\frac{3}{2} - \frac{3}{2}i \\ -\frac{3}{2} + \frac{3}{2}i \end{pmatrix} u \tag{6.117}$$

so that indeed $P_{i\gamma}^S G_2 : \mathbb{R} \to \mathrm{ran}(P_{i\gamma}^S)$ is injective for $\gamma = \pm 2$. In conclusion all items in Assumption 6.50 are satisfied. Since the closed loop system is exponentially stable, robust output regulation as described in Corollary 6.54 occurs. We remark, in particular, that the dynamic controller of Example 1.1 studied above can also be used to simultaneously reject disturbances $Pw(t)$ generated by the above exosystem; this feature is not pointed out in [12] where the controller was first introduced.

Example 6.75. Let $a > 0$, $r \neq 0$, $\tau_1 > \tau_2 > 0$ and consider the disturbance-free scalar delay differential equation

$$\dot{x}(t) = -ax(t) - b\big[x(t - \tau_1) + x(t - \tau_2)\big] + u(t) \tag{6.118a}$$

$$y(t) = rx(t), \quad t \geq 0 \tag{6.118b}$$

of Example 3.54 and Example 4.43. We assume that the system operator of (6.118) generates an exponentially stable C_0−semigroup as in Example 3.54 and Example 4.43. Then there are no transmission zeros of the plant on the imaginary axis.

Let $W_a = \mathcal{G} = H_{per}^{\alpha}(0, p)$ for $\alpha = \frac{5}{3}$ and $p > 0$. We can then easily solve the regulator equations (6.108) for $\Pi \in \mathcal{L}(W_a, Z)$ and $\Gamma \in \mathcal{L}(W_a, \mathbb{C})$ as in Section 3.5. Let the stabilizing operator L be as in Lemma 6.68 for $\gamma = \alpha + 1$. Then by Theorem 6.70 the dynamic controller (4.1) with $X = Z \times \mathcal{G}$ and parameters

$$F = \begin{pmatrix} A & B\Gamma \\ -LC & S|_{\mathcal{G}} + L\delta_0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & \Gamma \end{pmatrix} \quad and \quad G = \begin{pmatrix} 0 \\ L \end{pmatrix} \tag{6.119}$$

achieves the asymptotic tracking of all reference signals in $H_{per}^{\beta}(0, p)$ for (say) $\beta > 5$. Moreover, output regulation is conditionally robust with respect to certain admissible perturbations, as described in Theorem 6.70.

Example 6.76. Consider the controlled and observed weakly damped SISO string system described in Example 3.56. Assume that the reference signals to be asymptotically tracked are in some finite-dimensional Sobolev space $\mathcal{H} = H_{AP}(\mathbb{C}, f_n, \omega_n)$, with $(f_n)_{n \in I} \subset \mathbb{R}$, $(\omega_n)_{n \in I} \subset \mathbb{R}$ for some finite set I of indices. Let $\phi_n(x) = f_n^{-1} e^{i\omega_n x}$ for all $x \in \mathbb{R}$ and $n \in I$. It is clear that $(\phi_n)_{n \in I}$ constitutes an orthonormal basis in $\mathcal{H}$.

Assume that $H(i\omega_n) = B^* R(i\omega_n, A)B \neq 0$ for all $n \in I$. Let us take $W = \mathcal{H}$ and $S = S|_{\mathcal{H}}$ as

in Proposition 2.3, and define

$$G_0 = \sum_{n \in I} f_n^{-1} H(i\omega_n)^{-1} \phi_n \in \mathcal{L}(\mathbb{C}, \mathcal{H}) \tag{6.120}$$

$$\Pi_0 = \sum_{n \in I} R(-i\omega_n, A^*) B G_0^* \phi_n \langle \cdot, \phi_n \rangle \in \mathcal{L}(\mathcal{H}, Z) \tag{6.121}$$

where * *denotes the operator adjoint (recall that* Z *and* $\mathcal{H}$ *are both Hilbert spaces). Then it is a straightforward calculation to show that* $\Pi_0(\mathcal{H}) \subset \mathcal{D}(A^*)$ *and*

$$-\Pi_0 S|_{\mathcal{H}} = A^* \Pi_0 + B G_0^* \quad in \ \mathcal{H} \tag{6.122a}$$

$$B^* \Pi_0 = \delta_0 \quad in \ \mathcal{H} \tag{6.122b}$$

whence for $\Pi = \Pi_0^* \in \mathcal{L}(Z, \mathcal{H})$ *the following regulator equations are satisfied:*

$$S|_{\mathcal{H}} \Pi = \Pi A + G_0 B^* \quad in \ \mathcal{D}(A) \tag{6.123a}$$

$$\Pi B = \delta_0^* \quad in \ \mathbb{C} \tag{6.123b}$$

because $S|_{\mathcal{H}}^* = -S|_{\mathcal{H}}$.

A regulating Davison-type dynamic state feedback controller is then given on the state space $X = \mathcal{H}$ *by the equations*

$$\dot{x}(t) = S|_{\mathcal{H}} x(t) + G_0 e(t), \quad x(0) \in \mathcal{H} \tag{6.124a}$$

$$u(t) = -\delta_0 \Pi z(t) - \delta_0 x(t) \tag{6.124b}$$

In fact, since $D = 0$ *the resulting closed loop system operator* $\mathcal{A} = \begin{pmatrix} A - B\delta_0\Pi & -B\delta_0 \\ G_0 B^* & S|_{\mathcal{H}} \end{pmatrix}$ *satisfies*

$$\begin{pmatrix} I & 0 \\ \Pi & I \end{pmatrix} \begin{pmatrix} A - B\delta_0\Pi & -B\delta_0 \\ G_0 B^* & S|_{\mathcal{H}} \end{pmatrix} \begin{pmatrix} I & 0 \\ -\Pi & I \end{pmatrix} = \begin{pmatrix} A & -B\delta_0 \\ 0 & S|_{\mathcal{H}} - \delta_0^* \delta_0 \end{pmatrix} \tag{6.125}$$

Hence by similarity $\mathcal{A}$ *generates a strongly stable* C_0-*semigroup on* $Z \times X$*, because* A *generates a strongly stable* C_0-*semigroup on* Z *(see Example 3.56) and because* $S|_{\mathcal{H}} - \delta_0^* \delta_0$ *generates an exponentially stable* C_0-*semigroup on* $\mathcal{H}$ *(by Theorem 4.22 and the fact that* $\dim(\mathcal{H}) < \infty$*). Moreover, since* $P_{i\omega_n}^{S|_{\mathcal{H}}} G_0 u = f_n^{-1} H(i\omega_n)^{-1} \phi_n u$ *for all* $u \in \mathbb{C}$ *and all* $n \in I$*, Assumption 6.57 holds true. Finally, by similarity and the above triangular structure,* $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$*, because* $\sigma(A) \cap i\mathbb{R} = \emptyset$ *and* $\sigma(S|_{\mathcal{H}} - \delta_0^* \delta_0) \cap i\mathbb{R} = \emptyset$*. Lemma 6.59 and Corollary 6.26 then show that the controller* (6.124) *solves the EFRP for a plant (which is also subject to the state feedback* $-\delta_0\Pi$*) for all* $P \in \mathcal{L}(\mathcal{H}, Z)$

and all $Q \in \mathcal{L}(\mathcal{H}, H)$ in such a way that output regulation is conditionally robust with respect to those perturbations to A, B, C, G_0, Π and δ_0 for which the corresponding perturbation $\Delta_{\mathcal{A}}$ to the closed loop operator $\mathcal{A}$ satisfies $\|\Delta_{\mathcal{A}}\| < \min_{\lambda \in \sigma(S_{\mathcal{H}})} \|R(\lambda, \mathcal{A})\|^{-1} = \min_{n \in I} \|R(i\omega_n, \mathcal{A})\|^{-1}$.

Chapter 7

Practical output regulation

Since virtually all real world control systems are subject to unpredictable disturbances and uncertainties, exact output regulation is, strictly speaking, very often beyond reach in practice. For infinite-dimensional systems matters are made worse by the fact that all computer simulations must be conducted using finite-dimensional approximations, e.g. as in the finite element analysis of partial differential equations.

Fortunately, one can often achieve a satisfactory degree of accuracy in output regulation in practice using robust controllers, as is now well known. However, certain issues should be borne in mind when considering the application of robust controllers in output regulation problems. First of all, as we have seen in Chapter 6, a robust controller often utilizes a suitable reduplication of the exosystem operator S — or at least some operator S_a with similar properties — but this part of the controller cannot sustain any perturbations if output regulation is to be maintained (see [32] for a discussion of this topic). Secondly, the more general the reference/disturbance signals are, the more difficult it is to design error feedback controllers achieving even conditionally robust output regulation, because sufficient closed loop stability becomes more difficult to achieve. Finally, a robust controller which solves an output regulation problem often explicitly incorporates some solutions (Π, Γ) of the regulator equations (3.10) (see e.g. Subsection 6.5.1). However, in the most general setup of this thesis the regulator equations (3.10) are operator equations on infinite-dimensional spaces. In practice they can only be solved approximately — a fact that can introduce intolerable uncertainty in the control system, especially if the robustness margin (i.e. the degree of tolerance for perturbations) for the controller is small or unknown.

Because of the above issues, in applications it is often sensible to content oneself with *practical* output regulation instead of explicitly requiring *exact* output regulation. In practical output regulation the goal is to achieve asymptotic tracking/rejection of the exogenous signals with a given accuracy. More specifically, if $e(t)$ denotes the corresponding tracking error (in the presence of disturbances) and if $\epsilon > 0$ is the desired tracking accuracy, then the goal in practical output regulation is to have $\limsup_{t\to\infty}\|e(t)\| < M\epsilon$ where $M \geq 0$ only depends on the particular reference/disturbance signals for which practical regulation is desired, i.e. the initial condition of the exosystem. We have actually already touched upon this topic in Section 6.6 where we explicitly designed robust (EFRP) controllers which approximately regulate all reference signals in a given infinite-dimensional Sobolev space $H_{AP}(H, f_n, \omega_n)$.

For finite-dimensional *nonlinear* systems much research has been devoted to the practical output regulation problem, because it turns out to be considerably simpler and more convenient to solve than its exact counterpart. This is largely due to the fact that the so-called nonlinear regulator equations — whose solvability in part guarantees output regulation for nonlinear systems — are difficult to solve precisely [71]. A survey of various approximative methods for the solution of the nonlinear regulator equations and their use in practical output regulation can be found in [91], while some general existence results for the nonlinear practical output regulation problem can be found in [75].

On the other hand, only little research seems to have been reported on practical output regulation specifically for *linear* systems. This is quite surprising, because the linear problem is not a trivial one even for finite-dimensional systems — although in this case the above cited nonlinear theory is of course applicable, and although in this case robustness sometimes guarantees even exact output regulation. Vast majority of the knowledge related to practical output regulation for linear infinite-dimensional systems seems to exist in the form of model reduction techniques (see e.g. [16, 68] and the references therein). This is quite natural, because model reduction deals with one of the key sources of model uncertainty in practice, namely that of approximating infinite-dimensional systems by finite-dimensional ones. As regards other related research, we mention that practical output regulation of general $p-$periodic $(p > 0)$ signals for linear finite-dimensional systems has been studied in [36, 92] as "modified" repetitive control problems (see Chapter 1 for more details).

In this chapter we shall develop the mathematical foundations of practical output regulation for

exponentially stabilizable linear state space control systems, both finite-dimensional and infinite-dimensional. Instead of directly designing controllers which achieve practical output regulation with a desired accuracy[1], our approach here is to *assume* that there already exists a (hypothetical) exactly regulating controller, which we may not be able to construct in practice due to modelling errors et cetera. Under exponential closed loop stability this existence assumption can, as we have seen in the previous chapters, be reduced to the solvability of the regulator equations (3.10) or the extended regulator equations (4.3), depending on the controller type in question. Our idea is then to directly employ a perturbation analysis to the closed loop control system and the corresponding (extended) regulator equations which in a certain sense describe the system's *steady state* behaviour. Our main results in the present chapter are general upper bounds for the norms of additive, bounded, linear perturbations to the the parameters of the plant, the exosystem and the (hypothetical) controller, which solves the corresponding exact output regulation problem, such that practical output regulation with a given accuracy ϵ occurs. Our results cover in a unified way practical output regulation for the FRP, the EFRP and the FFRP, which have been studied in detail in the previous chapters.

We shall next review the contents of this chapter in more detail. However, we emphasize that the results of this chapter are difficult to compare to the existing literature, because our operator-theoretic approach seems to be entirely new even in the output regulation theory of linear finite-dimensional systems. The results of this chapter are based on those in [44, 50].

Section 7.1: We shall first recall how the output regulation problems FRP, EFRP and FFRP can be studied as output stabilization problems for certain triangular dynamical systems on products of Banach spaces. Thereafter in Theorem 7.2 we shall prove an abstract perturbation result which can be used to study practical output stability of such triangular systems.

Section 7.2: We shall show that the abstract perturbation result, Theorem 7.2, of Section 7.1 can readily be used in practical output regulation. In particular:

- In Corollary 7.7 we shall establish such upper bounds for the norms of the perturbations to the parameters A, B, C, D, P, Q, K and L of the closed loop feedforward control system (3.1) which guarantee that practical (FRP) output regulation with a desired

---

[1]As is usually done in the nonlinear systems literature, see e.g. [91], and as was done in the case study of Section 6.6 of this thesis.

accuracy $\epsilon > 0$ occurs.

- In Corollary 7.8 we shall establish such upper bounds for the norms of the perturbations to the parameters $\mathcal{A}, \mathcal{P}$, Q and $\mathcal{C}$ of the closed loop system (4.2) (see also the proof of Theorem 4.4) which guarantee that practical (EFRP) output regulation with a desired accuracy $\epsilon > 0$ occurs.

- In Corollary 7.9 we shall establish such upper bounds for the norms of the perturbations to the parameters $\mathcal{A}, \mathcal{P}_\Gamma$, Q, Γ and $\mathcal{C}$ of the closed loop system (5.2) (see also the proof of Theorem 5.3) which guarantee that practical (FFRP) output regulation with a desired accuracy $\epsilon > 0$ occurs.

Section 7.3: We shall consider two illustrative applications of the results of Section 7.2, namely:

- In Subsection 7.3.1 we shall study quantitatively the effect of bounded linear additive perturbations to the internal model of the exogenous signals, as utilized in the robust error feedback controllers of Subsection 6.5.1. It should be pointed out that the robustness results of Chapter 6 do not allow for perturbations in F if *exact* output regulation is to be maintained; however this example shows that *practical* output regulation can sometimes be achieved even if the critical part of F is subject to perturbations.

- In Subsection 7.3.2 we shall study practical periodic tracking/disturbance rejection in the sense of the FRP for exponentially stabilizable SISO systems and reference signals in the generalized Sobolev spaces $H(f_n, \omega_n)$, using Proposition 2.3. In Section 3.5 (see in particular (3.47)) we derived a series expansion for the operator L in a regulating feedforward control law $u(t) = Kz(t) + Lw(t)$. Given a desired asymptotic tracking accuracy $\epsilon > 0$, here we shall establish how many terms should be included in the *truncation* of this series expansion for L such that practical output regulation with accuracy ϵ occurs.

In general, the results of this chapter improve the existing ones in the following ways:

- The source of uncertainties and perturbations to the parameters of the plant, the exosystem and the controller is irrelevant here, as opposed to e.g. specific techniques for model reduction where uncertainty is a result of finite-dimensional approximation.

- Our results cover practical output regulation for several different controller configurations, as opposed to [36, 92] which only cover practical error feedback regulation.

- Our results allow for the use of approximate solutions of the regulator equations (3.10) in the controller, as opposed to [7, 12, 29, 80] where exact solutions are needed for exact output regulation. Moreover, here it is irrelevant how the regulator equations (3.10) are approximated.

- Our results allow for arbitrary bounded uniformly continuous reference/disturbance signals generated by the exosystem (2.2), as opposed to $p-$periodic signals in [36, 92] and trigonometric polynomial signals in [12].

- Our results can be used to establish practical output regulation with a desired accuracy in the case that the internal model of the exogenous signals in the system operator F of a robust controller is subject to perturbations. In the related existing work it is well-known that this internal model cannot be perturbed if *exact* output regulation is to be maintained; however, no general quantitative information on the control systems' dynamical behaviour seems to exist even for finite-dimensional systems, in the case that the internal model *is* perturbed [7, 24, 29, 32, 33]. For repetitive control systems such information does exist e.g. in [36, 92].

- In the literature related to the approximate solution of the regulator equations, see e.g. [50, 91], the plant data need not be explicitly known in the controller design, but the resulting (approximate) controller is *only* guaranteed to achieve practical output regulation for a plant whose parameters *are* explicitly known (i.e. they are at their nominal values). Our general results can be used to guarantee practical output regulation in the case that a controller employing approximate solutions of the regulator equations is applied to a plant which has uncertain parameters and which is subject to uncertain disturbances. To the author's knowledge this has not been possible in any related earlier work, linear or nonlinear.

However, it should be pointed out that, as in Chapter 6, throughout this chapter we shall only consider bounded perturbations to the control system's parameters. This may be restrictive in some applications, because certain parts of the control system can in practice be subject to unbounded perturbations too. Moreover, here we confine our attention to exponentially stable closed loop

systems only[2], which may restrict the applicability of the results of this chapter to the EFRP if the exosystem (2.2) is not finite-dimensional.

7.1 Practical output stability of triangular systems

Let X_1, X_2 and H be Banach spaces, let A_1 generate an exponentially stable C_0-semigroup $T_{A_1}(t)$ on X_1, let A_2 generate a uniformly bounded C_0-group $T_{A_2}(t)$ on X_2 and let $A_3 \in \mathcal{L}(X_2, X_1)$. Furthermore, let $C_1 \in \mathcal{L}(X_1, H)$, let $C_2 \in \mathcal{L}(X_2, H)$ and consider the following dynamical system on $X_1 \times X_2$:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad t \geq 0, \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} \in X_1 \times X_2 \qquad (7.1a)$$

$$e(t) = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad t \geq 0 \qquad (7.1b)$$

in the mild sense. The system (7.1) is said to be output stable if $\lim_{t \to \infty} e(t) = 0$ for all $x_1(0) \in X_1$ and all $x_2 \in X_2(0)$.

As we have seen in the previous chapters, in many output regulation problems the extended system consisting of the plant and the controller, with the exogenous signals present, can be described by such a triangular dynamical system (see Theorem 3.6, Theorem 4.4 and Theorem 5.3). For example, in the case of the FRP, we can choose $X_1 = Z$, $X_2 = W$, $A_1 = A + BK$, $A_2 = S$, $A_3 = BL + P$ etc. provided that $A + BK$ generates an exponentially stable C_0-semigroup. In such cases the output stability of the triangular system in question means asymptotic tracking of the reference signals in the presence of disturbances generated by the exosystem (2.2).

Now let us apply certain perturbations to the system (7.1). More specifically, let $\Delta_{A1} \in \mathcal{L}(X_1), \Delta_{A3} \in \mathcal{L}(X_2, X_1), \Delta_{C1} \in \mathcal{L}(X_1, H)$ and $\Delta_{C2} \in \mathcal{L}(X_2, H)$ and consider the perturbed dynamical system described by the equations

$$\begin{pmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{pmatrix} = \begin{pmatrix} A_1 + \Delta_{A1} & A_3 + \Delta_{A3} \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad t \geq 0, \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} \in X_1 \times X_2 \qquad (7.2a)$$

$$\tilde{e}(t) = \begin{pmatrix} C_1 + \Delta_{C1} & C_2 + \Delta_{C2} \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad t \geq 0 \qquad (7.2b)$$

---

[2]See Remark 7.6 for the motivation behind this choice.

in the mild sense. It is clear by the above that if we can prove a general perturbation result which gives upper bounds for the norms $\|\Delta_{A1}\|, \|\Delta_{A3}\|, \|\Delta_{C1}\|$ and $\|\Delta_{C2}\|$ such that $\limsup_{t\to\infty}\|\widetilde{e}(t)\| < M\epsilon$ where M only depends on the initial state $y_2(0)$ and $\epsilon > 0$ is given, then that result would immediately yield practical output regulation results for the FRP, the EFRP and the FFRP. The main result of this section is precisely such a general perturbation theorem. In order to prove it, we shall need the following addendum to Lemma 3.5, which is easy to prove using the techniques developed in this thesis; we leave the details to the reader.

