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Output Regulation Theory for Linear Systems with Infinite-Dimensional and Periodic Exosystems

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Abstract

In this thesis we consider the output regulation problem consisting of choosing a controller to asymptotically steer the output of a linear infinite-dimensional system to a given reference signal despite external disturbances. In particular we are interested in a situation where the considered reference and disturbance signals are nonsmooth polynomially bounded functions. The existing theory on this problem can only be used in the case where the signals to be tracked and rejected are smooth and polynomially bounded, or nonsmooth but uniformly bounded functions. The availability of more general reference and disturbance signals is useful in many applications such as the control of robot arms and disk drive systems.

For generating our reference and disturbance signals we consider two separate methods, namely, a time-invariant infinite-dimensional exosystem and a periodically time-dependent finite-dimensional exosystem. We will see that the chosen method has a considerable effect on the properties of the resulting control law as well as on the behavior of the controlled closed-loop system. One of the main differences in these respective theories of output regulation is that the control law designed based on the infinite-dimensional exosystem is guaranteed to be robust with respect to a class of perturbations preserving the stability of the closed-loop system.

The first main result of this thesis is the generalization of the well-known internal model principle of finite-dimensional control theory for distributed parameter systems with infinite-dimensional exosystems. On a general level this result states that in order for a controller to solve the robust output regulation problem related to a given signal generator, the controller must be able to reproduce the dynamics of this exosystem. In addition to its theoretical significance the internal model principle can also be applied in the construction of controllers solving the robust output regulation problem. Our proof of this result is based on a close connection between the behavior of the state of the closed-loop system and an associated Sylvester operator equation. In particular, the controllers achieving asymptotic tracking of the reference signals can be characterized using the solvability of certain constrained Sylvester equations, and the robustness of this property can be expressed as a condition involving equations of this type.

The second main contribution of this thesis consists of the development of the theory of output regulation for infinite-dimensional systems with periodically time-dependent exosystems. In particular this also includes designing nonautonomous controllers achieving asymptotic output tracking and disturbance rejection. Our treatment shows that it is possible to study the output regulation problem for a distributed parameter system together with a nonautonomous exosystem using methods similar to the ones familiar from case of a time-invariant signal generator. In particular, the solvability of the problem related to a given periodic exosystem can be characterized using a periodically time-dependent version of the well-known regulator equations if the associated Sylvester operator equation is replaced with an infinite-dimensional Sylvester differential equation.

Foreword

The research presented in this thesis was carried out at the Department of Mathematics at Tampere University of Technology during 2007–2011. During this time I have had the opportunity to work with several talented people who have influenced me greatly. Most of all I would like to thank my supervisor Professor Seppo Pohjolainen for introducing me to the world of academic research. His enthusiasm for mathematics has been a constant source of inspiration, and his encouraging guidance has been essential for the completion of this thesis. I am also grateful to Henri Pesonen and Heikki Orelma for both professional and personal support. I thank Timo Hämäläinen and Petteri Laakkonen for helpful discussions, and for the inspiring atmosphere in the research group on mathematical systems theory. Finally, I would also like to thank Professor Sirkka-Liisa Eriksson, Mikael Kurula, and Pertti Koivisto for their positive influence on my work.

The writing of this thesis was finished during my visit to University of Twente in the Netherlands where I had the great honor of working under the supervision of Professor Hans Zwart. I am deeply grateful to have been influenced by his expertise and insights into mathematical research as well as for the opportunity to get to know him on a personal level.

I would also like to express my gratitude to Professor George Weiss and Professor Olof Staffans for carefully examining my thesis and for the constructive criticism and suggestions that have greatly improved the presentation and the readability of the theoretic material.

Above all I am grateful to Netta, my parents Silva and Hannu, my family, and my friends for the constant support throughout my studies.

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Tampere, October 2011,

Lassi Paunonen

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List of Symbols

Chapter 1

Introduction

The main topics of this thesis are the state space theory and control of infinite-dimensional linear systems. These abstract mathematical structures can be used to model various kinds of phenomena involving heat and diffusion processes, delayed effects and vibrations. Examining these types of models is necessary for many engineering applications, but it is also essential to understanding many processes occuring in nature. In the course of this study we will also encounter a wide variety of interesting mathematical problems.