Lemma 7.1. *Under the above assumptions, if there exists* $\Pi \in \mathcal{L}(X_2, X_1)$, *with* $\Pi(\mathcal{D}(A_2)) \subset \mathcal{D}(A_1)$, *such that the following operator equations are satisfied:*

$$\Pi A_2 = A_1\Pi + A_3 \quad in\ \mathcal{D}(A_2) \tag{7.3a}$$

$$0 = C_1\Pi + C_2 \quad in\ X_2 \tag{7.3b}$$

then $\lim_{t\to\infty} e(t) = 0$ *for all initial states* $x_1(0) \in X_1$ *and* $x_2(0) \in X_2$.

The following is the main result of this section.

Theorem 7.2. *Assume that* $\Pi \in \mathcal{L}(X_2, X_1)$ *is such that* $\Pi(\mathcal{D}(A_2)) \subset \mathcal{D}(A_1)$ *and the operator equations (7.3) are satisfied. Let* $\|T_{A_1}(t)\| \le Me^{-\omega t}$ *for some* $M \ge 1$ *and* $\omega > 0$ *and all* $t \ge 0$. *Let* $\epsilon > 0$ *be given and let* $0 \le a < \frac{\omega}{MN}$ *where* $1 \le N = \sup_{t\in\mathbb{R}}\|T_{A_2}(t)\| < \infty$. *Then whenever the above perturbations satisfy*

$$\|\Delta_{A1}\| \le a \tag{7.4a}$$

$$\frac{MN}{\omega}\|\Delta_{A3}\| + \frac{M^2N^2a}{\omega^2}\left(1 - \frac{MNa}{\omega}\right)^{-1}\|A_3 + \Delta_{A3}\| < \frac{\epsilon}{3(1 + \|C_1 + \Delta_{C1}\|)} \tag{7.4b}$$

$$\|\Delta_{C1}\| < \frac{\epsilon}{3(1 + \|\Pi\|)} \tag{7.4c}$$

$$\|\Delta_{C2}\| < \frac{\epsilon}{3} \tag{7.4d}$$

we have $\limsup_{t\to\infty}\|\widetilde{e}(t)\| < \epsilon N\|y_2(0)\|$ *for every* $y_1(0) \in X_1$ *and every* $y_2(0) \in X_2$.

Proof. Let us define the linear Sylvester operator $\mathcal{T}_{A_1,A_2}$ on a subspace of $\mathcal{L}(X_2, X_1)$ by

$$\mathcal{D}(\mathcal{T}_{A_1,A_2}) = \{\, \Lambda \in \mathcal{L}(X_2, X_1) \mid \Lambda(\mathcal{D}(A_2)) \subset \mathcal{D}(A_1), \exists Y \in \mathcal{L}(X_2, X_1) :$$

$$Yx = A_1\Lambda x - \Lambda A_2 x\ \forall x \in \mathcal{D}(A_2)\,\} \tag{7.5}$$

$$\mathcal{T}_{A_1,A_2}\Lambda = Y \tag{7.6}$$

Since $T_{A_1}(t)$ is (by assumption) exponentially stable and since $T_{A_2}(t)$ is a uniformly bounded group, the operator equation (7.3a) has a unique solution for each $A_3 \in \mathcal{L}(X_2, X_1)$ (see [88, 90] and Section A.2). Thus the operator $\mathcal{T}_{A_1,A_2}$ is a closed bijection $\mathcal{D}(\mathcal{T}_{A_1,A_2}) \to \mathcal{L}(X_2, X_1)$ and $0 \in \rho(\mathcal{T}_{A_1,A_2})$ [3, 88]. Moreover, $\Pi = -\mathcal{T}_{A_1,A_2}^{-1}A_3$, and by Corollary 8 in [88] we must have

$$\|\mathcal{T}_{A_1,A_2}^{-1}\| = \sup_{\|A_3\|=1} \sup_{\|x\|=1} \left\| \int_0^\infty T_{A_1}(t)A_3 T_{-A_2}(t)x\,dt \right\| \leq \frac{MN}{\omega} \tag{7.7}$$

Next define the operator $\Delta \in \mathcal{L}(\mathcal{L}(X_2, X_1))$ such that $\Delta\Lambda = \Delta_{A1}\Lambda$ for each $\Lambda \in \mathcal{L}(X_2, X_1)$. Obviously we may consider the perturbed Sylvester operator $\mathcal{T}_{A_1+\Delta_{A1},A_2}$ as $\mathcal{T}_{A_1+\Delta_{A1},A_2}\Lambda = (A_1 + \Delta_{A1})\Lambda - \Lambda A_2 = \mathcal{T}_{A_1,A_2}\Lambda + \Delta\Lambda$ for each $\Lambda \in \mathcal{D}(\mathcal{T}_{A_1,A_2})$. Now $\|\Delta\| \leq \|\Delta_{A_1}\| \leq a < \frac{\omega}{MN}$, and $a\|\mathcal{T}_{A_1,A_2}^{-1}\| < 1$. Consequently, by Theorem IV.1.16 in [57] (p. 196) it is true that $0 \in \rho(\mathcal{T}_{A_1+\Delta_{A1},A_2})$ and that $\|\mathcal{T}_{A_1+\Delta_{A1},A_2}^{-1} - \mathcal{T}_{A_1,A_2}^{-1}\| \leq \frac{a\|\mathcal{T}_{A_1,A_2}^{-1}\|^2}{1-a\|\mathcal{T}_{A_1,A_2}^{-1}\|}$. Thus whenever the above inequalities (7.4) are satisfied, there exists a unique $\widetilde{\Pi} \in \mathcal{L}(X_2, X_1)$ such that $\widetilde{\Pi}(\mathcal{D}(A_2)) \subset \mathcal{D}(A_1)$ and $\widetilde{\Pi}A_2 = (A_1 + \Delta_{A1})\widetilde{\Pi} + (A_3 + \Delta_{A3})$ in $\mathcal{D}(A_2)$. Moreover, necessarily $\widetilde{\Pi} = -\mathcal{T}_{A_1+\Delta_{A1},A_2}^{-1}(A_3 + \Delta_{A3})$, so that

$$\|\widetilde{\Pi} - \Pi\| = \|\mathcal{T}_{A_1,A_2}^{-1}A_3 - \mathcal{T}_{A_1+\Delta_{A1},A_2}^{-1}(A_3 + \Delta_{A3})\| \tag{7.8}$$

$$\leq \|\mathcal{T}_{A_1,A_2}^{-1}(A_3 - A_3 - \Delta_{A3})\| + \|[\mathcal{T}_{A_1,A_2}^{-1} - \mathcal{T}_{A_1+\Delta_{A1},A_2}^{-1}](A_3 + \Delta_{A3})\| \tag{7.9}$$

$$\leq \frac{MN}{\omega}\|\Delta_{A3}\| + \frac{a\|\mathcal{T}_{A_1,A_2}^{-1}\|^2}{1 - a\|\mathcal{T}_{A_1,A_2}^{-1}\|}\|A_3 + \Delta_{A3}\| \tag{7.10}$$

$$\leq \frac{MN}{\omega}\|\Delta_{A3}\| + \frac{aM^2N^2}{\omega^2}\left(1 - a\frac{MN}{\omega}\right)^{-1}\|A_3 + \Delta_{A3}\| \tag{7.11}$$

$$< \frac{\epsilon}{3}\left(1 + \|C_1 + \Delta_{C1}\|\right)^{-1} \tag{7.12}$$

by (7.4b). Hence also

$$\|(C_1 + \Delta_{C1})\widetilde{\Pi} + (C_2 + \Delta_{C2})\| \leq \|(C_1 + \Delta_{C1})(\widetilde{\Pi} - \Pi)\| + \|(C_1 + \Delta_{C1} - C_1)\Pi\| \tag{7.13}$$

$$+ \|C_1\Pi + C_2\| + \|-C_2 + C_2 + \Delta_{C2}\| \tag{7.14}$$

$$\leq \|C_1 + \Delta_{C1}\|\frac{\epsilon}{3}\left(1 + \|C_1 + \Delta_{C1}\|\right)^{-1} \tag{7.15}$$

$$+ \|\Delta_{C1}\|\|\Pi\| + \|\Delta_{C2}\| \tag{7.16}$$

$$< \epsilon \tag{7.17}$$

because $C_1\Pi + C_2 = 0$. As in the proof of Lemma 3.5 we then deduce that for every $y_1(0) \in X_1$

and every $y_2(0) \in X_2$

$$\limsup_{t \to \infty} \|\tilde{e}(t)\| = \limsup_{t \to \infty} \|(C_1 + \Delta_{C1})y_1(t) + (C_2 + \Delta_{C2})y_2(t)\| \tag{7.18}$$

$$\leq \limsup_{t \to \infty} \|(C_1 + \Delta_{C1})T_{A_1 + \Delta_{A1}}(t)[y_1(0) - \tilde{\Pi}y_2(0)]\| \tag{7.19}$$

$$+ \limsup_{t \to \infty} \|\big[(C_1 + \Delta_{C1})\tilde{\Pi} + (C_2 + \Delta_{C2})\big]T_{A_2}(t)y_2(0)\| \tag{7.20}$$

$$\leq \limsup_{t \to \infty} \|C_1 + \Delta_{C1}\|Me^{(-\omega + M\|\Delta_{A1}\|)t}\|y_1(0) - \tilde{\Pi}y_2(0)\| \tag{7.21}$$

$$+ N\|(C_1 + \Delta_{C1})\tilde{\Pi} + (C_2 + \Delta_{C2})\|\|y_2(0)\| \tag{7.22}$$

$$< \epsilon N\|y_2(0)\| \tag{7.23}$$

since $\|T_{A_1 + \Delta_{A1}}(t)\| \leq Me^{(-\omega + M\|\Delta_{A1}\|)t} \to 0$ as $t \to \infty$, by Theorem III.1.3 in [28]. $\square$

Remark 7.3. The bounds (7.4) are not optimized; the right hand sides of the inequalities (7.4b) and (7.4c) are chosen as such to accommodate the possibility that one or more of the relevant operators is zero. We also emphasize that if some of the perturbation operators are zero, then the factors $\frac{\epsilon}{3}$ can obviously be modified to take into account these changes (see Section 7.3 for some examples).

Remark 7.4. It is clear from the proof of Theorem 7.2 that whenever the perturbations satisfy the bounds (7.4), we have that

$$\|\tilde{e}(t)\| = \|(C_1 + \Delta_{C1})T_{A_1 + \Delta_{A1}}(t)[y_1(0) - \tilde{\Pi}y_2(0)] \tag{7.24}$$

$$+ \big[(C_1 + \Delta_{C1})\tilde{\Pi} + (C_2 + \Delta_{C2})\big]T_{A_2}(t)y_2(0)\| \tag{7.25}$$

$$\leq \|C_1 + \Delta_{C1}\|\|T_{A_1 + \Delta_{A1}}(t)\|\|y_1(0) - \tilde{\Pi}y_2(0)\| + \epsilon N\|y_2(0)\| \tag{7.26}$$

for all $y_1(0), y_2(0)$, and $t \geq 0$. Moreover, $\|T_{A_1 + \Delta_{A1}}(t)\| \leq Me^{(-\omega + M\|\Delta_{A1}\|)t}$ for all $t \geq 0$. These inequalities can be used to determine the smallest $t \geq 0$ for which $\|\tilde{e}(t)\| < 2\epsilon N\|y_2(0)\|$ (say). Observe that Theorem IV.1.16 in [57] provides the useful estimates $\|\tilde{\Pi}\| = \|\mathcal{T}_{A_1 + \Delta_{A1}, A_2}^{-1}(A_3 + \Delta_{A3})\| \leq \frac{\|\mathcal{T}_{A_1, A_2}^{-1}\|}{1 - a\|\mathcal{T}_{A_1, A_2}^{-1}\|}\|A_3 + \Delta_{A3}\| \leq \frac{MN}{\omega}(1 - \frac{MNa}{\omega})^{-1}\|A_3 + \Delta_{A3}\|$.

Remark 7.5. The reason why we do not allow for perturbations to A_2 in Theorem 7.2 is because often in output regulation problems we may let $X_2 = W$ and $A_2 = S$. In output regulation problems it is reasonable to assume that the exosystem dynamics is not perturbed or uncertain, because otherwise we could — up to a certain point — enlarge the state space W of the exosystem

to accommodate these perturbations. For example, in the setting of Proposition 2.3, if the period length of periodic reference signals is not exactly known, we can at least in principle take $W = \mathcal{H} = AP(\mathbb{R}, H)$ which contains *all* continuous periodic $H-$valued functions (regardless of the period length).

Remark 7.6. The reason why we require *exponential* stability of $T_{A_1}(t)$ in Theorem 7.2 is twofold. First of all, in this case it is possible to find an upper bound for the norms of additive bounded perturbations to the generator A_1 such that they do not destroy exponential stability of $T_{A_1}(t)$. Secondly, in this case we can *quantitatively* specify how much the solution operator Π of the Sylvester operator equation $\Pi A_2 = A_1 \Pi + A_3$ in $\mathcal{D}(A_2)$ is perturbed if A_1 and A_3 undergo small bounded perturbations. This would not have been possible if we had only required that A_1 generates a *strongly* stable C_0-semigroup[3]. However, we must acknowledge that the requirement that $T_{A_1}(t)$ is exponentially stable limits the applicability of the results of this chapter in the EFRP whenever the exosystem (2.2) is infinite-dimensional. This is because in the case of the EFRP, $T_{A_1}(t)$ will represent the closed loop semigroup, which may be impossible to stabilize exponentially if $\dim(W) = \infty$ (see Chapter 4).

7.2 Practical output regulation

In this section we shall apply the abstract perturbation result (Theorem 7.2) to certain practical output regulation problems. We shall treat the FRP, the EFRP and the FFRP separately, but in a unified way.

7.2.1 Practical feedforward output regulation

In this subsection we assume that there exist operators K and L which solve the FRP in such a way that the closed loop system (i.e. $T_{A+BK}(t)$) is exponentially stable. Our aim is to apply Theorem 7.2 to obtain bounds for the norms of the perturbations to the parameters A, B, C, D, K, L, P and Q of the plant, the controller and the exosystem for which practical output regulation with a desired accuracy occurs. In the following we indicate the perturbed operators by primes, and all perturbations (denoted by Δ_R, with $R = A, B, C, D, K, L, P$ or Q) are assumed to be bounded,

---

[3]In fact, in this case even the solvability of the Sylvester operator equation $\Pi A_2 = A_1 \Pi + A_3$ in $\mathcal{D}(A_2)$ under small perturbations to A_1 and A_3 is not guaranteed [90].

linear and additive. For example, $A' = A + \Delta_A$ for some $\Delta_A \in \mathcal{L}(Z)$. Moreover, $\widetilde{y}(t)$ denotes the output of the perturbed closed loop system.

Corollary 7.7. *Assume that $u(t) = Kz(t) + Lw(t)$ solves the FRP such that $L = \Gamma - K\Pi$ where $\Pi \in \mathcal{L}(W, Z)$ and $\Gamma \in \mathcal{L}(W, H)$ solve the regulator equations (3.10). Let $\|T_{A+BK}(t)\| \le Me^{-\omega t}$ for some $M \ge 1$ and $\omega > 0$ and all $t \ge 0$. Let $\epsilon > 0$ be given, let $0 \le a < \frac{\omega}{M}$ and let A, B, C, D, K, L, P and/or Q be subject to perturbations. Then whenever these perturbations satisfy*

$$\|A' + B'K' - A - BK\| \le a \tag{7.27a}$$

$$\frac{M}{\omega}\|B'L' + P' - BL - P\| + \frac{M^2 a}{\omega^2}\left(1 - \frac{Ma}{\omega}\right)^{-1}\|B'L' + P'\| < \frac{\epsilon}{3(1 + \|C' + D'K'\|)} \tag{7.27b}$$

$$\|C' + D'K' - C - DK\| < \frac{\epsilon}{3(1 + \|\Pi\|)} \tag{7.27c}$$

$$\|D'L' - Q' - DL + Q\| < \frac{\epsilon}{3} \tag{7.27d}$$

we have $\limsup_{t \to \infty}\|\widetilde{y}(t) - Q'T_S(t)w(0)\| < \epsilon\|w(0)\|$ for every $z(0) \in Z$ and every $w(0) \in W$.

Proof. First observe that since Π and Γ solve the regulator equations (3.10), the operators Π and $L = \Gamma - K\Pi$ solve the following operator equations:

$$\Pi S = (A + BK)\Pi + BL + P \quad \text{in } \mathcal{D}(S) \tag{7.28a}$$

$$0 = (C + DK)\Pi + DL - Q \quad \text{in } W \tag{7.28b}$$

Now, as the control law $u(t) = Kz(t) + Lw(t)$ is applied to the plant, the extended system consisting of the plant and the exosystem on $Z \times W$ can be described (in the mild sense) as follows:

$$\begin{pmatrix} \dot{z}(t) \\ \dot{w}(t) \end{pmatrix} = \begin{pmatrix} A + BK & BL + P \\ 0 & S \end{pmatrix}\begin{pmatrix} z(t) \\ w(t) \end{pmatrix}, \quad t \ge 0, \quad \begin{pmatrix} z(0) \\ w(0) \end{pmatrix} \in Z \times W \tag{7.29a}$$

$$e(t) = \begin{pmatrix} C + DK & DL - Q \end{pmatrix}\begin{pmatrix} z(t) \\ w(t) \end{pmatrix}, \quad t \ge 0 \tag{7.29b}$$

This system is of the form (7.1) with $X_1 = Z$, $X_2 = W$, $A_1 = A + BK$, $A_2 = S$, $A_3 = BL + P$, $C_1 = C + DK$ and $C_2 = DL - Q$. In this notation, by the operator equations (7.28), we have $\Pi A_2 = A_1\Pi + A_3$ in $\mathcal{D}(A_2)$ and $C_1\Pi + C_2 = 0$ in X_2. The result then follows immediately from Theorem 7.2 because $T_S(t)$ is an isometry, i.e. $N = \sup_{t \in \mathbb{R}}\|T_S(t)\| = 1$. $\qquad\square$

7.2.2   Practical error feedback output regulation

In this subsection we shall study practical output regulation in the sense of the EFRP. It turns out that we can again apply Theorem 7.2 to obtain upper bounds for the norms of the perturbations to the parameters of the plant, the controller and the exosystem such that practical output regulation with a desired accuracy occurs. Let us recall some operators from Chapter 4:

$$\mathcal{A} = \begin{pmatrix} A & BJ \\ GC & F + GDJ \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} P \\ -GQ \end{pmatrix} \quad \text{and} \quad \mathcal{C} = \begin{pmatrix} C & DJ \end{pmatrix} \tag{7.30}$$

with obvious domains of definition. In the following we again indicate the perturbed operators by primes, and all perturbations (denoted by Δ_R, with $R = \mathcal{A}, \mathcal{P}, Q$ and $\mathcal{C}$) are assumed to be bounded, linear and additive. For example, $\mathcal{A}' = \mathcal{A} + \Delta_{\mathcal{A}}$ for some $\Delta_{\mathcal{A}} \in \mathcal{L}(Z \times X)$. Moreover, we denote by $\widetilde{y}(t)$ the output of the perturbed closed loop system.