Our main interest lies in the control of linear infinite-dimensional systems. This corresponds to a situation where the behavior of a real-life process can be affected via an *input* and its behavior can be observed through an *output* or a *measurement*. For an abstract system such as the one depicted in Figure 1.1 our main goal is to "choose the input u of the system in such a way that the output y behaves as desired despite external disturbance signals w".

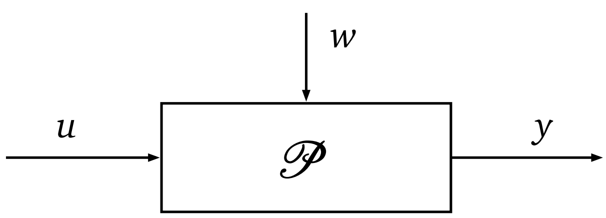

Figure 1.1: The system to be controlled.

This very general problem formulation covers a wide range of problems. First of all, if the system $\mathscr{P}$ is an infinite-dimensional linear system, the above types of control problems can be considered for mathematical models involving various types of linear partial differential equations and delay equations. For example, heating of a metal rod can be formulated as a control system where $\mathscr{P}$ is a one-dimensional partial differential equation. The output of the system can be the measurement of the temperature at a point or averaged over a part of the rod. The system can then be controlled, for example, by adjusting the temperature of one of its endpoints or by inserting heat into a section of the rod.

There are also many different ways of defining the "desired behavior" of the output, the appropriate one depending on the situation. In this thesis we are interested in so-called *output regulation*, where the output of the controlled system is required to approach a given reference signal asymptotically, i.e.,

$$\lim_{t\to\infty} \|y(t) - y_{ref}(t)\| = 0.$$

It is also common to set additional constraints or optimality criteria on the rate of convergence of the error or on the control signal.

One of the most useful additional requirements is that the control structure is robust with respect to small uncertainties and perturbations arising, for example, from modeling errors or changes within the system. In *robust output regulation* our goal is to choose the control law in such a way that the asymptotic tracking of the reference signal is achieved despite internal perturbations and changes in the system $\mathscr{P}$. It is clear that the necessarily finite accuracy of any mathematical model of a real world system makes robustness an essential property of any control structure used in applications. Engineers have understood for a long time that the robustness of a control law can be achieved by incorporating a *feedback* into the control scheme, as illustrated in Figure 1.2. However, recently it has been perceived that same kind of feedback structures also appear in nature, and that they explain many of the robustness properties observed in biological systems. Thermoregulation, i.e., the heat control within the human body, is an example of these types of robust control structures encountered in nature [29].

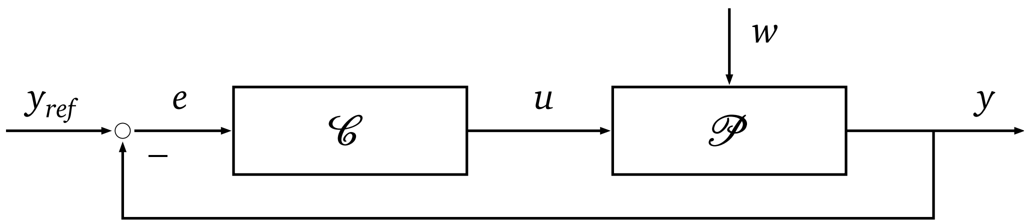

Figure 1.2: A control scheme incorporating feedback.

Our main goal in this thesis is to expand the classes of functions that can be considered as reference and disturbance signals in asymptotic output regulation, and to extend the theory of robust output regulation to cover these classes of signals. In the mathematical formulation of the control problem these functions are considered to be outputs of another dynamical system called the *exosystem* (or the *signal generator*). The traditional way of generating these signals is to choose an exosystem which is described by a system of ordinary differential equations. The exosystems of this type are capable of generating reference signals which are of the form

$$y_{ref}(t) = y_n(t)t^n + y_{n-1}(t)t^{n-1} + \cdots + y_1(t)t + y_0(t) \tag{1.1}$$

where $y_j(\cdot)$ are linear combinations of trigonometric functions. For many purposes this class of reference signals is sufficient, but in certain applications it is necessary or useful to be able to consider more general functions of time. The most serious drawback of using this type of exosystems is that we can only consider reference signals which are smooth, i.e., infinitely many times continuously differentiable.

This thesis is dedicated to methods of generating reference signals of the form (1.1), where the coefficient functions $y_j(\cdot)$ are general periodic functions, and to the study of the associated theories of output tracking and disturbance rejection. Signals of this type occur naturally in applications where periodic motion is present, for example in the control of robot arms and disk drive systems. These kinds of situations are explored in greater detail in Chapter 2.

1.1 A Mathematical Overview and Main Topics

Throughout this thesis we consider an infinite-dimensional linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \tag{1.2a}$$

$$y(t) = Cx(t) + Du(t) \tag{1.2b}$$

on a Banach space X. To ensure that this system has a well-defined state $x(t)$, the operator $A : \mathscr{D}(A) \subset X \to X$ is assumed to generate a strongly continuous semigroup. The rich theory of C_0-semigroups is one of our main tools in the development of the theory of output regulation. In this thesis we restrict our attention to the situation where the operators B, C and D are linear and bounded. It is also common to consider a case where these operators are allowed to be unbounded to include more general types of control and observation [55].

The exosystem is a linear differential equation on a linear space W. Even in the control of infinite-dimensional systems it is still usually assumed that the space W is finite-dimensional and that the signal generator is an autonomous system of ordinary differential equations. The reference and disturbance signals are in this case obtained as outputs of a system of the form

$$\dot{v}(t) = Sv(t), \qquad v(0) = v_0 \in W \tag{1.3a}$$

$$y_{ref}(t) = Fv(t), \tag{1.3b}$$

where $W = \mathbb{C}^q$ and where S and F are matrices of appropriate sizes. In this thesis we study two different ways of generalizing this class of exosystems. The first one is to allow the state space of the signal generator (1.3) to be an infinite-dimensional Hilbert space and to let S be an unbounded operator generating a strongly continuous group on W. Our second generalization, on the other hand, allows the exosystem to be a periodically time-dependent differential equation on a finite-dimensional space, i.e., of the form

$$\dot{v}(t) = S(t)v(t), \qquad v(0) = v_0 \in W \tag{1.4a}$$

$$y_{ref}(t) = F(t)v(t), \tag{1.4b}$$

where $W = \mathbb{C}^q$ and where $S(\cdot)$ and $F(\cdot)$ are periodic matrix-valued functions.

Given one of these exosystems, the system and the chosen controller can be written in a standard closed-loop form. We will see in Chapter 3 that for an autonomous finite or infinite-dimensional exosystem of the form (1.3) this closed-loop system is an infinite-dimensional linear system

$$\dot{x}_e(t) = A_e x_e(t) + B_e v(t), \qquad x_e(0) = x_{e0} \in X_e \tag{1.5a}$$

$$e(t) = C_e x_e(t) + D_e v(t) \tag{1.5b}$$

on a Banach space X_e. Here the operator $A_e : \mathcal{D}(A_e) \subset X_e \to X_e$ generates a semigroup $T_e(t)$, $v(t)$ is the state of the exosystem (1.3) and $e(t) = y(t) - y_{ref}(t)$ is the regulation error. For the purposes of the theory of output regulation this standard form is very useful. We will see that it is in particular possible to characterize the solvability of the output regulation problem using only the parameters of the closed-loop system (1.5). In this way, the theory of output regulation can be developed simultaneously for several controller types. The results obtained for the closed-loop system can subsequently be used to derive corresponding results for particular types of controllers.

In the case of a nonautonomous exosystem (1.4), the closed-loop system consisting of the plant and a periodically time-dependent controller can be written correspondingly as

$$\dot{x}_e(t) = A_e(t)x_e(t) + B_e(t)v(t), \qquad x_e(0) = x_{e0} \in X_e$$
$$e(t) = C_e(t)x_e(t) + D_e(t)v(t),$$

where $(A_e(t), \mathcal{D}(A_e(t)))$ is in general a periodic family of unbounded operators. A system of this form has a well-defined mild state if there exists a strongly continuous evolution family $U_e(t,s)$ associated to this family of operators.

The Reference and Disturbance Signals

It turns out that both the infinite-dimensional and the periodic exosystem are ideal for generating signals of the form (1.1) with general periodic coefficient functions $y_j(\cdot)$. There are, however, significant differences in the classes of signals produced by the two types of exosystems and in the corresponding theories of output regulation. Both of the two signal generators have their own advantages and weaknesses.

A particularly useful property of the infinite-dimensional signal generator is that it can be used to easily generate signals of predetermined smoothness. Furthermore, this property of the exosystem can be used in the corresponding theory of output regulation to establish a link between the smoothness properties of the reference and disturbance signals and the strictness of the conditions for the solvability of the output regulation problem. More precisely, we will learn that any additional level of smoothness in the exogeneous signals results in weaker conditions for the existence of a controller solving the output regulation problem.