Corollary 7.8. *Let the operators F, G and J in (4.1) be such that $\|T_{\mathcal{A}}(t)\| \leq M e^{-\omega t}$ for some $M \geq 1$ and $\omega > 0$ and all $t \geq 0$. Also assume that there exist $\Pi \in \mathcal{L}(W, Z)$ and $\Lambda \in \mathcal{L}(W, X)$ satisfying the extended regulator equations (4.3), so that the dynamic controller (4.1) with these parameters F, G, J solves the EFRP. Let $\epsilon > 0$ be given, let $0 \leq a < \frac{\omega}{M}$ and let the operators $\mathcal{A}, \mathcal{C}, Q$ and $\mathcal{P}$ be subject to perturbations. Then whenever the perturbations satisfy*

$$\|\Delta_{\mathcal{A}}\| \leq a \tag{7.31a}$$

$$\frac{M}{\omega}\|\Delta_{\mathcal{P}}\| + \frac{M^2 a}{\omega^2}\left(1 - \frac{Ma}{\omega}\right)^{-1}\|\mathcal{P}'\| < \frac{\epsilon}{3(1 + \|\mathcal{C}'\|)} \tag{7.31b}$$

$$\|\Delta_{\mathcal{C}}\| < \frac{\epsilon}{3(1 + \|\begin{pmatrix}\Pi \\ \Lambda\end{pmatrix}\|)} \tag{7.31c}$$

$$\|\Delta_Q\| < \frac{\epsilon}{3} \tag{7.31d}$$

we have $\limsup_{t \to \infty}\|\widetilde{y}(t) - Q'T_S(t)w(0)\| < \epsilon\|w(0)\|$ for every $z(0) \in Z$, $x(0) \in X$ and every $w(0) \in W$.

Proof. The extended system consisting of the plant, the controller and the exosystem is given by

$$\dot{z}(t) = Az(t) + BJx(t) + Pw(t) \tag{7.32a}$$

$$\dot{x}(t) = GCz(t) + (F + GDJ)x(t) - GQw(t) \tag{7.32b}$$

$$\dot{w}(t) = Sw(t) \tag{7.32c}$$

$$e(t) = y(t) - y_{ref}(t) = Cz(t) + DJx(t) - Qw(t) \tag{7.32d}$$

on the state space $Z \times X \times W$. If we let $\Theta(t) = \left(\begin{smallmatrix} z(t) \\ x(t) \end{smallmatrix} \right) \in Z \times X$ and define $\mathcal{A}, \mathcal{C}$ and $\mathcal{P}$ as in the above, then we can write the extended system (7.32) as

$$\begin{pmatrix} \dot{\Theta}(t) \\ \dot{w}(t) \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{P} \\ 0 & S \end{pmatrix} \begin{pmatrix} \Theta(t) \\ w(t) \end{pmatrix}, \quad t \geq 0, \quad \begin{pmatrix} \Theta(0) \\ w(0) \end{pmatrix} \in Z \times X \times W \qquad (7.33a)$$

$$e(t) = \begin{pmatrix} \mathcal{C} & -Q \end{pmatrix} \begin{pmatrix} \Theta(t) \\ w(t) \end{pmatrix}, \quad t \geq 0 \qquad (7.33b)$$

which is precisely of the desired form (7.1) for $X_1 = Z \times X$, $X_2 = W$, $A_1 = \mathcal{A}$, $A_2 = S$, $A_3 = \mathcal{P}$, $C_1 = \mathcal{C}$ and $C_2 = -Q$. Moreover, since Π and Λ satisfy the regulator equations (4.3), it is elementary to verify that $\left(\begin{smallmatrix} \Pi \\ \Lambda \end{smallmatrix} \right) S = \mathcal{A} \left(\begin{smallmatrix} \Pi \\ \Lambda \end{smallmatrix} \right) + \mathcal{P}$ in $\mathcal{D}(S)$ and $\mathcal{C} \left(\begin{smallmatrix} \Pi \\ \Lambda \end{smallmatrix} \right) - Q = 0$ in W. Hence the operator $\left(\begin{smallmatrix} \Pi \\ \Lambda \end{smallmatrix} \right)$ satisfies the operator equations (7.3). The result now follows immediately by Theorem 7.2. $\square$

7.2.3 Practical feedforward-error feedback output regulation

In this subsection we shall study practical output regulation in the sense of the FFRP using an analysis similar to that in Subsection 7.2.2. Let us recall the following operators from Theorem 5.3:

$$\mathcal{A} = \begin{pmatrix} A & BJ \\ GC & F + GDJ \end{pmatrix}, \quad \mathcal{P}_\Gamma = \begin{pmatrix} P + B\Gamma \\ G(D\Gamma - Q) \end{pmatrix} \quad \text{and} \quad \mathcal{C} = \begin{pmatrix} C & DJ \end{pmatrix} \qquad (7.34)$$

with obvious domains of definition. In the following we again indicate the perturbed operators by primes, and all perturbations (denoted by Δ_R, with $R = \mathcal{A}, \mathcal{P}_\Gamma, Q$ and $\mathcal{C}$) are assumed to be bounded, linear and additive. For example, $\mathcal{A}' = \mathcal{A} + \Delta_{\mathcal{A}}$ for some $\Delta_{\mathcal{A}} \in \mathcal{L}(Z \times X)$. Moreover, we denote by $\widetilde{y}(t)$ the output of the perturbed closed loop system.

Corollary 7.9. *Assume that the operators F, G and J in (5.1) have been chosen such that $\|T_{\mathcal{A}}(t)\| \leq Me^{-\omega t}$ for some $M \geq 1$ and $\omega > 0$ and all $t \geq 0$. Also assume that $\Pi \in \mathcal{L}(W, Z)$ and $\Gamma \in \mathcal{L}(W, H)$ satisfy the regulator equations (3.10), so that the controller (5.1) with these parameters F, G, J, Γ solves the FFRP. Let $\epsilon > 0$ be given, let $0 \leq a < \frac{\omega}{M}$ and let the operators*

$\mathcal{A}, \mathcal{C}, Q$ and $\mathcal{P}_\Gamma$ be subject to perturbations. If these perturbations satisfy

$$\|\Delta_\mathcal{A}\| \leq a \tag{7.35a}$$

$$\frac{M}{\omega}\|\Delta_{\mathcal{P}_\Gamma}\| + \frac{M^2 a}{\omega^2}\left(1 - \frac{Ma}{\omega}\right)^{-1}\|\mathcal{P}_\Gamma'\| < \frac{\epsilon}{3(1 + \|\mathcal{C}'\|)} \tag{7.35b}$$

$$\|\Delta_\mathcal{C}\| < \frac{\epsilon}{3(1 + \|\Pi\|)} \tag{7.35c}$$

$$\|D'\Gamma' - Q' - D\Gamma + Q\| < \frac{\epsilon}{3} \tag{7.35d}$$

we have $\limsup_{t\to\infty} \|\widetilde{y}(t) - Q'T_S(t)w(0)\| < \epsilon\|w(0)\|$ for every $z(0) \in Z$, $x(0) \in X$ and every $w(0) \in W$.

Proof. The extended system consisting of the plant, the controller and the exosystem is given by

$$\dot{z}(t) = Az(t) + BJx(t) + (P + B\Gamma)w(t) \tag{7.36a}$$

$$\dot{x}(t) = GCz(t) + (F + GDJ)x(t) + G(D\Gamma - Q)w(t) \tag{7.36b}$$

$$\dot{w}(t) = Sw(t) \tag{7.36c}$$

$$e(t) = y(t) - y_{ref}(t) = Cz(t) + DJx(t) + (D\Gamma - Q)w(t) \tag{7.36d}$$

on the state space $Z \times X \times W$. If we let $\Theta(t) = \left(\begin{smallmatrix} z(t) \\ x(t) \end{smallmatrix}\right) \in Z \times X$ and define $\mathcal{A}, \mathcal{C}$ and $\mathcal{P}_\Gamma$ as in the above, then we can write the extended system (7.36) as

$$\begin{pmatrix} \dot{\Theta}(t) \\ \dot{w}(t) \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{P}_\Gamma \\ 0 & S \end{pmatrix} \begin{pmatrix} \Theta(t) \\ w(t) \end{pmatrix}, \quad t \geq 0, \quad \begin{pmatrix} \Theta(0) \\ w(0) \end{pmatrix} \in Z \times X \times W \tag{7.37a}$$

$$e(t) = \begin{pmatrix} \mathcal{C} & D\Gamma - Q \end{pmatrix} \begin{pmatrix} \Theta(t) \\ w(t) \end{pmatrix}, \quad t \geq 0 \tag{7.37b}$$

which is precisely of the desired form (7.1) for $X_1 = Z \times X$, $X_2 = W$, $A_1 = \mathcal{A}$, $A_2 = S$, $A_3 = \mathcal{P}_\Gamma$, $C_1 = \mathcal{C}$ and $C_2 = D\Gamma - Q$. Moreover, since Π and Γ satisfy the regulator equations (3.10), it is elementary to verify that $\left(\begin{smallmatrix} \Pi \\ 0 \end{smallmatrix}\right)S = \mathcal{A}\left(\begin{smallmatrix} \Pi \\ 0 \end{smallmatrix}\right) + \mathcal{P}_\Gamma$ in $\mathcal{D}(S)$ and $\mathcal{C}\left(\begin{smallmatrix} \Pi \\ 0 \end{smallmatrix}\right) + D\Gamma - Q = 0$ in W. Hence the operator $\left(\begin{smallmatrix} \Pi \\ 0 \end{smallmatrix}\right)$ satisfies the operator equations (7.3). The result now follows by Theorem 7.2 because $\|\Pi\| = \|\left(\begin{smallmatrix} \Pi \\ 0 \end{smallmatrix}\right)\|$ in any reasonable product space norm. $\square$

7.3 Applications

In this section we shall present some applications of the above theory.

7.3.1 Practical error feedback regulation under perturbations to the internal model

As we have seen in Chapter 6, error feedback controllers (4.1) utilizing the internal model structure achieve a degree of robustness in output regulation if the closed loop system is exponentially stable *and* if none of the perturbations affect the system operator F of the controller. In the all-finite-dimensional case it is well-known that, more precisely, the reduplication of the maximal cyclic component of the exosystem matrix S in the controller's system matrix F cannot be perturbed if robust output regulation is to be maintained [24, 29, 32, 93]. In this subsection we show that *practical* output regulation with a prescribed accuracy can sometimes still be achieved if the perturbations to the critical parts of F are sufficiently small.

Let us consider the Francis-type error feedback controllers of Subsection 6.5.1 (cf. Lemma 6.52). Instead of those in (6.78) we are using the following perturbed parameters F', G, J in a dynamic controller (4.1) on the state space $X = Z \times W_a$:

$$F' = \begin{pmatrix} A + BK - G_1 C & P_a + B(\Gamma - K\Pi) + G_1 Q_a \\ -G_2 C & S_a + \Delta_S + G_2 Q_a \end{pmatrix}, \quad G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}, \quad J = \begin{pmatrix} K & \Gamma - K\Pi \end{pmatrix}$$

$$(7.38)$$

Our goal is to achieve practical output regulation of signals generated the exosystem (2.2) which utilizes the (unperturbed) system operator S, with $\sigma(S_a) = \sigma(S)$ as in Lemma 6.52.

Let the assumptions of Lemma 6.52 be satisfied for $\Delta_S = 0$ and assume that the closed loop operator $\mathcal{A} = \begin{pmatrix} A & BJ \\ GC & F' \end{pmatrix}$ generates an exponentially stable C_0−semigroup whenever $\Delta_S = 0^4$. More specifically, for $\Delta_S = 0$ we require $\|T_\mathcal{A}(t)\| \leq M e^{-\omega t}$ for some $M \geq 1, \omega > 0$ and all $t \geq 0$. Since this is a practical error feedback (EFRP) regulation problem, we can use Corollary 7.8 directly: If $0 \leq a < \frac{\omega}{M}$ and if the perturbation Δ_S to the operator S_a in F' *both* satisfy

$$\|\Delta_S\| \leq a \tag{7.39a}$$

$$\frac{M^2 a}{\omega^2} \left(1 - \frac{Ma}{\omega}\right)^{-1} \left\| \begin{pmatrix} P \\ -G_1 Q \\ -G_2 Q \end{pmatrix} \right\| < \frac{\epsilon}{(1 + \|C\|)} \tag{7.39b}$$

then $\limsup_{t\to\infty} \|\widetilde{y}(t) - QT_S(t)w(0)\| < \epsilon\|w(0)\|$ for every $z(0) \in Z$, $x(0) \in X$ and every $w(0) \in W$. Here $\widetilde{y}(t)$ is the output of the plant when it is subject to the perturbed control and $\epsilon > 0$ is the desired accuracy. Observe that in the above we may assume $\|\Delta_\mathcal{A}\| = \|\Delta_S\|$ because only

---

[4]It is possible that this closed loop stability assumption can only be met for a finite-dimensional exosystem (2.2) in practice; see Chapter 4.

the operator S_a in F' (in $\mathcal{A}$) is perturbed; similarly $\|\mathcal{C}'\| = \|\mathcal{C}\| = \|C\|$ because $D = 0$ under the assumptions of Lemma 6.52. Moreover, it is clearly possible to replace the factors $\frac{\epsilon}{3}$ in the inequalities (7.31) by ϵ since only $\mathcal{A}$ is subject to perturbations[5].

We shall now reward the patient reader with a real, albeit *very* simple, simulation example which demonstrates the use of the bounds (7.39) in practice; these bounds are new even in this simple finite-dimensional situation.

Example 7.10. *Consider the stable finite-dimensional SISO system*

$$\dot{z}(t) = -4z(t) + 2u(t) + 3, \quad z(0) = 1, \quad t \geq 0 \tag{7.40a}$$

$$y(t) = 5z(t) \tag{7.40b}$$

Our goal is to study a set point control problem where asymptotic tracking of the constant reference signal $y_{ref}(t) = 1$ as $t \to \infty$, in spite of the constant disturbance $\mathcal{U}_{dist}(t) = 3$, is desired.

We can formulate the above control problem in our framework by choosing $Z = H = W = \mathbb{C}$ and the parameters of the plant (1.1) as $A = -4, B = 2, C = 5, D = 0$. The parameters of the exosystem (2.2) are chosen as $S = 0, Q = 1, P = 3, w(0) = 1$.

According to the robustness theory of Chapter 6, for the nominal case $\Delta_S = 0$ we can regard the elements P_a and Q_a in the matrix $F = F'$ of (7.38) as design parameters (i.e. they need not coincide with the above choices of P and Q in the exosystem) as long as closed loop stability is achieved and the controller has the internal model structure. Since the plant is already stable, it is convenient to take $W_a = \mathbb{C}, P_a = 0, Q_a = Q, S_a = S, K = 0, G_1 = 0$ and $G_2 = -1$. Then we can choose the unperturbed operator $F = \begin{pmatrix} A & B\Gamma \\ -G_2 C & S+G_2 Q \end{pmatrix}$ such that Γ and Π satisfy the regulator equations

$$\Pi S = A\Pi + B\Gamma \tag{7.41a}$$

$$C\Pi = Q \tag{7.41b}$$

For this very simple problem Π and Γ are elementary to work out; we have $\Pi = -A^{-1}B\Gamma = \frac{1}{5}$ and $\Gamma = -[CA^{-1}B]^{-1}Q = \frac{2}{5}$. In conclusion we obtain the nominal controller (4.1) with state space

---

[5]Here the inequalities (7.31c) and (7.31d) are satisfied regardless of the choice of $\epsilon > 0$. Hence the $\frac{\epsilon}{3}$-argument in the equations (7.13)-(7.17) in the proof of Theorem 7.2 can be reduced to an $\epsilon-$argument utilizing the triangle inequality only once.

$X = \mathbb{C}^2$ *and the parameters*

$$F = \begin{pmatrix} -4 & \frac{4}{5} \\ 5 & -1 \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & \frac{2}{5} \end{pmatrix} \tag{7.42}$$

The closed loop operator is given by

$$\mathcal{A} = \begin{pmatrix} A & BJ \\ GC & F \end{pmatrix} = \begin{pmatrix} -4 & 0 & \frac{4}{5} \\ 0 & -4 & \frac{4}{5} \\ -5 & 5 & -1 \end{pmatrix} \tag{7.43}$$

which is exponentially stable such that $\|T_{\mathcal{A}}(t)\| \le Me^{-\omega t}$ *approximately for* $M = 1.5$ *and* $\omega = 1$. *Figure 7.1 shows a* MATLAB *simulation for the controller's initial state* $x(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$; *it is obvious that asymptotic tracking of the desired constant reference signal occurs in spite of the disturbance.*
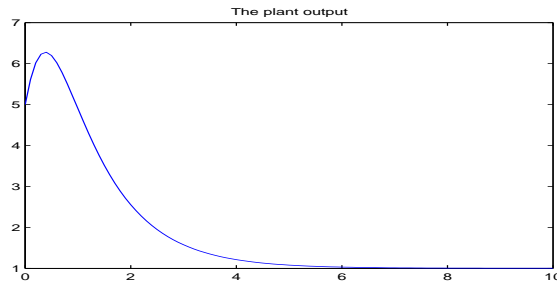

Figure 7.1: The nominal controller (7.42) solves the set point control problem.

Now let us fix the desired tracking accuracy as $\epsilon = \frac{1}{2}$ *and introduce nonzero perturbations* $\Delta_S \in \mathbb{C}$ *to the element* $S_a = 0$ *in the* $(2,2)-$*block of the matrix* F *in (7.42). We obtain the perturbed controller matrix* $F' = \begin{pmatrix} -4 & \frac{4}{5} \\ 5 & -1+\Delta_S \end{pmatrix}$. *For this particularly simple example it is easy apply the inequalities (7.39) and work out an upper bound for the constant* a *(as a function of* ϵ*) such that all perturbations satisfying* $|\Delta_S| \le a$ *are tolerated. We have that approximately* $a < 0.0115$ *is sufficient; this bound clearly also satisfies* $a < \frac{\omega}{M}$. *Figure 7.2 presents a* MATLAB *simulation of a critical case* $\Delta_S = i0.0115$ *for the same initial values as in the above. Clearly the absolute value of the tracking error in Figure 7.2 is bounded by* $\epsilon\|w(0)\| = \epsilon$, *and even* $\frac{\epsilon}{2}$, *which illustrates the fact that our bounds are not sharp (as has been indicated in Remark 7.3).*

Let us finally enlarge the perturbation to $\Delta_S = 0.4 + 0.5i$. *Now* $|\Delta_S|$ *is larger than the above bound for* a. *A* MATLAB *simulation for the same initial values as in the above is presented in*
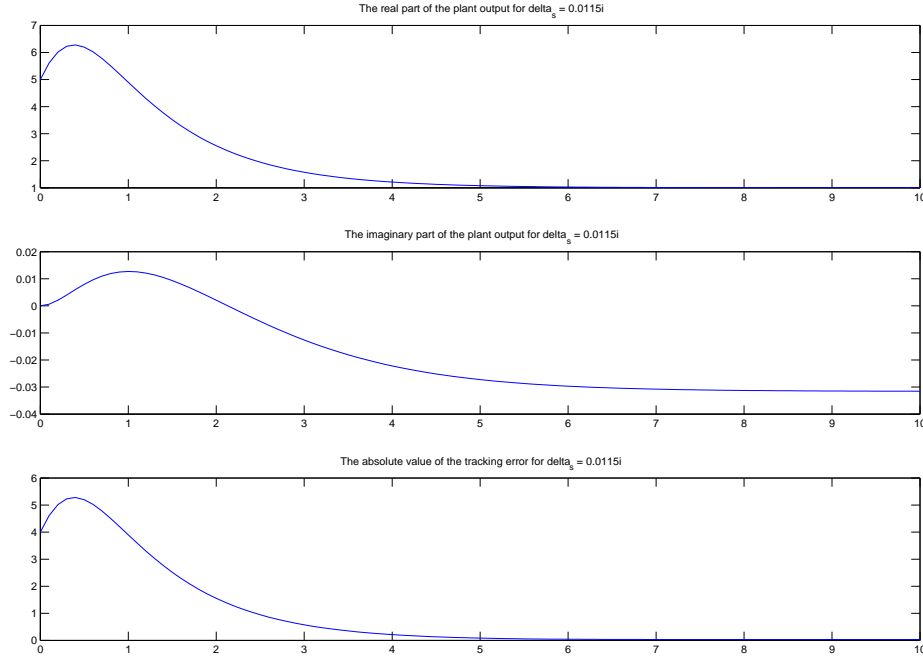
Figure 7.2: Small enough perturbations to the internal model result in practical output regulation with a desired accuracy.

Figure 7.3. In this case the closed loop system remains stable, but the absolute value of the tracking error $e(t)$ is not bounded by ϵ.

It is illustrative to view the above results in light of the inequality

$$\|\widetilde{y}(t) - y_{ref}(t)\| = \|\mathcal{C}\Theta'(t) - QT_S(t)w(0)\| \tag{7.44}$$

$$\leq \|\mathcal{C}\Theta'(t) - \mathcal{C}\Theta(t)\| + \|\mathcal{C}\Theta(t) - QT_S(t)w(0)\| \tag{7.45}$$

where $\Theta(t) = \left(\begin{smallmatrix} z(t) \\ x(t) \end{smallmatrix}\right) \in Z \times X$, $t \geq 0$, is the state of the closed loop system and $\Theta'(t)$ denotes its perturbation when $\Delta_S \neq 0$. Roughly stated, small bounded perturbations Δ_S to the operator S_a in F' result in small perturbations to the closed loop state; hence the first term in (7.45) is small on the whole nonnegative real axis. Since by assumption the nominal plant achieves output regulation, the second term in (7.45) decays to zero as $t \to \infty$. Altogether $\limsup_{t \to \infty} \|\mathcal{C}\Theta'(t) - QT_S(t)w(0)\|$ is small regardless of the initial states $\Theta(0)$ and $w(0)$, which is precisely what is required for practical output regulation.
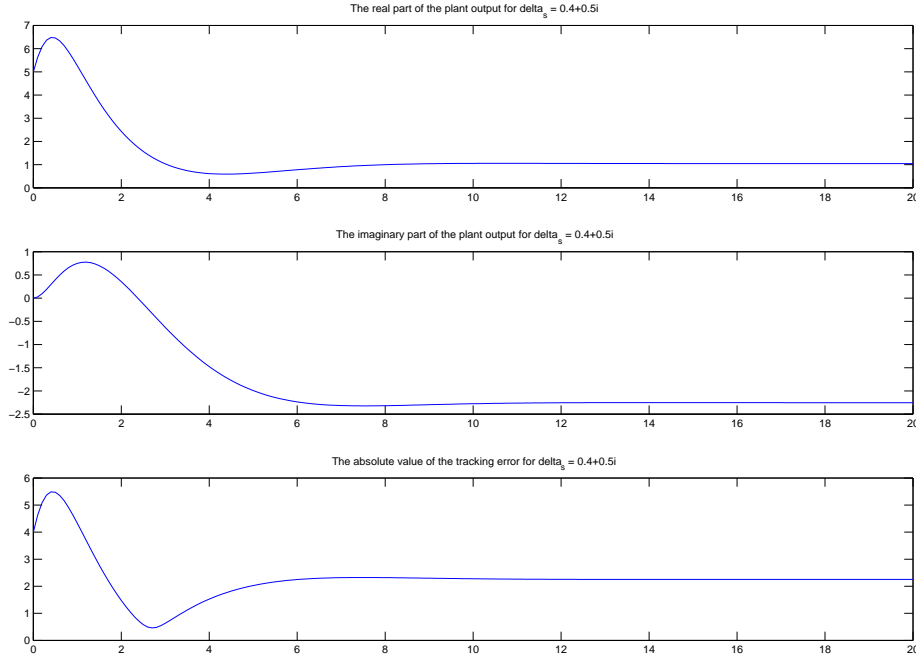
Figure 7.3: Large enough perturbations to the internal model destroy practical output regulation with a desired accuracy.

7.3.2 Practical feedforward regulation of periodic signals in generalized Sobolev spaces

In this subsection we shall consider a practical aspect of the feedforward regulation problem discussed in Section 3.5, where the reference signals are in some of the generalized Sobolev spaces $\mathcal{H} = H(f_n, \omega_n)$. As in Section 3.5, here we shall also assume that the plant is a SISO system, that $A + BK$ generates an exponentially stable C_0-semigroup on Z, and that $H_K(i\omega_n) = (C + DK)R(i\omega_n, A + BK)B + D \neq 0$ for all $n \in I$. Moreover, as in Section 3.5 here we take $W = \mathcal{H}$, $S = S|_{\mathcal{H}}$, $Q = \delta_0 \in \mathcal{L}(\mathcal{H}, \mathbb{C})$, $P \in \mathcal{L}(\mathcal{H}, Z)$ and $w(0) = y_{ref} \in W$ in accordance with Proposition 2.3.

In Theorem 3.35 we showed that under the above assumptions the control law $u(t) = Kz(t) + Lw(t)$ where

$$L = \sum_{n \in I} H_K(i\omega_n)^{-1}[1 - H_d(n)]\langle \cdot, \phi_n \rangle \tag{7.46}$$

solves the FRP if and only if $L \in \mathcal{L}(\mathcal{H}, \mathbb{C})$. This in turn was shown to be equivalent to the condition (3.55). Here $\phi_n(x) = e^{i\omega_n x}$ for each $n \in I$ and $x \in \mathbb{R}$, and $\langle \cdot, \cdot \rangle$ is the L^2 inner product on $\mathcal{H}$.

Unfortunately, if the index set I is not finite, then in practice the series expansion (7.46) for L cannot be precisely evaluated, and we cannot apply the control law $u(t) = Kz(t) + Lw(t)$ which would be required for exact output regulation. On the other hand, we can use the obvious truncation approximation L_N for L defined by

$$L_N = \sum_{|n| \leq N} H_K(i\omega_n)^{-1}[1 - H_d(n)]\langle \cdot, \phi_n \rangle, \quad N \in \mathbb{N} \tag{7.47}$$

An important question immediately arises: How big must N be in order that we achieve practical output regulation with some given accuracy $\epsilon > 0$? The following result answers this question. As before, we let $\widetilde{y}(t)$ denote the output of the plant subject to a perturbed control law.

Corollary 7.11. *Let* $\|T_{A+BK}(t)\| \leq Me^{-\omega t}$ *for some* $M \geq 1$ *and* $\omega > 0$ *and all* $t \geq 0$. *Let* $\epsilon > 0$ *be given and let* $L \in \mathcal{L}(\mathcal{H}, \mathbb{C})$ *and* $L_N \in \mathcal{L}(\mathcal{H}, \mathbb{C})$ *be as in the above. If* $u(t) = Kz(t) + L_N w(t)$ *where* N *is such that*

$$\sqrt{\sum_{|n|>N} |H_K(i\omega_n)|^{-2}|1 - H_d(n)|^2 f_n^{-2}} < \min\left\{ \frac{\epsilon\omega}{2M(1 + \|B\|)(1 + \|C + DK\|)}, \frac{\epsilon}{2(1 + \|D\|)} \right\} \tag{7.48}$$

then $\limsup_{t \to \infty} \|\widetilde{y}(t) - y_{ref}(t)\| < \epsilon\|y_{ref}\|_{\mathcal{H}}$ *for every* $z(0) \in Z$ *and every* $y_{ref} \in \mathcal{H}$.

Proof. Since in this case we may let $a = 0$ in Corollary 7.7, it is sufficient to show that $\frac{M}{\omega}\|BL_N - BL\| < \frac{\epsilon}{2(1+\|C+DK\|)}$ and $\|DL_N - DL\| < \frac{\epsilon}{2}$; observe that since only L is subject to perturbations in the entire control system, we may trivially change the factor $\frac{\epsilon}{3}$ to $\frac{\epsilon}{2}$ in the bounds (7.27) in analogy with the procedure of Subsection 7.3.1. Hence it is sufficient to show that

$$\|L_N - L\|_{\mathcal{L}(\mathcal{H}, \mathbb{C})} < \min\left\{ \frac{\epsilon\omega}{2M(1 + \|B\|)(1 + \|C + DK\|)}, \frac{\epsilon}{2(1 + \|D\|)} \right\} \tag{7.49}$$

A direct calculation shows that for every $y_{ref} \in \mathcal{H}$

$$|L_N y_{ref} - L y_{ref}| \leq \sum_{|n|>N} |H_K(i\omega_n)|^{-1}|1 - H_d(n)|f_n^{-1}f_n|\langle y_{ref}, \phi_n \rangle| \tag{7.50}$$

$$\leq \sqrt{\sum_{|n|>N} |H_K(i\omega_n)|^{-2}|1 - H_d(n)|^2 f_n^{-2}} \sqrt{\sum_{|n|>N} f_n^2 |\widehat{y_{ref}}(n)|^2} \tag{7.51}$$

$$< \min\left\{ \frac{\epsilon\omega}{2M(1 + \|B\|)(1 + \|C + DK\|)}, \frac{\epsilon}{2(1 + \|D\|)} \right\}\|y_{ref}\|_{\mathcal{H}} \tag{7.52}$$

by the Schwartz inequality and the fact that

$$\sqrt{\sum_{n \in I} |\langle y_{ref}, e^{i\omega_n \cdot}\rangle_{L^2}|^2 f_n^2} = \sqrt{\sum_{n \in I} |\widehat{y_{ref}}(n)|^2 f_n^2} = \|y_{ref}\|_{\mathcal{H}} \qquad (7.53)$$

This establishes the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 7.12. The bound (7.48) in Corollary 7.11 is not optimized. For instance, if $B \neq 0$, $C \neq 0$ and $D = 0$, then it is clearly sufficient that

$$\sqrt{\sum_{|n|>N} |H_K(i\omega_n)|^{-2}|1 - H_d(n)|^2 f_n^{-2}} < \frac{\epsilon\omega}{M\|B\|\|C\|} \qquad (7.54)$$

as can be easily seen by modifying the proof of Theorem 7.2, in particular the argument in (7.13)-(7.17), appropriately.

In the case that $K = 0$ and $P = 0$ (i.e. the exponentially stable disturbance-free case) the above procedure results in the remarkably simple approximative feedforward control law $u(t) = \sum_{|n|\leq N} \frac{\widehat{y_{ref}}(n)}{H(i\omega_n)} e^{i\omega_n t}$, $t \geq 0$, for the practical output regulation of any $y_{ref} \in \mathcal{H}$. We conclude this discussion by pointing out that uncertainties in the values $H_K(i\omega_n)$ and $H_d(n)$, for $|n| \leq N$, can also be easily accommodated in the above bounds because $\|\widetilde{L}_N - L\| \leq \|\widetilde{L}_N - L_N\| + \|L_N + L\|$. Here the operator $\widetilde{L}_N$ incorporates these uncertainties and $\|\widetilde{L}_N - L_N\|$ needs to be estimated.

Chapter 8

Solving the regulator equations

By now the reader has without doubt realized that the solvability of the regulator equations (3.10) and the extended regulator equations (4.3) is crucial for the existence and construction of controllers achieving output regulation of bounded uniformly continuous exogenous signals. We have conducted a preliminary study of the solution of the regulator equations (3.10) in the SISO case in Section 3.5 in order to convince the reader that our treatment of infinite-dimensional exosystems is indeed useful. On the other hand, in this chapter we shall present considerably more general methods for the solution of these regulator equations (3.10). Since the extended regulator equations (4.3) play the role of the regulator equations (3.10) for an extended system with zero control and zero feedthrough (see the proof of Theorem 4.4), the same methods apply for the extended regulator equations (4.3) with obvious changes. Moreover, as seen in Chapter 4 and Chapter 6, the solvability of the regulator equations (3.10) guarantees the *existence* of such operators F, G and J for which the extended regulator equations (4.3) also have a solution. Hence the equations (4.3) do not have to be explicitly solved in many output regulation applications; restricting our attention to the equations (3.10) is thus justified.

The regulator equations (3.10) first appeared in the finite-dimensional work of Francis and Wonham [29, 30, 31], and their solvability has also been studied in that setting by Hautus [37] (among others). For infinite-dimensional linear systems (1.1), with $D = 0$ and $\dim(H) < \infty$, and for finite-dimensional exosystems (2.1), Byrnes et al. [12] showed that the solvability of the regulator equations (3.10) is guaranteed (under a certain additional assumption on the structure of $\sigma(A)$) *for every* $P \in \mathcal{L}(W, Z)$ *and every* $Q \in \mathcal{L}(W, H)$ whenever no eigenvalue of S is a transmission

zero of the plant, i.e. $i\omega \in \sigma(S)$ implies $\det(H(i\omega)) \neq 0$ for all $\omega \in \mathbb{R}$. Here $H(\lambda) = CR(\lambda, A)B$ for each $\lambda \in \rho(A)$. It should be emphasized, however, that in a slightly different form the above-cited result of Byrnes et al. also appears in the earlier work of Schumacher (cf. Proposition 3.2 of [80]) where the regulator equations (3.10) are written in a geometric form.

On the other hand, if the exosystem is not finite-dimensional, then the nonexistence of transmission zeros on $\sigma(S)$ is in general *not* sufficient for the solvability of the regulator equations (3.10). This was first suggested in [10, 11] where some examples of output regulation for infinite-dimensional exosystems were studied. In Section 3.5 of this thesis we made the formal arguments in [10, 11] mathematically rigorous, whereas in Section 3.6 we illustrated the necessity of nonexistence of certain system zeros on $\sigma(S)$ for output regulation of bounded uniformly continuous signals. The key implication of these results is that also the high frequency (i.e. $|\omega| \to \infty$) behaviour of the transfer function of the (stabilized) plant must in a sense be "compatible" with that of the exogenous signals to be regulated, in order that the regulator equations (3.10) possess a solution.

The purpose of the present chapter is to extend the solvability criteria of Section 3.5 for the two (separate) cases in which $\sigma(S)$ is not necessarily discrete or $H \neq \mathbb{C}$, under the following standing assumption which covers all of our cases but which can sometimes also be weakened in applications[1]:

Assumption 8.1. *There exists $K \in \mathcal{L}(Z, H)$ such that $i\mathbb{R} \subset \rho(A + BK)$.*

Clearly Assumption 8.1 holds whenever the pair (A, B) is exponentially stabilizable by K, because then $\sigma(A + BK) \subset \{ z \in \mathbb{C} \mid \Re(z) < -\epsilon \}$ for some $\epsilon > 0$. However, the exponential stabilizability of the pair (A, B) is not necessary for the above kind of removability of the imaginary spectrum. In fact, there exist pairs (A, B) which can only be stabilized strongly by a bounded feedback K, such that $\sigma(A + BK) \cap i\mathbb{R} = \emptyset$ but $\pm i\infty$ are points of accumulation for $\sigma(A + BK)$. Such pairs (A, B) also satisfy Assumption 8.1; we refer the reader to Section 6.7 for an example of this phenomenon. Finally, we point out that $K = 0$ is also possible in Assumption 8.1, and we emphasize that $A + BK$ does not have to generate a strongly stable C_0-semigroup under Assumption 8.1.

---

[1] In particular, the *whole* imaginary spectrum need not always be removable in such applications in which the exosystem is very simple (e.g. finite-dimensional).

As was done in Section 3.5, also here we will explicitly use the feedback operator K of Assumption 8.1 in the solution of the regulator equations (3.10). This approach is quite natural, because often K can be chosen as a stabilizing state feedback for the plant, and such an operator is also needed for output regulation purposes. This was the case, for example, in Section 3.5 where the regulator equations (3.10) were solved for exponentially stabilizable SISO systems and periodic signals in $H(f_n, \omega_n)$, with the help of an exponentially stabilizing state feedback operator K. However, it is often also necessary to transform the obtained solvability criteria to such conditions which only utilize the original data or to such conditions which are readily derivable from these. Before outlining the contents of this chapter, we present a result which is very useful in this respect throughout this chapter[2].

Theorem 8.2. *Let $s \in \rho(A) \cap \rho(A + BK)$, and denote $H(s) = CR(s, A)B + D$ and $H_K(s) = (C + DK)R(s, A + BK)B + D$ as usual. If $H(s)^{-1} \in \mathcal{L}(H)$, then $H_K(s)^{-1} \in \mathcal{L}(H)$ and*

$$H_K(s)^{-1} = (I - KR(s, A)B)H(s)^{-1} \tag{8.1}$$

Proof. We have to verify that $H_K(s)[I - KR(s, A)B]H(s)^{-1} = I$, or, equivalently, that

$$H_K(s)[I - KR(s, A)B] = H(s) \tag{8.2}$$

In order to do that, first observe that $R(s, A) = R(s, A + BK) - R(s, A + BK)BKR(s, A)$. Hence

$$H_K(s)[I - KR(s, A)B] = [(C + DK)R(s, A + BK)B + D][I - KR(s, A)B] \tag{8.3}$$

$$= (C + DK)R(s, A + BK)B + D \tag{8.4}$$

$$- (C + DK)R(s, A + BK)BKR(s, A)B - DKR(s, A)B \tag{8.5}$$

$$= (C + DK)[R(s, A + BK) - R(s, A + BK)BKR(s, A)]B \tag{8.6}$$

$$- DKR(s, A)B + D \tag{8.7}$$

$$= (C + DK)R(s, A)B - DKR(s, A)B + D \tag{8.8}$$

$$= H(s) \tag{8.9}$$

and the proof is complete. $\square$

This chapter is organized as follows.

---

[2]This result was suggested to the author by Prof. Ruth Curtain (personal communication).

Section 8.1: We shall present such a method for the solution of the regulator equations (3.10) which applies
for SISO plants (1.1) (i.e. $H = \mathbb{C}$) without any additional assumptions on the exosystem
(2.2), in particular on $\sigma(S)$. The results of this section generalize those of Section 3.5 and
[10, 11, 12]. They are contained in [40].

Section 8.2: We shall present two results showing how the regulator equations (3.10) can be solved if the
plant (1.1) is not a SISO system. In this case we have to assume that either $\sigma(S)$ is discrete
or $W = H_{AP}(H, f_n, \omega_n)$ (see Chapter 2). Our results generalize those of Section 3.5 and
those in [10, 11, 12] by using the ideas of Section 8.1. They have not been submitted for
publication.

Section 8.3: We shall discuss some shortcuts and simplifications for the solution of the regulator equations
(3.10). The results of this section have been pointed to the author by Prof. Ruth Curtain
(personal communication).

8.1 A general method for SISO plants

In this section we shall present such a method for the solution of the regulator equations (3.10)
which applies for SISO plants (1.1) (i.e. $H = \mathbb{C}$) only, but without any additional assumptions on
the exosystem (2.2), in particular on $\sigma(S)$. Our strategy is based on the general decomposition
$W = \overline{\cup_{n \in \mathbb{N}} W_n}$ of the state space W of the exosystem (2.2) into the maximal spectral subspaces
$W_n = M([-in, in])$, $n \in \mathbb{N}$, as given in Lemma A.8[3]. We shall first solve the regulator equations
(3.10) in $W_n \subset \mathcal{D}(S)$ for each $n \in \mathbb{N}$. This is relatively easy to do because of the boundedness
of $S_n = S|_{W_n}$. Lemma A.8 then guarantees that in a sense $S_n \to S$ as $n \to \infty$. If the limiting
behaviour, as $n \to \infty$, of the corresponding sequence of solutions $(\Pi_n)_{n \in \mathbb{N}}$ and $(\Gamma_n)_{n \in \mathbb{N}}$ of the
regulator equations (3.10) in W_n is sufficiently good, then $\Pi = \lim_{n \to \infty} \Pi_n$ and $\Gamma = \lim_{n \to \infty} \Gamma_n$
(strong limits) solve the regulator equations (3.10) in the entire spaces $\mathcal{D}(S)$ and W.

The following two lemmata form the basis of our approach. The first one follows easily from
well-known results [8, 88, 90], but we reproduce a proof here for the sake of completeness.

Lemma 8.3. *Let $W_n = M([-in, in])$ and $S_n = S|_{W_n}$ for all $n \in \mathbb{N}$ be as in Lemma A.8. Let
$n \in \mathbb{N}$ and let $\Delta \in \mathcal{L}(W_n, Z)$ be arbitrary. Let γ_n denote a smooth contour enclosing $\sigma(S_n)$ in*

---

[3]See also Proposition A.9 for a concrete example of how the elements of W_n are constructed.

such a way that it and its interior do not contain any points in $\sigma(A + BK)$, and γ_n is traversed counterclockwise. Then the bounded linear operator $\Pi_n : W_n \to Z$ defined by

$$\Pi_n w = \frac{1}{2\pi i} \oint_{\gamma_n} R(\lambda, A + BK) \Delta R(\lambda, S_n) w d\lambda \quad \forall w \in W_n \tag{8.10}$$

is the unique solution of the operator equation $\Pi S_n = (A + BK)\Pi + \Delta$ in W_n.