The periodic exosystem, on the other hand, turns out to be a very natural way of generating signals of the form (1.1) with periodic coefficient functions $y_j(\cdot)$. These types of exosystems are also very well suited to generating small classes of signals consisting of, for example, only a few chosen periodic functions and their linear combinations. This is a considerable advantage if we are only interested in tracking a small number of specific signals. By definition, a controller solving the output regulation problem will be able to track all the signals generated by the exosystem. It can therefore be argued that any extraneous signals generated by the exosystem are reflected in the controller as complexity which is unnecessary for the purposes of tracking the signals we were originally interested in.

The Steady State Behaviour of the Closed-Loop System

Many of the main results presented in this thesis are based on a relationship between the closed-loop system and a corresponding Sylvester-type operator equation. In the case of a finite or infinite-dimensional exosystem of the form (1.3), this equation is an infinite-dimensional Sylvester operator equation

$$\Sigma S = A_e \Sigma + B_e. \tag{1.6}$$

We will see in Chapter 3 that for a given operator Σ the state of the closed-loop system can be written in the form

$$x_e(t) = T_e(t)(x_{e0} - \Sigma v_0) + \Sigma v(t) \tag{1.7}$$

if (and in fact, only if) Σ is a solution of the above Sylvester equation. The importance of this formula comes from the fact that it allows us to easily study the behaviors of the state of the closed-loop system and of the regulation error. In particular, if the closed-loop system is stable, the first term on the right-hand side of (1.7) decays to zero with time and we can see that the remaining asymptotic part of the state of the closed-loop system satisfies

$$x_e(t) \sim \Sigma v(t).$$

This behavior is determined only by the solution Σ of the Sylvester equation and the state $v(t)$ of the exosystem. We will see that this observation allows us to easily characterize the controllers capable of steering the output of the system to the signals generated by an infinite-dimensional exosystem of the form (1.3).

This connection between the closed-loop system and the Sylvester equation is also essential in our study of the problem of robust output regulation. In this problem we consider controllers which achieve asymptotic output tracking despite perturbations to the operators of the system (1.2). For any such perturbations preserving the stability of the closed-loop system and the solvability of the associated Sylvester equation, we have as above that the state of the perturbed closed-loop system can be written in the form similar to (1.7). Furthermore, since the closed-loop system is stable, the steady-state behavior of its state is given by

$$x_e(t) \sim \Sigma' v(t),$$

where Σ' is the solution of the perturbed Sylvester equation. This way we can see that in order to design a controller solving the robust output regulation problem, it is sufficient to choose the controller parameters in such a way that for a suitable class of perturbations this perturbed steady state behavior still produces the desired output.

In Chapter 7 we will see that these results also have their analogs in the theory of output regulation related to the periodic exosystem (1.4). However, in the case of a nonautonomous signal generator we need to replace the Sylvester operator equation with an infinite-dimensional *Sylvester differential equation*

$$\dot{\Sigma}(t) + \Sigma(t)S(t) = A_e(t)\Sigma(t) + B_e(t).$$

We will see that for a periodic operator-valued function $\Sigma(\cdot)$ the state of the nonautonomous closed-loop system can in this case be written in the form

$$x_e(t) = U_e(t,0)(x_{e0} - \Sigma(0)v_0) + \Sigma(t)v(t)$$

if (and again, only if) the function $\Sigma(\cdot)$ is the solution of the Sylvester differential equation. Using this formula it is again easy to see that if the closed-loop is stable, then the first term decays asymptotically and the remaining behavior of the closed-loop system

$$x_e(t) \sim \Sigma(t)v(t)$$

is completely determined by the solution $\Sigma(\cdot)$ of the Sylvester differential equation and the state $v(t)$ of the periodic exosystem (1.4). This property of stable closed-loop systems further allows us to characterize the controllers solving the output regulation problem related to an infinite-dimensional system and a periodic exosystem.

The Internal Model Principle

A classical result by Francis and Wonham [11] states roughly that in order for a controller to solve the robust output regulation problem related to a given exosystem, it is both necessary and sufficient that this controller contains a multiple copy of the dynamics of the exosystem in question. This result was first established for finite-dimensional linear systems, and even though it has also been studied in the cases of infinite-dimensional systems with finite and infinite-dimensional signal generators, it has not yet been completely generalized to such systems. The obvious difficulty is that the formulation of the internal model principle utilizes finite-dimensional concepts such as the invariant factors, minimal polynomials, and the Jordan canonical form.

In this thesis we investigate the relationship between the dynamics of the controller and the solvability of the robust output regulation problem for infinite-dimensional systems with infinite-dimensional time-invariant exosystems. Our main result is a very direct generalization of the classical internal model principle to distributed parameter systems. Furthermore, our treatment also establishes precise conditions for the equivalences between three alternative definitions for the internal model used in the literature. Although these conditions have been used for the same purpose — to characterize the controllers solving the robust output regulation problem — the redefinition of the internal model for infinite-dimensional systems has led to very different types of conditions, and the relationships between the concepts are not at all obvious.

1.2 Literature Review

In this section we will present a brief account of the history of the output regulation problem for distributed parameter systems. Output regulation and robust output regulation of these types of systems have been studied since the early 1980's beginning with the work of Schumacher [52]. The research on this topic has been active ever since [5, 45, 14, 48, 6, and references therein]. In the following we will concentrate on the development of the parts of the theory related to the main topics of this thesis.

Output Regulation for Infinite-Dimensional Exosystems

Most of the theory developed for robust output regulation for distributed parameter systems only considers reference and disturbance signals which are generated by finite-dimensional exosystems. As we already saw, more general classes of reference and disturbance signals can be considered if also the exosystem is allowed to be infinite-dimensional. The generation of signals using these types of exosystems and the corresponding theory of output regulation have been studied recently in [26, 24, 23, 15]. In these references the signal generator is constructed in such a way that it is capable of generating bounded and uniformly continuous signals. These types of signals are indeed very general in the context of output regulation of infinite-dimensional systems, where the properties of the system often set some limitations to the classes of signals one can hope to track [25]. Still, this type of exosystem has the drawback that it can only generate uniformly bounded signals. In many engineering applications it is necessary to generate signals which have a growth rate of t, or t^n for some $n \in \mathbb{N}$.

Robust Output Regulation and the Internal Model Principle

At a very early stage in the study of output regulation of infinite-dimensional systems, also some of the robustness and structural stability results by Francis and Wonham [11, 12, 57] were extended to infinite-dimensional systems by Bhat [5], whose main focus were time-delay systems. A part of this theory was later generalized by Immonen [23, 24] to distributed parameter systems and a certain class of infinite-dimensional exosystems. His approach in generalizing the internal model principle was the use of properties of Sylvester equations, which we have already seen to have a close connection to the behavior of the closed-loop system. Immonen was able to extract the property of the controller which guarantees that any perturbations of the system's parameters also lead to the correct output at the dynamic steady state of the closed-loop system. He formulated this *internal model structure* of the dynamic error feedback controller using properties of certain Sylvester equations involving the parameters of the controller.

Unfortunately, the internal model structure has a disadvantage that the conditions in the definition are difficult to verify for actual controllers. Hämäläinen and Pohjolainen [15] later found more easily verifiable sufficient conditions for a controller to have this property. The origin of these conditions is in the proof of the internal model principle in a paper by Francis and Wonham [12], and they are given in terms of the parameters of the error feedback controller and the spectrum of the exosystem. Although Hämäläinen and Pohjolainen called these conditions an *internal model*, they were only used as sufficient conditions for the controller to solve the robust output regulation problem. In particular, the authors did not discuss whether or not they are also necessary for the controller to have this property. To distinguish them from other alternative definitions of the internal model we refer to these conditions as *$\mathcal{G}$-conditions* in this thesis.

Output Regulation for Periodic Exosystems and the Sylvester Differential Equation

Control of distributed parameter systems with time-dependent exosystems falls under the category of control of general time-dependent infinite-dimensional linear systems. This problem has been studied in the literature [4, 7], but in general the interest for these types of systems has been smaller than towards the infinite-dimensional autonomous linear systems or — the more general alternative — nonlinear systems. The particular problem of output regulation for a time-invariant system with a nonautonomous signal generator has been studied recently in the finite-dimensional case [60, 20]. In these references the authors present results on the solvability of the output regulation problem using the solution of the Sylvester differential equation.