Proof. By our assumptions on K the contour γ_n can be chosen so that the operator Π_n is well defined. It is also evident that $\Pi_n \in \mathcal{L}(W_n, Z)$ and that $\Pi_n(W_n) \subset \mathcal{D}(A + BK)$. For every $\lambda \in \rho(A + BK)$ we have $(A + BK)R(\lambda, A + BK) = \lambda R(\lambda, A + BK) - I$. Hence

$$(A + BK)\Pi_n w = \frac{1}{2\pi i} \oint_{\gamma_n} \lambda R(\lambda, A + BK) \Delta R(\lambda, S_n) w d\lambda - \frac{1}{2\pi i} \oint_{\gamma_n} \Delta R(\lambda, S_n) w d\lambda \tag{8.11}$$

$$= \Pi_n S_n w - \Delta w \quad \forall w \in W_n \tag{8.12}$$

because similarly $R(\lambda, S_n)S_n = \lambda R(\lambda, S_n) - I$ for each $\lambda \in \rho(S_n)$, and so

$$\Pi_n S_n w = \frac{1}{2\pi i} \oint_{\gamma_n} \lambda R(\lambda, A + BK) \Delta R(\lambda, S_n) w d\lambda - \frac{1}{2\pi i} \oint_{\gamma_n} R(\lambda, A + BK) \Delta w d\lambda \tag{8.13}$$

$$= \frac{1}{2\pi i} \oint_{\gamma_n} \lambda R(\lambda, A + BK) \Delta R(\lambda, S_n) w d\lambda \quad \forall w \in W_n \tag{8.14}$$

where the fact that $R(\lambda, A + BK)\Delta w$ is analytic inside an on γ_n has been used. The uniqueness of the solution follows from the boundedness of S_n and the fact that $\sigma(S_n) \cap \sigma(A + BK) \subset \sigma(S) \cap \sigma(A + BK) \subset \sigma(A + BK) \cap i\mathbb{R} = \emptyset$, according to the results in [3, 90]. $\qquad\square$

Lemma 8.4. *Let the sequences $(W_n)_{n \in \mathbb{N}}$ and $(S_n)_{n \in \mathbb{N}}$ be as in Lemma 8.3. Let $n \in \mathbb{N}$. Define the linear operators $F_n \in \mathcal{L}(W_n)$ and $P_n \in \mathcal{L}(W_n, \mathbb{C})$ by*

$$F_n w = \frac{1}{2\pi i} \oint_{\gamma_n} (C + DK) R(\lambda, A + BK) B R(\lambda, S_n) w d\lambda + Dw \quad \forall w \in W_n \tag{8.15}$$

$$P_n w = \frac{1}{2\pi i} \oint_{\gamma_n} (C + DK) R(\lambda, A + BK) P R(\lambda, S_n) w d\lambda \quad \forall w \in W_n \tag{8.16}$$

where the contour γ_n is as in Lemma 8.3. If F_n is boundedly invertible, i.e. $F_n^{-1} \in \mathcal{L}(W_n)$, then the following define operators in $\mathcal{L}(W_n, \mathbb{C})$ and $\mathcal{L}(W_n, Z)$ respectively:

$$L_n w = (Q - P_n) F_n^{-1} w \quad \forall w \in W_n \tag{8.17}$$

$$\Pi_n w = \frac{1}{2\pi i} \oint_{\gamma_n} R(\lambda, A + BK)(BL_n + P) R(\lambda, S_n) w d\lambda \quad \forall w \in W_n \tag{8.18}$$

Moreover, Π_n and $\Gamma_n = L_n + K\Pi_n \in \mathcal{L}(W_n, \mathbb{C})$ solve the regulator equations (3.10) in W_n.

Proof. Clearly F_n is indeed a bounded linear operator $W_n \to W_n$ because W_n is invariant for $R(\lambda, S_n)$ and because the plant is assumed to be SISO. Similarly $P_n \in \mathcal{L}(W_n, \mathbb{C})$ and $\Pi_n \in \mathcal{L}(W_n, Z)$. Since $S_n \in \mathcal{L}(W_n)$ and since $L_n \in \mathcal{L}(W_n, \mathbb{C})$, by Lemma 8.3 the operator $\Pi = \Pi_n$ satisfies the equation $\Pi S_n = (A + BK)\Pi + BL_n + P$ in $\mathcal{D}(S_n) = W_n$.

On the other hand, for each $w \in W_n$ we have that

$$(C + DK)\Pi_n w + DL_n w = \frac{1}{2\pi i} \oint_{\gamma_n} (C + DK)R(\lambda, A + BK)(BL_n + P)R(\lambda, S_n)w d\lambda \qquad (8.19)$$

$$+ DL_n w \qquad (8.20)$$

$$= L_n \Big[\frac{1}{2\pi i} \oint_{\gamma_n} (C + DK)R(\lambda, A + BK)BR(\lambda, S_n)w d\lambda + Dw \Big] \qquad (8.21)$$

$$+ \frac{1}{2\pi i} \oint_{\gamma_n} (C + DK)R(\lambda, A + BK)PR(\lambda, S_n)w d\lambda \qquad (8.22)$$

$$= L_n F_n w + P_n w \qquad (8.23)$$

$$= (Q - P_n)F_n^{-1}F_n w + P_n w \qquad (8.24)$$

$$= Qw \qquad (8.25)$$

since the plant (1.1) is assumed to be a SISO system. Consequently Π_n and $\Gamma_n = L_n + K\Pi_n$ solve the regulator equations (3.10) in W_n. $\qquad \square$

Before proceeding any further, we hasten to show that the bounded invertibility of the operator F_n in Lemma 8.4 holds provided that the system, which is obtained after the application of the state feedback K to the plant, does not have transmission zeros on $\sigma(S_n)$.

Proposition 8.5. *Let K and the sequences $(W_n)_{n\in\mathbb{N}}$ and $(S_n)_{n\in\mathbb{N}}$ be as in the above. Let $n \in \mathbb{N}$ and assume that $H_K(s) = (C + DK)R(s, A + BK)B + D \neq 0$ for all $s \in \sigma(S_n)$. Then the operator $F_n \in \mathcal{L}(W_n)$ defined in (8.15) is boundedly invertible and thus the regulator equations (3.10) are solvable in W_n.*

Proof. By definition, for each $w \in W_n$, we have that

$$F_n w = \frac{1}{2\pi i} \oint_{\gamma_n} (C + DK)R(\lambda, A + BK)BR(\lambda, S_n)w d\lambda + Dw \qquad (8.26)$$

$$= \frac{1}{2\pi i} \oint_{\gamma_n} \big[(C + DK)R(\lambda, A + BK)B + D \big] R(\lambda, S_n)w d\lambda \qquad (8.27)$$

$$= \frac{1}{2\pi i} \oint_{\gamma_n} H_K(\lambda)R(\lambda, S_n)w d\lambda \qquad (8.28)$$

because $w = \frac{1}{2\pi i} \oint_{\gamma_n} R(\lambda, S_n) w d\lambda$ for all $w \in W_n$. Since H_K is an analytic scalar function in the neighbourhood of $\sigma(S_n)$, for which $H_K(s) \neq 0$ for all $s \in \sigma(S_n)$, by the standard results in operational calculus, e.g. Theorem V.8.2 in [86], F_n is a bijection on W_n. Since $F_n \in \mathcal{L}(W_n)$, by the Open Mapping Theorem (see e.g. Theorem IV.5.5 in [86]) F_n must be boundedly invertible on W_n. $\qquad\square$

Corollary 8.6. *Let K and the sequences $(W_n)_{n\in\mathbb{N}}$ and $(S_n)_{n\in\mathbb{N}}$ be as in the above. Let $n \in \mathbb{N}$ and assume that $\sigma(S_n) \subset \rho(A) \cap \rho(A + BK)$. If $H(s) = CR(s, A)B + D \neq 0$ for all $s \in \sigma(S_n)$, then the operator $F_n \in \mathcal{L}(W_n)$ defined in (8.15) is boundedly invertible and thus the regulator equations (3.10) are solvable in W_n.*

Proof. This result follows from Proposition 8.6 and Theorem 8.2. $\qquad\square$

It is easy to see that with the definitions and notation of Lemma 8.4 we have $\Pi_{n+1}|_{W_n} = \Pi_n$ and $\Gamma_{n+1}|_{W_n} = \Gamma_n$, provided that the assumptions of Lemma 8.4 are satisfied for every $n \in \mathbb{N}$. In this case we can define a linear operator $\Pi_0 : \cup_{n\in\mathbb{N}} W_n \to Z$ by $\Pi_0 w = \Pi_n w$ for $w \in W_n$ and a linear operator $\Gamma_0 : \cup_{n\in\mathbb{N}} W_n \to \mathbb{C}$ by $\Gamma_0 w = \Gamma_n w$ for $w \in W_n$. As we shall see below, this construction allows us to solve the regulator equations (3.10) in the dense subspace $\cup_{n\in\mathbb{N}} W_n$ of W. It is then only a matter of continuous extension (if it exists) to solve these equations in $\mathcal{D}(S)$ and W respectively:

Theorem 8.7. *Let the assumptions of Lemma 8.4 hold for all $n \in \mathbb{N}$. Assume (in the above notation) that $\sup_{n\in\mathbb{N}} \|\Gamma_n\| < \infty$ and $\sup_{n\in\mathbb{N}} \|\Pi_n\| < \infty$. Define $\Pi \in \mathcal{L}(W, Z)$ and $\Gamma \in \mathcal{L}(W, \mathbb{C})$ as the unique continuous extensions of the operators Π_0 and Γ_0 above. Then Π and Γ solve the regulator equations (3.10).*

Proof. It is clear that for every $w \in \cup_{n\in\mathbb{N}} W_n \subset \mathcal{D}(S)$ we have

$$\Pi S w = \Pi S_k w = \Pi_k S_k w = A\Pi_k w + B\Gamma_k w + Pw = A\Pi w + B\Gamma w + Pw \tag{8.29}$$

$$Q w = C\Pi_k w + D\Gamma_k w = C\Pi w + D\Gamma w \tag{8.30}$$

for some $k \in \mathbb{N}$. Then for every $t \geq 0$ and every $w \in \cup_{n \in \mathbb{N}} W_n \subset \mathcal{D}(S)$ we also have that

$$\Pi T_S(t)w - T_A(t)\Pi w = \Big|_{\tau=0}^{t} T_A(t-\tau)\Pi T_S(\tau)w d\tau \tag{8.31}$$

$$= \int_0^t \frac{d}{d\tau} T_A(t-\tau)\Pi T_S(\tau)w d\tau \tag{8.32}$$

$$= \int_0^t T_A(t-\tau)[\Pi S - A\Pi]T_S(\tau)w d\tau \tag{8.33}$$

$$= \int_0^t T_A(t-\tau)(B\Gamma + P)T_S(\tau)w d\tau \tag{8.34}$$

because $T_S(\tau)w \in \mathcal{D}(S)$ and $\Pi T_S(\tau)w \in \mathcal{D}(A)$ for every $\tau \geq 0$. Since $\cup_{n \in \mathbb{N}} W_n$ is dense in W, we have by the proof of Lemma 3.5 that

$$\Pi T_S(t)w = T_A(t)\Pi w + \int_0^t T_A(t-\tau)(B\Gamma + P)T_S(\tau)w d\tau \quad \forall w \in W \quad \forall t \geq 0 \tag{8.35}$$

We next show that $\Pi(\mathcal{D}(S)) \subset \mathcal{D}(A)$. Let $w \in \mathcal{D}(S)$. Then for $h > 0$

$$\frac{T_A(h)\Pi w - \Pi w}{h} = \frac{T_A(h)\Pi w - \Pi T_S(h)w}{h} + \frac{\Pi T_S(h)w - \Pi w}{h} \tag{8.36}$$

$$= -\frac{\int_0^h T_A(h-\tau)(B\Gamma + P)T_S(\tau)w d\tau}{h} + \frac{\Pi T_S(h)w - \Pi w}{h} \tag{8.37}$$

which by the boundedness of Π shows that $\Pi w \in \mathcal{D}(A)$; also observe that the function $t \to (B\Gamma + P)T_S(t)w$ is continuously differentiable (because $w \in \mathcal{D}(S)$) so that the convolution in (8.37) is differentiable (cf. Proposition 1.3.6 in [2]). Taking the limit $h \to 0^+$, we see that $A\Pi w = -(B\Gamma + P)w + \Pi Sw$ for each $w \in \mathcal{D}(S)$.

By continuity and density, also the equation (8.30) must hold in W. Consequently Π and Γ solve the regulator equations (3.10). □

Remark 8.8. Theorem 8.7 immediately also gives the operator $L = \Gamma - K\Pi$ (as a continuous extension of the operators L_n) which is important in the solution of the FRP (see Chapter 3) and the EFRP (see Chapter 4).

At this stage the reader may wonder when (if ever) and how the continuous extension required in Theorem 8.7 can actually be done. Therefore, it is worthwhile to conclude this section with a discussion on this topic. There are three issues which should be pointed out in this context:

1. If $H_K(i\omega) \neq 0$ for all $\omega \in \mathbb{R}$, then the assumptions $\sup_{n \in \mathbb{N}} \|\Gamma_n\| < \infty$ and $\sup_{n \in \mathbb{N}} \|\Pi_n\| < \infty$ of Theorem 8.7 are also *necessary* for the solvability of the regulator equations (3.10). In

fact, if $\Pi \in \mathcal{L}(W, Z)$ and $\Gamma \in \mathcal{L}(W, \mathbb{C})$ solve the equations (3.10), then for all $n \in \mathbb{N}$

$$\Pi S_n = (A + BK)\Pi + BL + P \quad \text{in } W_n \tag{8.38a}$$

$$Q = (C + DK)\Pi + DL \quad \text{in } W_n \tag{8.38b}$$

for $L = \Gamma - K\Pi \in \mathcal{L}(W, \mathbb{C})$. But the equations (8.38) have a unique solution by the above. Indeed, we have $L|_{W_n} = L_n$ and $\Pi|_{W_n} = \Pi_n$ as in (8.17) and (8.18). Since Π and L are bounded on W, we must have $\sup_{n \in \mathbb{N}} \|\Gamma_n\| < \infty$ and $\sup_{n \in \mathbb{N}} \|\Pi_n\| < \infty$.

2. If it is sufficient to consider approximations of the reference/disturbance signals in the sense that we can take $W = W_n$ for some $n \in \mathbb{N}$, then no continuous extension has to be done at all. In practice one always has to accept compromises in terms of output regulation accuracy, and therefore an approximation of the exogenous signals is usually viable. A sensible and useful way to approximate general bounded uniformly continuous functions is provided by Lemma A.8 and Proposition A.9 — the idea is to utilize convolutions with suitable Fejér kernels if $W \hookrightarrow BUC(\mathbb{R}, \mathbb{C})$. This approximation procedure allows us to explicitly construct the function spaces W_n. Moreover, according to Proposition 8.5 and Lemma 8.4 in this case it is sufficient for the solvability of the regulator equations (3.10) that $H_K(i\omega) \neq 0$ for $\omega \in [-n, n]$. In many cases this condition can be verified using knowledge of the original plant data only; see Corollary 8.6.

3. If $A + BK$ generates an exponentially stable C_0-semigroup, if we take $W = \mathcal{H} = H(f_n, \omega_n)$ for some sequence $(f_n)_{n \in I}$, if $Q = \delta_0 \in \mathcal{L}(\mathcal{H}, \mathbb{C})$ and if $H_K(i\omega_n) \neq 0$ for all $n \in I$, then it is easy to see that the methods of Section 3.5 apply directly. Indeed, if we denote $\phi_n(x) = e^{i\omega_n x}$ for all $n \in I$ and $x \in \mathbb{R}$ and if we set $H_d(n) = (C + DK)R(i\omega_n, A + BK)P\phi_n$, then the operator L_n of (8.17) is given by

$$L_n w = L_n \sum_{k \in I_n} \langle w, \phi_k \rangle_{L^2} \phi_k = \sum_{k \in I_n} H_K(i\omega_k)^{-1}[1 - H_d(k)]\langle w, \phi_k \rangle_{L^2} \tag{8.39}$$

for all $w \in W_n = \mathrm{span}\{ e^{i\omega_k \cdot} \mid k \in I_n \}$ (the index set I_n is finite). This precisely what we obtained in Section 3.5; in particular, the condition (3.55) for the sequence $(f_n)_{n \in I}$ and the signals guarantees the uniform boundedness of $\|L_n\|$. Exponential stability of $T_{A+BK}(t)$ subsequently guarantees the existence of a bounded Π, as in Theorem 3.35. In conclusion, verifiable sufficient conditions exist for $\|\Pi_n\|$ and $\|\Gamma_n\|$ to be uniformly bounded in certain interesting applications.

In contrast to Section 3.5, in Theorem 8.7 there are no restrictions on $\sigma(S)$. However, if $\sigma(S)$ is discrete or if W is some generalized Sobolev space $H_{AP}(H, f_n, \omega_n)$ of functions, then, as we shall see in Section 8.2 below, the above uniform boundedness requirement for $\|\Pi_n\|$ and $\|\Gamma_n\|$ reduces to the strong convergence of certain series of operators. It is remarkable that in this case we can dispense with the assumption $H = \mathbb{C}$, which was also posed in Section 3.5 for the sake of simplicity.

8.2 A general method for exosystems with additional structure

In this section we shall present two results showing how the regulator equations (3.10) can be solved if the exosystem (2.2) has some additional structure, but the plant (1.1) is not necessarily a SISO system. Our method generalizes that in Section 3.5 while employing ideas from Section 8.1. Recall that according to the standing Assumption 8.1 of this chapter the imaginary spectrum of A can be removed by the state feedback operator $K \in \mathcal{L}(Z, H)$.

Theorem 8.9. *Assume that $\sigma(S) = \{ i\omega_n \mid n \in I \}$, $I \subset \mathbb{Z}$, is a discrete set and that $H_K(i\omega_n) = (C + DK)R(i\omega, A + BK)B + D$ is a boundedly invertible operator $H \to H$ for all $n \in I$. Let P_n denote the spectral projection corresponding to $i\omega_n \in \sigma(S)$ for every $n \in I$. If the sequences $(L_N)_{N \geq 1} \subset \mathcal{L}(W, H)$ and $(\Pi_N)_{N \geq 1} \subset \mathcal{L}(W, Z)$ of operators defined by*

$$L_N w = \sum_{|n| \leq N} H_K(i\omega_n)^{-1} \big[Q - (C + DK)R(i\omega_n, A + BK)P \big] P_n w, \quad \forall w \in W, \forall N \in \mathbb{N} \quad (8.40)$$

$$\Pi_N w = \sum_{|n| \leq N} R(i\omega_n, A + BK)(BL_N + P)P_n w, \quad \forall w \in W, \forall N \in \mathbb{N} \quad (8.41)$$

are uniformly bounded in N, then the operators $\Pi = \lim_{N \to \infty} \Pi_N$ and $\Gamma = \lim_{N \to \infty} [K\Pi_N + L_N]$ (limits in the strong operator topology) exist in $\mathcal{L}(W, Z)$ and $\mathcal{L}(W, H)$ repectively, and they solve the regulator equations (3.10).

Proof. Since $\sigma(S)$ is discrete, by Gelfand's Theorem (Corollary 4.4.8 in [2]) for every $n \in I$ we have that $P_n S = i\omega_n P_n = SP_n$. Moreover, since for every $n \in \mathbb{N}$ the interval $[-in, in] \subset i\mathbb{R}$ is compact and since $\sigma(S)$ is discrete, the maximal spectral subspace (cf. Lemma A.8) $W_n = M([-in, in]) = \text{span}\{ \text{ran } P_k \mid k \in I_n \}$ for some finite set I_n of indices. Additionally, $w = \sum_{k \in I_n} P_k w$ for each $w \in W_n$ (see Theorem 9.1 in [86]). Then it is not difficult to see that for each $n \in \mathbb{N}$ there exists a

(smallest) $N_0 \in \mathbb{N}$ such that for every $N \geq N_0$ we have that

$$\Pi_N S w = \Pi_N S \sum_{k \in I_n} P_k w = \sum_{k \in I_n} i\omega_k R(i\omega_k, A + BK)(BL_N + P)P_k w \tag{8.42}$$

$$= \sum_{k \in I_n} \left[I + (A + BK)R(i\omega_k, A + BK) \right] (BL_N + P)P_k w \tag{8.43}$$

$$= (BL_N + P) \sum_{k \in I_n} P_k w + (A + BK) \sum_{k \in I_n} R(i\omega_k, A + BK)(BL_N + P)P_k w \tag{8.44}$$

$$= (BL_N + P)w + (A + BK)\Pi_N w, \quad \forall w \in W_n \tag{8.45}$$

and that

$$(C + DK)\Pi_N w + DL_N w = (C + DK)\Pi_N \sum_{k \in I_n} P_k w + DL_N \sum_{k \in I_n} P_k w \tag{8.46}$$

$$= \sum_{k \in I_n} \left[(C + DK)R(i\omega_k, A + BK)B + D \right] L_N P_k w \tag{8.47}$$

$$+ \sum_{k \in I_n} (C + DK)R(i\omega_k, A + BK)PP_k w \tag{8.48}$$

$$= Q \sum_{k \in I_n} P_k w - \sum_{k \in I_n} (C + DK)R(i\omega_k, A + BK)PP_k w \tag{8.49}$$

$$+ \sum_{k \in I_n} (C + DK)R(i\omega_k, A + BK)PP_k w \tag{8.50}$$

$$= Qw, \quad \forall w \in W_n \tag{8.51}$$

This shows that Π_N and $\Gamma_N = L_N + K\Pi_N$ solve the regulator equations (3.10) in W_n. We can now define a linear operator $\Pi_0 : \cup_{k \in \mathbb{N}} W_k \to Z$ by $\Pi_0 w = \Pi_{N_0} w$ for $w \in W_n$ and a linear operator $L_0 : \cup_{k \in \mathbb{N}} W_k \to H$ by $L_0 w = L_{N_0} w$ for $w \in W_n$. By our assumptions and Lemma A.8 the operators Π_0 and L_0 have unique continuous extensions $\Pi \in \mathcal{L}(W, Z)$ and $L \in \mathcal{L}(W, H)$. Clearly this means that $\Pi = \lim_{N \to \infty} \Pi_N$ and $\Gamma = \lim_{N \to \infty} [K\Pi_N + L_N]$ in the strong operator topology. That Π and $\Gamma = L + K\Pi \in \mathcal{L}(W, H)$ solve the regulator equations (3.10) now follows precisely as in the proof of Theorem 8.7. $\qquad \square$

Remark 8.10. Section 3.5 illustrates the use of Theorem 8.9 for SISO systems. In Section 3.5 the convergence of L_N as $N \to \infty$ is guaranteed by condition (3.55), i.e. by a suitable state space topology for W, while the exponential stability of $T_{A+BK}(t)$ guarantees the convergence of Π_N as $N \to \infty$.

In a completely analogous manner we can prove the following result in which $\sigma(S)$ need not be a discrete set, but, on the other hand, some additional structure is required of the elements of W.

Theorem 8.11. *Let $W = \mathcal{H} = H_{AP}(H, f_n, \omega_n)$ for some sequences $(f_n)_{n \in I}$ and $(\omega_n)_{n \in I}$, where $I \subset \mathbb{Z}$ (see Chapter 2). Assume that $S = S|_{\mathcal{H}}$ generates the isometric left translation C_0-group $T_S(t)|_{\mathcal{H}}$ on $\mathcal{H}$, and that $H_K(i\omega_n) = (C + DK)R(i\omega_n, A + BK)B + D : H \to H$ is boundedly invertible for all $n \in I$. For every $n \in I$ define the bounded linear operator $P_n : \mathcal{H} \to \mathcal{H}$ by $P_n f = \hat{f}(n)e^{i\omega_n \cdot}$ for each $f = \sum_{n \in I} \hat{f}(n)e^{i\omega_n \cdot} \in \mathcal{H}$. If the sequences $(L_N)_{N \geq 1} \subset \mathcal{L}(\mathcal{H}, H)$ and $(\Pi_N)_{N \geq 1} \subset \mathcal{L}(\mathcal{H}, Z)$ of operators defined by*

$$L_N f = \sum_{\{\, i\omega_n \,|\, |n| \leq N \,\}} H_K(i\omega_n)^{-1}\big[Q - (C + DK)R(i\omega_n, A + BK)P\big]P_n f, \quad \forall f \in \mathcal{H}, N \in \mathbb{N} \quad (8.52)$$

$$\Pi_N f = \sum_{\{\, i\omega_n \,|\, |n| \leq N \,\}} R(i\omega_n, A + BK)(BL_N + P)P_n f, \quad \forall f \in \mathcal{H}, N \in \mathbb{N} \quad (8.53)$$

are uniformly bounded in N, then the operators $\Pi = \lim_{N \to \infty} \Pi_N$ and $\Gamma = \lim_{N \to \infty}[K\Pi_N + L_N]$ (limits in the strong operator topology) exist in $\mathcal{L}(\mathcal{H}, Z)$ and $\mathcal{L}(\mathcal{H}, H)$ repectively, and they solve the regulator equations (3.10).

Proof. Observe first that by construction every $f \in \mathcal{H}$ is an almost periodic function, which is uniquely determined by the series expansion $\sum_{n \in I} \hat{f}(n)e^{i\omega_n \cdot}$, because the sequence $(\|\hat{f}(n)\|)_{n \in I} \in \ell^1$. Consequently, the operators P_n can be given in terms of the Fourier-Bohr transformation [38, 63], i.e. $(P_n f)(t) = e^{i\omega_n t} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(s)e^{-i\omega_n s}ds$. But according to [63] (p. 22) we have that

$$\hat{f}(n) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T+x}^{T+x} f(t)e^{-i\omega_n t}dt \quad (8.54)$$

uniformly for $x \in \mathbb{R}$. Hence

$$P_n f = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} T_S(s)|_{\mathcal{H}} f e^{-i\omega_n s}ds \quad (8.55)$$

and consequently for every $f \in \mathcal{D}(S|_{\mathcal{H}})$ we have

$$P_n S|_{\mathcal{H}} f = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} T_S(s)|_{\mathcal{H}} S|_{\mathcal{H}} f e^{-i\omega_n s}ds \quad (8.56)$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} T_{S|_{\mathcal{H}} - i\omega_n I}(s)[S|_{\mathcal{H}} - i\omega_n I]f ds + i\omega_n P_n f \quad (8.57)$$

$$= i\omega_n P_n f \quad (8.58)$$

This shows that $S|_{\mathcal{H}}P_n f = i\omega_n P_n f = P_n S|_{\mathcal{H}}f$ for each $f \in \mathcal{D}(S|_{\mathcal{H}})$. The rest of the proof parallels that of Theorem 8.9. We can define, using Lemma A.8, $W_N = M(\{\, i\omega_n \mid |n| \leq N \,\})$, so that $\{\, i\omega_n \mid |n| \leq N \,\} \subset i\mathbb{R}$ is a finite set for all $N \in \mathbb{N}$, and $\cup_{N \geq 1} W_N$ is dense in $W = \mathcal{H}$. $\qquad\square$

Remark 8.12. In Theorem 8.9 and Theorem 8.11 it is not assumed that $\dim(H) < \infty$.

Remark 8.13. Theorem 8.9 and Theorem 8.11 immediately also give the operator $L = \Gamma - K\Pi$ which is important in the solution of the FRP (see Chapter 3) and the EFRP (see Chapter 4).

Remark 8.14. The method we have used in this section has been shown to work for finite-dimensional exogenous systems in [12, 80]. In addition to that, C. Byrnes, D. Gilliam, V. Shubov and J. Hood have applied a similar eigenfunction decomposition method in the solution of the regulator equations for a boundary controlled heat plant driven by a wave equation [10, 11]. They obtained series expansions for the solution operators Π and Γ (similar to those in the above) for this special case, and they presented criteria for the convergence of these series; we point out that in [12, 80] no such convergence problems can arise because of the finite-dimensionality of the exosystem.

Remark 8.15. If in Theorem 8.9 and Theorem 8.11 the operator $A + BK$ generates an exponentially stable C_0−semigroup and if the family of operators L_N is uniformly bounded (i.e. the limit $L \in \mathcal{L}(W,H)$ exists), then also the family Π_N is uniformly bounded and $\Pi = \int_0^\infty T_{A+BK}(t)(BL + P)T_S(-t)dt$ (strong convergence) according to the results of Appendix A.2.

Remark 8.16. In the case that $H = \mathbb{C}$ a very useful method for establishing the uniform boundedness of the operators L_N in Theorem 8.11 is the following. Using the Schwartz inequality we can deduce that

$$|L_N g| \leq \sum_{n \in I_N} |H_K(i\omega_n)^{-1}[Q - (C + DK)R(i\omega_n, A + BK)P]P_n g| \tag{8.59}$$

$$= \sum_{n \in I_N} |H_K(i\omega_n)^{-1}[Q - (C + DK)R(i\omega_n, A + BK)P]e^{i\omega_n \cdot}||\hat{g}(n)| \tag{8.60}$$

$$\leq \sqrt{\sum_{n \in I_N} |H_K(i\omega_n)^{-1}[Q - (C + DK)R(i\omega_n, A + BK)P]e^{i\omega_n \cdot}|^2 f_n^{-2}} \sqrt{\sum_{n \in I_N} |\hat{g}(n)|^2 f_n^2}$$

$$\tag{8.61}$$

for all $g \in W = \mathcal{H} = H_{AP}(\mathbb{C}, f_n, \omega_n)$. Here $I_N = \{\, i\omega_n \mid |n| \leq N \,\}$. Since, according to Proposition 2.16, $\sqrt{\sum_{n \in I_N} |\hat{g}(n)|^2 f_n^2}$ converges to $\|g\|_{\mathcal{H}}$ as $N \to \infty$, the uniform boundedness of L_N can be

achieved by an appropriate choice of $(f_n)_{n \in I}$, i.e. by *an appropriate choice of the reference signal space topology.* In particular, the sequence $(f_n)_{n \in I}$ should be chosen such that

$$\sum_{n \in I_N} |H_K(i\omega_n)^{-1}[Q - (C + DK)R(i\omega_n, A + BK)P]e^{i\omega_n \cdot}|^2 f_n^{-2} < \infty \qquad (8.62)$$

which is essentially the idea presented already in the condition (3.55) of Section 3.5: The signals must be smooth enough with respect to the high-frequency damping rate of the plant.

Remark 8.17. If in Theorem 8.11 we have $Q = \delta_0$ and $P = 0$ but possibly $H \neq \mathbb{C}$, then

$$L_N g = \sum_{\{ i\omega_n \,||n| \leq N \}} H_K(i\omega_n)^{-1} \hat{g}(n), \quad \forall g \in \mathcal{H}, N \in \mathbb{N}. \qquad (8.63)$$

In this case the argument of Remark 8.16 relying on the Schwartz inequality can again be utilized to deduce the uniform boundedness of L_N. We immediately see that whenever

$$(\|H_K(i\omega_n)^{-1}\| f_n^{-1})_{n \in I} \in \ell^2 \qquad (8.64)$$

the family L_N of operators is uniformly bounded on $H_{AP}(H, f_n, \omega_n)$. Again, this just amounts to an appropriate choice of topology for the signal space. Observe that the assumption $Q = \delta_0$ is not very restrictive from the point of view of reference signals, and the assumption $P = 0$ is not restrictive if the ultimate goal is to design a conditionally robust regulator; see e.g. the equations (6.108).

Remark 8.18. Theorem 8.2 can be readily utilized to transform the above square summability criteria to verifiable conditions for the original plant data.

8.3 Discussion

The results of the previous two sections on the solution of the regulator equations (3.10) are by no means exhaustive: Although they are suitable for rather general problems, often in practice there are shortcuts which can be taken if additional assumptions are posed. A particularly useful shortcut is at our disposal if the feedthrough operator D of the plant is a *design parameter*, as has been pointed to the author by Prof. Ruth Curtain (personal communication). The purpose of this section is to illustrate the use of this shortcut using her arguments. The key idea is that whenever D is boundedly invertible on H, the regulator equations (3.10) reduce to the *unconstrained* Sylvester

operator equation

$$\Pi S = (A - BD^{-1}C)\Pi + BD^{-1}Q + P \quad \text{in } \mathcal{D}(S) \tag{8.65}$$

which can be readily solved using e.g. the well-known results of Appendix A.2 provided that $A - BD^{-1}C$ has certain desirable properties. These properties are described below.

If $A - BD^{-1}C$ generates an exponentially stable C_0-semigroup, then the unconstrained equation (8.65) has a unique solution regardless of $P \in \mathcal{L}(W, Z)$ and $Q \in \mathcal{L}(W, H)$. This is the case, in particular, if one of the following two conditions holds:

- A generates an exponentially stable C_0-semigroup such that $\|T_A(t)\| \leq Me^{-\omega t}$ for some $M \geq 1$ and $\omega > 0$ such that $\|BD^{-1}C\| < \frac{\omega}{M}$. This condition can always be met if D can be chosen freely (take e.g. $D = cI$ for a sufficiently large $c > 0$).

- The pair (A, B) is exponentially stabilizable, the pair (A, C) is exponentially detectable and $H(s)^{-1} \in H^\infty$. Here, of course, $H(s) = CR(s, A)B + D$ for $s \in \rho(A)$ and H^∞ is the standard Hardy space of bounded holomorphic functions [17]. Note that $H(s)^{-1} = D^{-1}[D - CR(s, A - BD^{-1}C)B]D^{-1}$. It should be pointed out that in this case D need not even be a design parameter; we only need its bounded invertibility.

If $A - BD^{-1}C$ generates a strongly stable C_0-semigroup, then the operator equation (8.65) does not necessarily always have a solution. However, if $S \in \mathcal{L}(W)$, if $\sigma(A) \cap \sigma(S) = \emptyset$, and if $H(i\omega) = CR(i\omega, A)B + D$ is boundedly invertible for all $i\omega \in \sigma(S)$, then $\sigma(A - BD^{-1}C) \cap \sigma(S) = \emptyset$ and thus the Sylvester equation (8.65) has a unique solution for all $P \in \mathcal{L}(W, Z)$ and all $Q \in \mathcal{L}(W, H)$. We point out that if D is a design parameter and if $B = C^*$ — which represents the case of collocated actuators and sensors for a Hilbert state space Z — then we can take $D = I$ and use e.g. the results of [5, 18] to verify that $A - BD^{-1}C$ generates a strongly stable C_0-semigroup on Z.

If $A - BD^{-1}C$ generates an analytic semigroup[4], then results similar to the ones given above hold even if no stability properties for the semigroup generated by $A - BD^{-1}C$ were known. Moreover, in this case S need not be bounded: If $\sigma(A) \cap \sigma(S) = \emptyset$, and if $H(i\omega) = CR(i\omega, A)B + D$ is boundedly invertible for all $i\omega \in \sigma(S)$, then $\sigma(A - BD^{-1}C) \cap \sigma(S) = \emptyset$ and thus the Sylvester equation (8.65) has a unique solution for all $P \in \mathcal{L}(W, Z)$ and all $Q \in \mathcal{L}(W, H)$.

---

[4]This is always the case whenever A generates an analytic semigroup, according to a well-known bounded perturbation result (cf. Proposition III.1.12 of [28]).

A nice additional feature in some of the above results is that a stabilizing state feedback operator $K = -D^{-1}C$, which is often also needed in the controller design process, is readily found. Moreover, by Theorem 3.6, a control law solving the FRP is then given by $u(t) = Kz(t) + (\Gamma - K\Pi)w(t) = -D^{-1}Cz(t) + [D^{-1}(Q - C\Pi) + D^{-1}C\Pi]w(t) = -D^{-1}Cz(t) + D^{-1}Qw(t) = -D^{-1}[y(t) - y_{ref}(t)] = -D^{-1}e(t)$. Therefore, neither direct state feedback nor the solution operator Π of (8.65) need be explicitly found in the controller design process. We only need the existence of Π solving (8.65), and it is quite tempting to conjecture that even this requirement may be superficial in applications.

Unfortunately, in many problems of practical interest the operator D is fixed and noninvertible. For example, in many partial differential equations there is no feedthrough at all, i.e. $D = 0$. In such cases we can still often utilize the methods of Section 3.5, Section 8.1 and Section 8.2 in the solution of the regulator equations (3.10). Nonetheless, any shortcut should be utilized whenever such is available. We conclude this section by pointing out that in certain special cases the regulator equations (3.10) can be solved as a coupled system of two point boundary value problems subject to additional constraints, provided that $D = 0$ (see [12] for more details).

Chapter 9

Conclusions

In the previous chapters we have developed a robust state space output regulation theory for linear infinite-dimensional systems and bounded uniformly continuous exogenous signals. Here we conclude the discussion with a summary of the main contributions of the present thesis, and we describe some open problems as well as new research directions.

9.1   Summary of the main contributions of the thesis

The simplest exosystem generating a given class of signals

We have constructed the simplest possible exosystems capable of generating signals (functions) in certain Banach subspaces of bounded uniformly continuous functions. These exosystems always utilize the generator of an isometric C_0-group, plus one bounded observation operator for the reference signals and one for the disturbance signals. A distinguishing feature in our approach is the following: While in most of the related earlier work the exogenous signals are assumed to be generated by some arbitrary finite-dimensional exosystem, here we have assumed a given class of exogenous signals for which we have constructed (i.e. realized) an exosystem capable of generating them. In our work the exosystem can also be — and usually is — infinite-dimensional.

Complete solution of three output regulation problems

We have generalized the finite-dimensional geometric output regulation theory for infinite-dimensional linear systems and bounded uniformly continuous exogenous signals.

For feedforward (state feedback) controllers the geometric output regulation theory gives rise to the regulator equations (3.10). We have shown how these regulator equations completely describe the steady state behaviour of an appropriately stabilized plant, subject to a feedforward control and bounded uniformly continuous exogenous reference/disturbance inputs. In particular, the solvability of the regulator equations has been shown in this thesis to completely characterize the solvability of the simple feedforward output regulation problem, under certain assumptions.

Generalizing a well-known finite-dimensional argument, we have shown how an error feedback output regulation problem can be formulated — and solved — as a feedforward output regulation problem for an extended system. This approach gives rise to the extended regulator equations (4.3). We have shown that under certain assumptions the solvability of the extended regulator equations completely characterizes the solvability of the error feedback output regulation problem. Various explicit choices for the controllers' parameters have been given, such that the extended regulator equations have a solution provided that the regulator equations have a solution.

The above generalization of a finite-dimensional argument has also been shown in this thesis to yield a complete characterization of the solvability of a hybrid feedforward-feedback output regulation problem, in terms of the solvability of the regulator equations.

Strong stabilization of the exosystem and the closed loop system

We have shown that under certain assumptions the exosystem's system operator S must (in a sense) be embedded in the controller's system operator F in order that error feedback output regulation is possible. However, this necessary embedding of S in F is a cause of severe stabilizability problems for the closed loop system, if the exosystem is infinite-dimensional. We have provided new proofs for certain well-known negative results about the lack of *exponential* stabilizability of the exosystem operator S, and we have proved some new positive results about the *strong* stabilizability of this operator. Taking into account this lack of exponential closed loop stabilizability, we have provided sufficient conditions for the strong stabilizability of the closed loop system for two particular error feedback controllers generalizing certain finite-dimensional ones due to Francis [29] and Davison

[24].

Robustness, conditional robustness and the Internal Model Principle

We have studied robustness and conditional robustness (i.e. robustness on condition that the closed loop stability is also robust) in error feedback output regulation. Our results show that the *unique* solvability of the extended regulator equations with respect to certain parameters implies a degree of conditional robustness in output regulation. We have generalized the celebrated Internal Model Principle for infinite-dimensional systems and possibly infinite-dimensional exosystems using state space techniques, but without relying on any purely finite-dimensional concepts. This result describes the necessary and sufficient structure of such exponentially stabilizing error feedback controllers which also achieve robust output regulation. We have also shown how two common error feedback controllers achieve (conditionally) robust output regulation, by utilizing this structure, even in infinite dimensions.