In the case of finite-dimensional systems the Sylvester differential equation is a system of ordinary differential equations. For distributed parameter systems, however, the equation becomes a considerably more interesting time-dependent operator equation with unbounded families of operators. The equations of this particular type have been studied very little in the literature. The finite-dimensional Sylvester differential equation has been considered in [60, 28, 20], and its infinite-dimensional version has been studied in the special case where $A(t) \equiv A$ and $B(t) \equiv B$ are generators of strongly continuous semigroups [9, 8]. For time-dependent families of operators some results are known for time-dependent Riccati equations [4].

1.3 Organization and Main Contributions

The main contributions of this thesis are short-listed in the following.

- The generalization of the internal model principle of robust output regulation to distributed parameter systems with infinite-dimensional signal generators.
- Theory of output regulation and methods of controller design for infinite-dimensional systems with periodically time-dependent signal generators.
- A robust controller for infinite-dimensional nondiagonal exosystems and new easily verifiable sufficient conditions for the solvability of the robust output regulation problem.

- An infinite-dimensional exosystem capable of generating polynomially bounded signals. Classification of the smoothness of the generated signals based on the choices of the initial states of the exosystem.

- Sufficient conditions for the solvability of infinite-dimensional Sylvester differential equations.

- Theory of robust output regulation for systems with infinite-dimensional exosystems allowing unbounded solutions of the regulator equations.

We will now outline the organization of this thesis and highlight the main results presented in each of the chapters. Certain parts of the theory in this thesis have been published in [36, 40, 35, 41, 37, 39], and some parts of it extend the results presented in these papers. In the writing of all these publications the author of this thesis did the mathematical work and wrote the manuscripts.

The theory presented in the publication [38] is also closely related to the topic of this thesis. In this reference the author of this thesis presents classes of perturbations preserving strong and polynomial stablities of semigroups generated by certain types of Riesz-spectral operators frequently encountered in applications.

Chapter 2

In this chapter we consider the generation of the reference and disturbance signals used later in the thesis. We begin by discussing applications where the output regulation of general periodic signals is necessary. In the main part of the chapter we present two alternative approaches to generating signals of the form (1.1) with general continuous periodic coefficient functions $y_j(\cdot)$, namely, the autonomous infinite-dimensional exosystem and the nonautonomous periodic exosystem. We analyze in detail the classes of signals generated by the exosystems. The main results of the chapter include, in particular, demonstration of the close connection between the smoothness properties of the signals generated by the infinite-dimensional exosystem and the corresponding choices of the initial states of the signal generator. The periodic exosystem introduced in this chapter generalizes the one used in [60] to achieve generation of nonsmooth signals. The chapter is concluded by a discussion concerning the advangates and disadvantages of each of these two methods for generating the same types of signals. The constructions of the exosystems in this chapter are based on those presented in the publications [36, 40, 35, 41, 39]. The results on the properties of the classes of signals generated by the exosystems have not been previously published.

Chapter 3

In this chapter we formulate the output regulation problem for distributed parameter systems and infinite-dimensional exosystems. The main result of the chapter is the characterization of the controllers solving the problem using the solution of the associated Sylvester equation. This result extends the corresponding theorem for finite-dimensional and diagonal infinite-dimensional exosystems. The result also generalizes the existing theory by allowing the solution of the regulator equations to be an unbounded operator. We show that the level of this unboundedness is directly related to the smoothness of the admissible reference and disturbance signals. In the final section of the chapter we formulate the robust output regulation problem for infinite-dimensional exosystems and discuss the properties of the controllers solving this problem. The results of the chapter extend the ones published in [35].

Chapter 4

In this chapter we present the main result of the thesis, the p-copy internal model principle for infinite-dimensional systems with infinite-dimensional exosystems. This result is new even for distributed parameter systems with finite-dimensional signal generators. The proof of this result also establishes the equivalence of the p-copy internal model and two other alternative definitions for the internal model — the internal model structure and the $\mathcal{G}$-conditions — along with precise conditions for their equivalence.

The most important one of these equivalences is the one stating that the controller stabilizing the closed-loop system solves the robust output regulation problem if and only if it satisfies the $\mathcal{G}$-conditions. This establishes the fact that these conditions can indeed be used as an alternative definition of the internal model. In particular, the $\mathcal{G}$-conditions are useful in the case of an infinite-dimensional output space, since in this case the p-copy internal model no longer guarantees robust output regulation. The results concerning this relationship generalize the ones in [46, 15] where the authors showed that for more restricted classes of signal generators the $\mathcal{G}$-conditions imply that the controller solves the robust output regulation problem, but the reverse implication was not studied. Our results also show that the internal model structure is equivalent to the controller solving the robust output regulation problem under considerably weaker assumptions than the ones used in [22, 23].

The results presented in this chapter are based on those appearing in publications [36, 40, 35].

Chapter 5

In this chapter we design an observer-based error feedback controller solving the robust output regulation problem related to a distributed parameter system and an infinite-dimensional exosystem. We use the results presented in Chapter 4 to choose the form of the controller in such a way that it satisfies the $\mathscr{G}$-conditions. Subsequently, we choose the parameters of the controller to stabilize the closed-loop system strongly. The stabilization of the closed-loop system is achieved using pole placement of an infinite-spectrum to stabilize the internal model in the controller. This method also allows us to derive easily verifiable sufficient conditions for the solvability of the robust output regulation problem based on the level of smoothness of the considered reference and disturbance signals. The chapter is concluded with an example where we design a robust error feedback controller to steer the output of a scalar system to the signals generated by an infinite-dimensional exosystem.

The results on the stabilization of the closed-loop system generalize the ones presented in [15], where the signals were generated by an infinite-dimensional diagonal exosystem. The use of pole placement in the stabilization of the internal model is a new approach. As stated above, this method allows us to derive simple sufficient conditions for the solvability of the robust output regulation problem. Furthermore, our method of constructing the signal generator now allows relating the smoothness of the reference and disturbance signals to the conditions for the solvability of the problem in a very concrete new way.

The results in this chapter have not been previously published.

Chapter 6

As we already remarked, the literature offers very little results on the solvability of the infinite-dimensional Sylvester differential equation. Because of this, we dedicate Chapter 6 to the study of this time-dependent operator differential equation. The results presented in this chapter are subsequently used in Chapter 7 to study the output regulation problem related to the periodic exosystem. We also use the opportunity to recall the definition of the strongly continuous evolution families related to nonautonomous abstract Cauchy problems. The main results of this section are the conditions for the unique classical solvability of the infinite-dimensional Sylvester differential equation, and the conditions for the existence of a unique periodic classical and mild solutions of the equation with periodic coefficients.

The results presented in this chapter are based on the ones appearing in publications [41, 39, 37].

Chapter 7

In this chapter we formulate the periodic output regulation problem consisting of finding a controller to asymptotically steer the output of a linear time-invariant distributed parameter system to the signals generated by a periodic exosystem. The main result of the chapter is the characterization of periodic controllers solving the problem using the solution of an infinite-dimensional Sylvester differential equation. In particular we will show that analogously to the case of a time-invariant signal generator considered in Chapter 3, the state of the closed-loop system can be expressed using a periodic solution of the associated Sylvester differential equation. This connection between the Sylvester differential equation and the closed-loop system is a significant new result previously unknown for periodic exosystems. In particular it allows us to analyze the asymptotic behavior of the regulation error and to prove the characterization of the controllers solving the periodic output regulation problem.

The results in this chapter generalize the ones presented in [60, 20] to infinite-dimensional systems and periodic exosystems capable of generating nonsmooth signals. On the other hand, the theory presented in the chapter also generalizes the corresponding theory of output regulation of infinite-dimensional systems with autonomous exosystems considered in Chapter 3. Consequently, the results presented here generalize the theory in [24, 15, 6] and others.

The results presented in this chapter are based on the ones in publications [41, 39].

Chapter 8

In this chapter we generalize the static state feedback law and dynamic error feedback controller used in the control of finite and infinite-dimensional systems with time-invariant exosystems to solve the periodic output regulation problem. We use the general results obtained for the closed-loop system in Chapter 7 to derive conditions for the solvability of the problem using these particular types of controllers. The conditions are given using the solvability of certain constrained Sylvester differential equations. The use of these results is illustrated with an example where we steer the output of a scalar delay system to the signals generated by a periodic exosystem.

The controller types used in this chapter and the conditions for their existence generalize the control laws and corresponding results for finite-dimensional [19] and infinite-dimensional [6, 24] systems with autonomous signal generators.

The results presented in this chapter are based on the ones appearing in publications [41, 39].