Practical output regulation

We have developed the mathematical foundations of practical output regulation, i.e. *approximate* asymptotic tracking/rejection of exosystem-generated signals, for linear infinite-dimensional systems. By a direct perturbation analysis, we have obtained general upper bounds for the norms of additive, bounded, linear perturbations to the the parameters of the plant, the exosystem and the (hypothetical) controller, which solves the corresponding exact output regulation problem, such that practical output regulation with a given accuracy occurs. Our approach covers in a unified way practical state space output regulation for the three exactly regulating controllers studied elsewhere in this thesis.

Solution of the regulator equations

We have solved the regulator equations in the following two separate cases: That of a SISO plant and that in which the spectrum of the exosystem's generator is a discrete set (this occurs e.g. in all repetitive control problems). The approach of the latter case has also been shown to extend to another closely related separate case in which the spectrum of the exosystem generator is not discrete but the signals are known to be in certain Banach spaces.

Applications

In order to illustrate the developed output regulation theory, as applications we have presented numerous case studies and examples throughout this thesis. These applications illustrate several new phenomena which arise from the possible infinite-dimensionality of the exogenous system; they are not present in the related earlier work. Such new phenomena include (i) the complex relationship between the system zeros and the (non)existence of a regulating controller, (ii) the necessary degree of smoothness of the exogenous signals for the existence of a regulating controller, and (iii) the robustness-enhancing effect of additional smoothness of the exogenous signals related to their internal model in a regulating error feedback controller.

9.2   Open problems and new research directions

In the author's view the most severe limitation in the results of this thesis lies in the extensive use of bounded control, observation and feedback operators in the plant, the controller and the exosystem. Although bounded control and observation operators can often be used as good approximations of the reality, it is well-known that many infinite-dimensional systems utilizing e.g. boundary control and/or point observation can only be formulated as a plant (1.1) with *unbounded* operators B and C. Therefore, a particularly important direction for future research is the generalization of the output regulation theory developed in the present thesis for unbounded control and observation operators in the plant. On the other hand, allowing for unbounded control and observation in the error feedback controllers would probably facilitate exponential closed loop stabilization even for infinite-dimensional exosystems. This would likely result in improved robustness properties in output regulation, too.

Another direction for future research, which is closely related to the above, is to consider more general exogenous signals than those which are bounded and uniformly continuous. It is evident that an exogenous system utilizing the left translation operators and point evaluation at the origin can be also used to generate signals which are not bounded and uniformly continuous; this method of generating signals appears to be strikingly universal. However, the lack of strong continuity of the left translation semigroup, as well as the possible unboundedness of the point evaluation operators, can be mathematically very difficult to handle in this case.

In the author's view one of the most interesting — and also fruitful — directions for future

research is a more thorough study of robustness in error feedback output regulation under the assumption of strong closed loop stability. In this thesis we have provided some sufficient conditions for conditionally robust output regulation, but we have not properly addressed e.g. the issue of persistence of strong closed loop stability under perturbations. We have referred to a result due to Casarino and Piazzera [13] which can be used to guarantee the persistence of strong closed loop stability under perturbations, but there are also at least two other ways to proceed here. The first way is to attempt to use the recent theory of strongly stable systems due to Oostveen [70]. His strongly stable systems utilize a strongly stable semigroup and suitably continuous input-output, input-state and state-output mappings to achieve a degree of robustness in the strong stability of the system. The second way is to attempt to apply the concept of polynomial stability, which is currently under development (see the preprint [4] and the references therein), to achieve a degree of robustness of closed loop stability with respect to certain perturbations. The concept of polynomial stability of a C_0-semigroup is intriguing because it allows for a uniform polynomial decay rate for all sufficiently smooth initial conditions. In this respect it resembles exponential stability, which is known to have nice robustness properties.

The last important direction for future research consists of the development of frequency domain analogues of the state space results of this thesis. Especially for the repetitive control applications, where frequency domain methods are prominent and also have proven to be quite effective, it would be vital to establish results taking into account (i) the smoothness of the exogenous signals to be regulated, and (ii) the lack of exponential closed loop stability. We have shown in this thesis that in the state space domain it is crucial to take these issues into account, in order to achieve tracking/rejection of periodic signals.

Appendix A

A.1 A collection of results from spectral theory

In this section we shall provide an array of well-known facts from spectral theory and harmonic analysis, for the reader's convenience. As before, $sp_C(\cdot)$ denotes the Carleman spectrum.

Proposition A.1. *Let X be a Banach space and let A generate a $C_0-semigroup$ on X. Let $\rho_\infty(A)$ denote the connected component of $\rho(A)$ which is unbounded to the right. If $X_0 \subset X$ is a closed subspace and if $\lambda \in \overline{\rho_\infty(A)} \cap \sigma(A|_{X_0})$, then $\lambda \in \sigma(A)$.*

Proof. See Corollary IV.2.16 in [28]. $\square$

Proposition A.2. *Let X be a Banach space and let A generate a $C_0-semigroup$ on X. If $(\lambda_n)_{n\in\mathbb{N}} \subset \rho(A)$ is such that $\lim_{n\to\infty} \lambda_n = \lambda$, then $\lambda \in \sigma(A)$ if and only if $\lim_{n\to\infty} \|R(\lambda_n, A)\| = \infty$.*

Proof. See Proposition IV.1.3 (iii) in [28]. $\square$

Proposition A.3. *Let A generate a uniformly bounded $C_0-group$ $T_A(t)$ on a Banach space X. If $\sigma(A) \subset \{0\}$, then $T_A(t) = I$ for all $t \in \mathbb{R}$. Consequently, each isolated point of $\sigma(A)$ is an eigenvalue.*

Proof. This is the well-known Gelfand Theorem, see Corollary 4.4.8 and Corollary 4.4.9 in [2]. $\square$

Proposition A.4. *Let X be a Banach space. A function $f \in BUC(\mathbb{R}, X)$ has empty Carleman spectrum, i.e. $sp_C(f) = \emptyset$, only if $f \equiv 0$.*

Proof. This is the well-known Wiener Tauberian Theorem, cf. [89] p. 38. $\square$

Proposition A.5. *Let X be a Banach space and let $f, g_n \in BUC(\mathbb{R}, X)$, $n \in \mathbb{N}$, such that $\|g_n - f\|_\infty \to 0$ as $n \to \infty$. Then*

   1. $sp_C(f)$ is closed.

   2. $sp_C(f) = sp_C(f(\cdot + h))$ for each $h \in \mathbb{R}$.

   3. If $sp_C(g_n) \subset \Lambda$ for each $n \in \mathbb{N}$, then $sp_C(f) \subset \overline{\Lambda}$.

   4. If Z is another Banach space and $P \in \mathcal{L}(X, Z)$, then $sp_C(Pf(\cdot + h)) \subset sp_C(f)$ for each $h \in \mathbb{R}$.

Proof. This result is essentially contained in Theorem 1.15 in [38]. $\qquad\square$

Proposition A.6. *Let A generate a uniformly bounded C_0-group $T_A(t)$ on a Banach space X, let $y \in X$ and define the function $f : \mathbb{R} \to X$ by $f(t) = T_A(t)y$ for every $t \in \mathbb{R}$. Then $isp_C(f) \subset \sigma(A)$.*

Proof. This is Remark 4.6.2 in [2]. $\qquad\square$

Proposition A.7. *Let A generate C_0-semigroup $T_A(t)$ on a Banach space X. If for some $x_0 \in X$ the function $x : \mathbb{R} \to X$ such that $x(0) = x_0$ satisfies $x(t) = T_A(t - s)x(s)$ for each $t \geq s$ and is uniformly bounded, then $isp_C(x) \subset \sigma_A(A) \cap i\mathbb{R}$.*

Proof. See Proposition 3.7 in [89]. $\qquad\square$

Lemma A.8. *The generator S of an isometric C_0-group $T_S(t)$ on a Banach space W is decomposable, i.e. for every compact $\Delta \subset i\mathbb{R}$ there exists a (maximal) spectral subspace $M(\Delta) \subset W$ which is closed and invariant for $T_S(t)$, the restriction $S|_{M(\Delta)} \in \mathcal{L}(M(\Delta))$, and $\sigma(S|_{M(\Delta)}) \subset \Delta$. Moreover, $\bigcup_{n \in \mathbb{N}} M([-in, in])$ is dense in W.*

Proof. See pp. 399-400 of [90] or [66]. For a constructive argument we refer the reader to the proof of Theorem IV.3.16 in [28]. $\qquad\square$

In order to illustrate the fundamental result of Lemma A.8, we shall construct a decomposition for the generator $-S|_{\mathcal{H}}$ of the strongly continuous right translation group $T_S(-t)|_{\mathcal{H}}$ on some scalar function space $\mathcal{H} \hookrightarrow BUC(\mathbb{R}, \mathbb{C})$. We shall rely on a functional calculus for $T_S(-t)|_{\mathcal{H}}$ on the convolution algebra $(L^1(\mathbb{R}), *)$, as constructed in Section IV.3.c in [28]. For each $f \in L^1(\mathbb{R})$ define

the operator $\hat{f}(T) = \int_{-\infty}^{\infty} f(t) T_S(-t)|_{\mathcal{H}} y \, dt$ for $y \in \mathcal{H}$. Here the integral is understood in the sense of Bochner. Next define

$$\mathcal{H}_n = \{\, y \in \mathcal{H} \mid \hat{f}(T)y = 0 \text{ for all } f \in L^1(\mathbb{R}) \text{ satisfying } \hat{f} \equiv 0 \text{ on } [-n,n] \,\} \qquad \text{(A.1)}$$

where $\hat{f}$ denotes the Fourier transform of $f \in L^1(\mathbb{R})$. It has been shown in Section IV.3.c in [28] that each $\mathcal{H}_n$ is $T_S(t)|_{\mathcal{H}}-$invariant, $S|_{\mathcal{H}_n}$ is bounded (in fact $\sigma(S|_{\mathcal{H}_n}) \subset [-in, in]$) and $\overline{\cup_{n=1}^{\infty} \mathcal{H}_n} = \mathcal{H}$.

The following result demonstrates *how* the functions in $\mathcal{H}$ can be approximated by those in $\mathcal{H}_n$.

Proposition A.9. *For each $n \in \mathbb{N}$ and $t \in \mathbb{R}$ let $g_n(t) = \frac{1}{2\pi n} \left(\frac{\sin(\frac{nt}{2})}{\frac{t}{2}} \right)^2 \in L^1(\mathbb{R})$ (the Fejér kernel). Let $y \in \mathcal{H}$ and $n \in \mathbb{N}$. Then $y_n = \widehat{g_n}(T)y \in \mathcal{H}_n$ and $\lim_{n \to \infty} \|y_n - y\|_{\mathcal{H}} = 0$.*

Proof. By the definition of $\mathcal{H}_n$ we must show that $\hat{f}(T)y_n = 0$ for all $f \in L^1(\mathbb{R})$ satisfying $\hat{f} \equiv 0$ on $[-n,n]$. First of all, for every such f by Lemma C.12 in [28] we have $\widehat{f * g_n} = \hat{f}\widehat{g_n} \equiv 0$ (recall from [58] that $\mathrm{supp}(\widehat{g_n}) \subset [-n,n]$). According to a well-known inversion theorem this implies $f * g_n \equiv 0$ almost everywhere. Then according to Lemma 3.17 in [28] we have $\hat{f}(T)y_n = \hat{f}(T)\widehat{g_n}(T)y = \widehat{f * g_n}(T)y = 0$ for all $f \in L^1(\mathbb{R})$ satisfying $\hat{f} \equiv 0$ on $[-n,n]$. Consequently $y_n \in \mathcal{H}_n$.

In order to establish $\lim_{n \to \infty} \|y_n - y\|_{\mathcal{H}} = 0$ we first observe that by construction $\mathcal{H}$ is a translation-invariant space on which the translation group in strongly continuous. This implies that $\mathcal{H}$ is a homogenous Banach space in the sense of Katznelson ([58] p. 127) such that for every $f \in \mathcal{H}$ we have $\lim_{t \to 0} \sup_{x \in \mathbb{R}} \int_x^{x+1} |f(s-t) - f(s)| ds \leq k \lim_{t \to 0} \sup_{x \in \mathbb{R}} \int_x^{x+1} \|T_S(-t)|_{\mathcal{H}} f - f\|_{\mathcal{H}} ds = k \lim_{t \to 0} \|T_S(-t)|_{\mathcal{H}} f - f\|_{\mathcal{H}} = 0$ (here $k > 0$ is obtained from the relation $\mathcal{H} \hookrightarrow BUC(\mathbb{R}, \mathbb{C})$). Excercise 14 on p. 130 of [58] shows that $\lim_{n \to \infty} \|g_n * y - y\|_{\mathcal{H}} = 0$, because the Fejér kernel is a summability kernel in $L^1(\mathbb{R})$. Now since $T_S(-t)y = y(\cdot - t)$, it is clear that $y_n(x) = [\widehat{g_n}(T)y](x) = \int_{-\infty}^{\infty} g_n(t) y(x-t) dt$ for every $x \in \mathbb{R}$. In other words, $y_n = g_n * y$. The proof is complete. $\qquad \square$

It is evident from the above proof that approximation of functions in $\mathcal{H}$ by those in $\mathcal{H}_n$ can be performed by convolving the original function with a suitable Fejér kernel.

A.2 A collection of results for the operator Sylvester equation

Throughout this section we assume that A and $-B$ are the generators of C_0-semigroups $T(t)$ and $S(t)$ on Banach spaces E and F, respectively, and that $C \in \mathcal{L}(F, E)$. We next collect from [3, 88, 90] some well-known results related to the solvability of the operator Sylvester equation $AX - XB = C$, for the reader's convenience. Recall from e.g. [90] that an operator $X \in \mathcal{L}(F, E)$ is a solution of this operator equation if for each $f \in \mathcal{D}(B)$ we have $Xf \in \mathcal{D}(A)$ and $AXf - XBf = Cf$. For such solution operators X the following assertions hold true.

- If one of the operators A, B is bounded (or, more generally, analytic) and if $\sigma(A) \cap \sigma(B) = \emptyset$, then the operator equation $AX - XB = C$ has a unique solution [3, 90].

- If for every $C \in \mathcal{L}(F, E)$ the operator equation $AX - XB = C$ has a unique solution, then necessarily $\sigma(A) \cap \sigma(B) = \emptyset$. The converse result does not hold in general, however [3, 90].

- If the (growth) types of the semigroups $T(t)$ and $S(t)$ are ω_A and ω_B, respectively, such that $\omega_A + \omega_B < 0$ (this is the case in particular if $T(t)$ is exponentially stable and if $S(t)$ is uniformly bounded), then the bounded linear operator $X : F \to E$ defined by

$$X = - \int_0^\infty T(t)CS(t)dt \quad \text{(strong convergence)} \tag{A.2}$$

is the unique solution of the operator equation $AX - XB = C$ [88, 90].

- If $T(t)$ and $S(t)$ are uniformly bounded semigroups, then a necessary condition for the solvability of the operator equation $AX - XB = C$ for a given $C \in \mathcal{L}(F, E)$ is the uniform boundedness of the family $(T_y)_{y>0}$ of operators defined by

$$T_y = - \int_0^\infty e^{-yt}T(t)CS(t)dt \quad \text{(strong convergence)} \tag{A.3}$$

For the case of reflexive E the uniform boundedness of the family $(T_y)_{y>0}$ is also sufficient for the solvability of the operator equation $AX - XB = C$ [88].

A.3 A brief introduction to bi-continuous semigroups

Following [59], in this section we shall provide a brief introduction to bi-continuous semigroups. Such semigroups constitute a generalization of C_0-semigroups, and they are very important e.g.

in the study of various operator algebras. An important subclass of bi-continuous semigroups is that of implemented semigroups [1]; this particular class of bi-continuous semigroups is utilized in the present thesis in the context of operator Sylvester equations.

Let us first fix some notation. For a linear space X, the value of a functional $f : X \to \mathbb{C}$ at $x \in X$ is in this section denoted by $\langle x, f \rangle$. Recall that for linear spaces X, Y and Z such that $X \subset Y$ and a linear operator $A : \mathcal{D}(A) \subset Y \to Z$, the restriction (or part) $A|_X$ of A in X has domain $\mathcal{D}(A|_X) = \{ x \in \mathcal{D}(A) \cap X \mid Ax \in X \}$ and $A|_X x = Ax$ for every $x \in \mathcal{D}(A|_X)$.

In the context of bi-continuous semigroups it is customary to make the following standing assumption.

Assumption A.10. *Let $(X, \|\cdot\|)$ be a Banach space with topological dual X' and let τ be a locally convex topology on X induced by a family P_τ of seminorms, with the following properties:*

1. *The space (X, τ) is sequentially complete on $\|\cdot\|-$bounded sets, i.e. every $\|\cdot\|-$bounded sequence, which is Cauchy in the $\tau-$topology, converges in (X, τ).*

2. *The topology τ is coarser than the $\|\cdot\|-$topology; hence we may without loss of generality assume that $p(x) \leq \|x\|$ for every $x \in X$ and every $p \in P_\tau$.*

3. *The space $(X, \tau)'$ is norming for $(X, \|\cdot\|)$, i.e.*

$$\|x\| = \sup\{\, |\langle x, \phi \rangle| \mid \phi \in (X, \tau)' \text{ and } \|\phi\|_{(X, \|\cdot\|)'} \leq 1 \,\} \quad \forall x \in X \tag{A.4}$$

Definition A.11. A linear operator A on a Banach space X is called a Hille-Yosida operator (of type ω) if there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and

$$\|R(\lambda, A)^k\| \leq \frac{M}{(\lambda - \omega)^k} \tag{A.5}$$

for all $k \in \mathbb{N}$, $\lambda > \omega$ and some $M \geq 1$.

Clearly Hille-Yosida operators are also closed operators [28].

Definition A.12. A set Y is bi-dense in X if for every $x \in X$ there exists a $\|\cdot\|-$bounded sequence $(y_n)_{n \geq 0} \subset Y$ which is $\tau-$convergent to x.

We shall need the following equicontinuity concepts.

Definition A.13. Let $(T(t))_{t \geq 0} \subset \mathcal{L}(X)$. Then

- $T(t)$ is (globally) bi-equicontinuos if for every $\|\cdot\|$−bounded sequence $(x_n)_{n \geq 0} \subset X$ which is τ-convergent to $x \in X$ we have $\tau - \lim_{n \to \infty} T(t)(x - x_n) = 0$ uniformly for all $t \geq 0$.

- $T(t)$ is locally bi-equicontinuos if for every $t_0 \geq 0$ and every $\|\cdot\|$−bounded sequence $(x_n)_{n \geq 0} \subset X$ which is τ−convergent to $x \in X$ we have $\tau - \lim_{n \to \infty} T(t)(x - x_n) = 0$ uniformly for all $t_0 \geq t \geq 0$.

We can now define bi-continuous semigroups.

Definition A.14. An operator family $(T(t))_{t \geq 0} \subset \mathcal{L}(X)$ (or shortly $T(t)$) is called a bi-continuous semigroup (with respect to τ and of type ω) if

1. $T(0) = I$ and $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$.

2. $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ and some constants $M \geq 1$ and $\omega \in \mathbb{R}$.

3. $T(t)$ is strongly continuous in (X, τ), i.e. the map $\mathbb{R}_+ \ni t \to T(t)x \in X$ is τ−continuous for each $x \in X$.

4. $T(t)$ is locally bi-equicontinuous.

We call $T(t)$ uniformly bounded if we can take $\omega = 0$ in Definition A.14.

Definition A.15. The generator $A : \mathcal{D}(A) \subset X \to X$ of a bi-continuous semigroup $T(t)$ on X is the unique operator on X such that its resolvent $R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x \, dt$ for all $x \in X$ and for all $\lambda \in \{ \mu \in \mathbb{C} \mid \Re(\mu) > \omega \}$ for some $\omega > 0$. Here the Laplace integral is understood as the norm limit (as $a \to \infty$) of a τ−Riemann integral over $[0, a]$ (see [59] for details).

The subsequent results are just parts of Proposition 1.16, Theorem 1.17, Proposition 1.18 and Theorem 1.28 of [59].

Theorem A.16. *Let A generate the bi-continuous semigroup $T(t)$ on X. Then*

$$Ax = \tau - \lim_{t \searrow 0} \frac{T(t)x - x}{t}, \quad \forall x \in \mathcal{D}(A) \tag{A.6a}$$

$$\mathcal{D}(A) = \left\{ x \in X \mid \tau - \lim_{t \searrow 0} \frac{T(t)x - x}{t} \text{ exists in } X \right\} \tag{A.6b}$$

Theorem A.17. *Let A generate the bi-continuous semigroup $T(t)$ of type ω on X. Then the following assertions hold true.*

1. *If $x \in \mathcal{D}(A)$ then $T(t)x \in \mathcal{D}(A)$ for all $t \geq 0$, $T(t)$ is continuously differentiable in t with respect to the topology τ, and $\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax$ for all $t \geq 0$.*

2. *An element $x \in X$ belongs to $\mathcal{D}(A)$ and $Ax = y$ if and only if $T(t)x - x = \int_0^t T(s)yds$ for all $t \geq 0$.*

3. *The operator A is bi-closed, i.e. for all sequences $(x_n)_{n \in \mathbb{N}} \subset \mathcal{D}(A)$ with $(x_n)_{n \in \mathbb{N}}$ and $(Ax_n)_{n \in \mathbb{N}}$ norm-bounded, $\tau - \lim_{n \to \infty} x_n = x \in X$ and $\tau - \lim_{n \to \infty} Ax_n = y \in X$ we have $x \in \mathcal{D}(A)$ and $Ax = y$.*

4. *A is a bi-densely defined Hille-Yosida operator of type ω.*

5. *The subspace $X_0 = \overline{\mathcal{D}(A)}^{\|\cdot\|} \subset X$ is $T(t)-$invariant and the restrictions $T(t)|_{X_0}$ constitute a C_0-semigroup (generated by the part $A|_{X_0}$ of A).*

As mentioned in the above, the so-called implemented semigroups constitute an important subclass of bi-continuous semigroups.

Definition A.18. Let X and Y be Banach spaces and let A and B be generators of C_0-semigroups $T(t)$ and $S(t)$ on X and Y, respectively. On the Banach space $\mathcal{L}(Y, X)$ we define the semigroup $\mathcal{U}(t)$ implemented by A and B according to the rule $\mathcal{U}(t)Z = T(t)ZS(t)$ for all $Z \in \mathcal{L}(Y, X)$ and for all $t \geq 0$.

The following result (cf. Proposition 3.16 of [59] and Remark 2.15 of [1]) is crucial in the output regulation results of this thesis. It reveals a deep relation between operator Sylvester equations and bi-continuous semigroups.

Theorem A.19. *Let X and Y be Banach spaces and let A and B be generators of C_0-semigroups $T(t)$ and $S(t)$ on X and Y, respectively. Let $\mathcal{U}(t)$ be the semigroup on $\mathcal{L}(Y, X)$ implemented by A and B. Then $\mathcal{U}(t)$ is a bi-continuous semigroup in the strong operator topology $\tau = \tau_{stop}$. Moreover, the generator $\mathcal{T}$ of $\mathcal{U}(t)$ is the operator defined by*

$$\mathcal{D}(\mathcal{T}) = \{ Z \in \mathcal{L}(Y, X) \mid Z(\mathcal{D}(B)) \subset \mathcal{D}(A), \exists P \in \mathcal{L}(Y, X) : Pu = AZu + ZBu \forall u \in \mathcal{D}(B) \} \tag{A.7a}$$

$$\mathcal{T}Z = P \tag{A.7b}$$

Remark A.20. The relation (A.7b) is just the operator Sylvester equation $AZ + ZB = P$.

Bibliography

[1] Jochen Alber. On implemented semigroups. *Semigroup Forum*, 63(3):371–386, 2001.

[2] Wolfgang Arendt, Charles J.K. Batty, Matthias Hieber, and Frank Neubrander. *Vector-valued Laplace transforms and Cauchy problems.* Birkhäuser, 2001.

[3] Wolfgang Arendt, Frank Räbiger, and Ahmed Sourour. Spectral properties of the operator equation $AX + XB = Y$. *Q. J. Math., Oxf. II. Ser.*, 45(178):133–149, 1994.

[4] Ándras Bátkai, Klaus-Jochen Engel, Jan Prüss, and Roland Schnaubelt. Polynomial stability of operator semigroups. Preprint, 2004. Available online at `www.mathematik.uni-halle.de/reports/sources/2004/04-13report.ps`.

[5] Charles J.K. Batty and Vu Quoc Phong. Stability of individual elements under one-parameter semigroups. *Trans. Am. Math. Soc.*, 322(2):805–818, 1990.

[6] Charles J.K. Batty, Jan van Neerven, and Frank Räbiger. Local spectra and individual stability of uniformly bounded C_0-semigroups. *Trans. Am. Math. Soc.*, 350(5):2071–2085, 1998.

[7] Mahabaleshwar Bhat. *Regulator theory for evolution systems.* PhD thesis, University of Toronto, 1976.

[8] Rajendra Bhatia and Peter Rosenthal. How and why to solve the operator equation $AX - XB = Y$. *Bull. Lond. Math. Soc.*, 29(1):1–21, 1997.

[9] Garrett Birkhoff and Saunders MacLane. *A survey of modern algebra.* Macmillan Company, 1965.

[10] Christopher Byrnes, David Gilliam, Victor Shubov, and Jeffrey Hood. An example of output regulation for a distributed parameter system with infinite dimensional exosystem. In *Electronic Proceedings of MTNS, 2002.* http : //www.nd.edu/~mtns/papers/22618_2.pdf.

[11] Christopher Byrnes, David Gilliam, Victor Shubov, and Jeffrey Hood. Examples of output regulation for distributed parameter systems with infinite dimensional exosystem. In *Proceedings of the 40th IEEE Conference on Decision and Control, 2001.*

[12] Christopher Byrnes, István Laukó, David Gilliam, and Victor Shubov. Output regulation for linear distributed parameter systems. *IEEE Trans. Autom. Control*, 45(12):2236–2252, 2000.

[13] Valentina Casarino and Susanna Piazzera. On the stability of asymptotic properties of perturbed C_0-semigroups. *Forum Math.*, 13(1):91–107, 2001.

[14] G. Chen, S.A. Fulling, F.J. Narcowich, and S. Sun. Exponential decay of energy of evolution equations with locally distributed damping. *SIAM J. Appl. Math.*, 51(1):266–301, 1991.

[15] Roger L. Cooke. Almost-periodic functions. *Am. Math. Mon.*, 88:515–526, 1981.

[16] Ruth Curtain. Model Reduction for Control Design for Distributed Parameter Systems. In *Ralph C. Smith and Michael A. Demetriou (ed.), Research Directions in Distributed Parameter Systems (Frontiers in Applied Mathematics 27)*, pages 95–120. 2003.

[17] Ruth Curtain and Hans Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory.* Springer-Verlag, 1995.

[18] Ruth Curtain and Hans Zwart. Riccati equations and normalized coprime factorizations for strongly stabilizable infinite-dimensional systems. *Syst. Control Lett.*, 28(1):11–22, 1996.

[19] E. Davison and A. Goldenberg. Robust control of a general servomechanism problem: The servo compensator. *Automatica*, 11:461–471, 1975.

[20] Edward J. Davison. The feedforward control of linear multivariable time-invariant systems. *Automatica*, 9:561–573, 1973.

[21] Edward J. Davison. A generalization of the output control of linear multivariable systems with unmeasurable arbitrary disturbances. *IEEE Trans. Autom. Control*, 20:788–792, 1975.

[22] Edward J. Davison. Multivariable turning regulators: The feedforward and robust control of a general servomechanism problem. *IEEE Trans. Autom. Control*, 21:35–47, 1976.

[23] Edward J. Davison. Multivariable turning regulators: The feedforward and robust control of a general servomechanism problem. *IEEE Trans. Autom. Control*, 21:35–47, 1976.

[24] Edward J. Davison. The robust control of a servomechanism problem for linear time-invariant multivariable systems. *IEEE Trans. Autom. Control*, 21:25–34, 1976.

[25] Edward J. Davison. The steady-state invertibility and feedforward control of linear time-invariant systems. *IEEE Trans. Autom. Control*, 21:529–534, 1976.

[26] J. Diestel and J. Uhl. *Vector measures.* Mathematical Surveys. No.15. American Mathematical Society, 1977.

[27] Nelson Dunford and Jacob T. Schwartz. *Linear Operators. I. General theory.* Pure and Applied Mathematics. Vol. 6. Interscience Publishers, 1958.

[28] Klaus-Jochen Engel and Rainer Nagel. *One-parameter semigroups for linear evolution equations.* Springer-Verlag, 2000.

[29] Bruce A. Francis. The linear multivariable regulator problem. *SIAM Journal of Control and Optimization*, 15:486–505, 1977.

[30] Bruce A. Francis and William M. Wonham. The internal model principle for linear multivariable regulators. *Appl. Math. Optimization*, 2:170–194, 1975.

[31] Bruce A. Francis and William M. Wonham. The role of transmission zeros in linear multivariable regulators. *Int. J. Control*, 22:657–681, 1975.

[32] Bruce A. Francis and William M. Wonham. The internal model principle of control theory. *Automatica*, 12:457–465, 1976.

[33] Timo Hämäläinen. *Robust Low-Gain Regulation of Stable Infinite-Dimensional Systems.* PhD thesis, Tampere University of Technology, 2005.

[34] Timo Hämäläinen and Seppo Pohjolainen. Robust control and tuning problem for distributed parameter systems. *Int. J. Robust Nonlinear Control*, 6(5):479–500, 1996.

[35] Timo Hämäläinen and Seppo Pohjolainen. A finite-dimensional robust controller for systems in the CD-algebra. *IEEE Trans. Autom. Control*, 45(3):421–431, 2000.

[36] Shinji Hara, Yutaka Yamamoto, Tohru Omata, and Michio Nakano. Repetitive control system: A new type servo system for periodic exogenous signals. *IEEE Trans. Autom. Control*, 33(7):659–668, 1988.

[37] M. Hautus. Linear matrix equations with applications to the regulator problem. In *Landau, I. D. (ed.), Outils et Modeles Mathematique Pour l'Automatique*, page 399. 1983.

[38] Yoshiyuki Hino, Toshiki Naito, Nguyen Van Minh, and Jong Son Shin. *Almost periodic solutions of differential equations in Banach spaces.* Taylor & Francis, 2002.

[39] Jie Huang. A simple proof of the robust linear regulator. In *Proceedings of the American Control Conference, 1995.*

[40] Eero Immonen. A feedforward-feedback controller for infinite-dimensional systems and regulation of bounded uniformly continuous signals. To appear in International Journal of Robust and Nonlinear Control.

[41] Eero Immonen. Function generators, the operator equation $\Pi S = A\Pi + BQ$ and inherited dynamics in inhomogenous cauchy problems. To appear in Journal of Differential Equations.

[42] Eero Immonen. On sylvester operator equations, complete trajectories, regular admissibility, and stability of c_0-semigroups. To appear in Electronic Journal of Differential Equations.

[43] Eero Immonen. On the internal model structure for infinite-dimensional systems: Two common controller types and repetitive control. Submitted to SIAM Journal on Control and Optimization.

[44] Eero Immonen. Practical output regulation for linear systems. Submitted to Automatica.

[45] Eero Immonen. Robust state space output regulation theory for linear infinite-dimensional systems. Submitted to SIAM Journal on Control and Optimization.

[46] Eero Immonen. Some properties of infinite-dimensional systems capable of asymptotically tracking a bounded uniformly continuous signal. Submitted to Mathematics and Control of Signals and Systems.

[47] Eero Immonen. On the concept of point value in the infinite-dimensional realization theory. *J. Math. Anal. Appl.*, 300(1):79–101, 2004.

[48] Eero Immonen and Seppo Pohjolainen. Error feedback output regulation of bounded uniformly continuous signals for infinite-dimensional linear systems. In *Proceedings of the 44th IEEE Conference on Decision and Control, 2005*.

[49] Eero Immonen and Seppo Pohjolainen. Feedback and feedforward output regulation of bounded uniformly continuous signals for infinite-dimensional systems. To appear in SIAM Journal on Control and Optimization.

[50] Eero Immonen and Seppo Pohjolainen. On solvability and approximation of the regulator equations. To appear in Proceedings of the MMAR 2005.

[51] Eero Immonen and Seppo Pohjolainen. Output regulation of periodic signals for dps : an infinite-dimensional signal generator. To appear in IEEE Transactions on Automatic Control.

[52] Eero Immonen and Seppo Pohjolainen. Output regulation of periodic signals for dps : an infinite-dimensional signal generator. In *Proceedings of the 43rd IEEE Conference on Decision and Control, 2004*.

[53] Eero Immonen and Seppo Pohjolainen. Output regulation of periodic signals for dps : an infinite-dimensional signal generator. In *Proceedings of the Mathematical Theory of Networks and Systems, 2004*.

[54] Eero Immonen and Seppo Pohjolainen. What periodic reference signals can an exponentially stabilizable linear control system asymptotically track? To appear in SIAM Journal on Control and Optimization.

[55] Eero Immonen, Seppo Pohjolainen, and Timo Hämäläinen. On the realization of periodic functions. *Syst. Control Lett.*, 54(3):225–235, March 2005.

[56] Rafael José Iorio and Valria de Magalhaes Iorio. *Fourier analysis and partial differential equations*. Cambridge University Press, 2001.

[57] Tosio Kato. *Perturbation theory for linear operators*. Springer-Verlag, 1966.

[58] Yitzhak Katznelson. *An introduction to harmonic analysis*. Dover Publications, 1976.

[59] Franziska Kuhnemund. *Bi-Continuous Semigroups on Spaces with Two Topologies: Theory and Applications.* PhD thesis, Eberhard-Karls-Universitat Tubingen, 2001. Available online at `http://w210.ub.uni-tuebingen.de/dbt/volltexte/2001/236/`.

[60] H.G. Kwatny and K.C. Kalnitsky. On alternative methodologies for the design of robust linear multivariable regulators. *IEEE Trans. Autom. Control*, 23:930–933, 1978.

[61] N. Levan and L. Rigby. Strong stabilizability of linear contractive control systems on Hilbert space. *SIAM J. Control Optimization*, 17:23–35, 1979.

[62] Nhan Levan. The stabilizability problem: A Hilbert space operator decomposition approach. *IEEE Trans. Circuits Syst.*, 25:721–727, 1978.

[63] B.M. Levitan and V.V. Zhikov. *Almost periodic functions and differential equations.* Cambridge University Press, 1982.

[64] Miao Li, Xiao-Hui Gu, and Fa-Lun Huang. Robustness with respect to small time-varied delay for linear dynamical systems on banach spaces. *Stud. Math.*, 163(3):289–305, 2004.

[65] Zheng-Hua Luo, Bao-Zhu Guo, and Omer Morgul. *Stability and stabilization of infinite dimensional systems with applications.* Springer-Verlag, 1999.

[66] Yurij I. Lyubich. *Introduction to the theory of Banach representations of groups.* Birkhäuser, 1988.

[67] J.M. Maciejowski. *Multivariable feedback design.* Addison-Wesley Publishing Company, 1989.

[68] N. Mahadevan, K.A. Hoo, and K. Adzlevski. Model reduction and control of distributed parameter processes. In *Proceedings of the American Control Conference, 1999.*

[69] Manfred Morari and Evanghelos Zafiriou. *Robust process control.* Prentice-Hall International, 1989.

[70] Job Oostveen. *Strongly stabilizable distributed parameter systems.* SIAM, 2000.

[71] A. Pavlov, N. van de Wouw, and H. Nijmeijer. The local approximate output regulation problem: Convergence region estimates. *Int. J. Robust Nonlinear Control*, 15(1):1–13, 2005.

[72] Seppo Pohjolainen. A feedforward controller for distributed parameter systems. *Int. J. Control*, 34:173–184, 1981.

[73] Seppo Pohjolainen. Robust controller for systems with exponentially stable strongly continuous semigroups. *J. Math. Anal. Appl.*, 111:622–636, 1985.

[74] Seppo A. Pohjolainen. Robust multivariable PI-controller for infinite dimensional systems. *IEEE Trans. Autom. Control*, 27:17–30, 1982.

[75] Chunjiang Qian and Wei Lin. Practical output tracking of nonlinear systems with uncontrollable unstable linearization. *IEEE Trans. Autom. Control*, 47:21–36, 2002.

[76] Richard Rebarber and Stuart Townley. Robustness with respect to delays for exponential stability of distributed parameter systems. *SIAM J. Control Optimization*, 37(1):230–244, 1999.

[77] Shigui Ruan and Junjie Wei. On the zeros of transcendental functions with applications to stability of delay differential equations with two delays. *Dyn. Contin. Discrete Impuls. Syst., Ser. A, Math. Anal.*, 10(6):863–874, 2003.

[78] Walter Rudin. *Functional analysis. 2nd ed.* McGraw-Hill, 1991.

[79] J.M. Schumacher. A direct approach to compensator design for distributed parameter systems. *SIAM J. Control Optimization*, 21(6):823–836, 1983.

[80] J.M. Schumacher. Finite-dimensional regulators for a class of infinite-dimensional systems. *Syst. Control Lett.*, 3:7–12, 1983.

[81] Allen Schweinsberg. Similarity orbits and the range of the generalized derivation $X \rightarrow MX - XN$. *Trans. Am. Math. Soc.*, 324(1):201–211, 1991.

[82] O.A. Sebakhy and W.M. Wonham. A design procedure for multivariable regulators. *Automatica*, 12:467–478, 1976.

[83] H. Shim, H. Kim, and C. Chung. Design and experiment of add-on track following controller for optical disc drives based on robust output regulation. In *Proceedings of the American Control Conference, 2004*.

[84] Olof Staffans. *Well-posed linear systems.* Cambridge University Press, 2005.

[85] B. Sz.-Nagy and C. Foias. *Harmonic analysis of operators on Hilbert spaces.* North-Holland Publishing Company, 1970.

[86] Angus E. Taylor and David C. Lay. *Introduction to functional analysis.* John Wiley & Sons., 1980.

[87] Hiroyuki Ukai and Tetsuo Iwazumi. General servomechanism problems for distributed parameter systems. *Int. J. Control*, 42:1195–1212, 1985.

[88] Quoc Phong Vu. The operator equation $AX - XB = C$ with unbounded operators A and B and related abstract Cauchy problems. *Math. Z.*, 208(4):567–588, 1992.

[89] Quoc Phong Vu. On the spectrum, complete trajectories, and asymptotic stability of linear semi-dynamical systems. *J. Differ. Equations*, 105(1):30–45, 1993.

[90] Quoc Phong Vu and Elena Schüler. The operator equation $AX - XB = C$, admissibility, and asymptotic behavior of differential equations. *J. Differ. Equations*, 145(2):394–419, 1998.

[91] Jin Wang and Jie Huang. Neural network enhanced output regulation in nonlinear systems. *Automatica*, 37(8):1189–1200, 2001.

[92] George Weiss and Martin Häfele. Repetitive control of MIMO systems using H^∞ design. *Automatica*, 35(7):1185–1199, 1999.

[93] W.Murray Wonham. *Linear multivariable control: a geometric approach. 2nd ed.* Springer-Verlag, 1979.

[94] Cheng-Zhong Xu and Gauthier Sallet. On spectrum and Riesz basis assignment of infinite-dimensional linear systems by bounded linear feedbacks. *SIAM J. Control Optimization*, 34(2):521–541, 1996.

[95] Yutaka Yamamoto. Learning control and related problems in infinite-dimensional systems. In *Trentelman, H. L. (ed.) et al., Essays on control: perspectives in the theory and its applications. Proceedings of the plenary lectures and mini-courses of the European Control Conference, ECC '93, held in Groningen, Netherlands, June 28-July 1, 1993. Boston, MA: Birkhäuser. Prog. Syst. Control Theory. 14, 191-222.* 1994.

[96] Yutaka Yamamoto and Shinji Hara. Relationships between internal and external stability for infinite- dimensional systems with applications to a servo problem. *IEEE Trans. Autom. Control*, 33(11):1044–1052, 1988.