Chapter 9

The last chapter of this thesis contains concluding remarks and discussion on the results presented in the thesis. In particular, we compare the theories of output regulation presented for the two types of exosystems capable of generating similar reference and disturbance signals. We also discuss possibilities and difficulties of formulating the robust output regulation problem related to a periodic exosystem.

1.4 Notation and Definitions

If X and Y are Banach spaces and $A : X \to Y$ is a linear operator, we denote by $\mathcal{D}(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ the domain, kernel and range of A, respectively. The space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$. If $A : \mathcal{D}(A) \subset X \to X$, then $\sigma(A)$, $\sigma_p(A)$ and $\rho(A)$ denote the spectrum, the point spectrum and the resolvent set of A, respectively. For $\lambda \in \rho(A)$ the resolvent operator is given by $R(\lambda,A) = (\lambda I - A)^{-1}$. If the operator A generates a strongly continuous semigroup on X, then this semigroup is usually denoted by $T_A(t)$.

We denote the dual space of a Banach space X by $X^* = \mathcal{L}(X,\mathbb{C})$. The dual pairing between elements in a Banach space and its dual is denoted by

$$\langle x, x^* \rangle = x^*(x), \qquad x \in X, \ x^* \in X^*.$$

The same notation is used for the inner product on a Hilbert space X. The adjoint operator of a linear operator $A : \mathcal{D}(A) \subset X \to Y$ between Banach spaces X and Y is denoted by $A^* : \mathcal{D}(A^*) \subset Y^* \to X^*$.

We use notation $\{x_k\}_{k\in I} = \{ x_k \mid k \in I \}$ for a set containing elements x_k for some indices $k \in I \subset \mathbb{Z}$. If the set I of indices is a list $I = \{1,\dots,n\}$ for some $n \in \mathbb{N}$ we also denote $\{x_k\}_{k=1}^n = \{ x_k \mid k \in \{1,\dots,n\} \}$. Ordered sequences of elements x_k for $k \in \{1,\dots,n\}$ and for $k \in \mathbb{Z}$ are denoted by $(x_k)_{k=1}^n$ and $(x_k)_{k\in\mathbb{Z}}$, respectively. We use notation $\overline{\Omega}$ for a closure of a set $\Omega \subset \mathbb{C}$. For $p \in [1,\infty)$ the spaces of p-summable sequences with elements in X are denoted by

$$\ell^p(X) = \left\{ (x_k)_{k\in\mathbb{Z}} \;\middle|\; \sum_{k\in\mathbb{Z}} \|x_k\|^p < \infty \right\}.$$

For an interval $[a,b] \subset \mathbb{R}$ and a Banach space X the spaces of continuous functions and continuously differentiable functions $f : [a,b] \to X$ are denoted by $C([a,b],X)$ and $C^1([a,b],X)$, respectively. We denote by $C([a,b],\mathcal{L}(X,Y))$ and $C([a,b],\mathcal{L}_s(X,Y))$ the spaces of the operator-valued functions continuous in uniform and strong operator topologies of $\mathcal{L}(X,Y)$, respectively. Analogously, the notations $C^1([a,b],\mathcal{L}(X,Y))$ and $C^1([a,b],\mathcal{L}_s(X,Y))$ stand for the spaces of functions continuously differentiable with respect to the uniform and strong operator topologies, respectively. The space of τ-periodic continuous functions is defined as

$$C_\tau(\mathbb{R},X) = \left\{ f : \mathbb{R} \to X \;\middle|\; f \text{ is continuous, } f(t+\tau) = f(t) \;\; \forall t \in \mathbb{R} \right\},$$

and $C^1_\tau(\mathbb{R},X)$ denotes the space of τ-periodic continuously differentiable functions. The spaces

$$C_\tau(\mathbb{R},\mathscr{L}(X,Y)), \qquad C_\tau(\mathbb{R},\mathscr{L}_s(X,Y)), \qquad C^1_\tau(\mathbb{R},\mathscr{L}(X,Y)), \qquad C^1_\tau(\mathbb{R},\mathscr{L}_s(X,Y))$$

of operator-valued τ-periodic functions are defined in the analogous way. For an operator-valued function $A(\cdot) \in C_\tau(\mathbb{R},\mathscr{L}_s(X,Y))$ we define

$$\|A\|_\infty = \sup_{t\in[0,\tau]} \|A(t)\| < \infty.$$

The notation $L^2(a,b)$ stands for the Hilbert space of square integrable functions on the interval $[a,b]$, and $L^1_{\text{loc}}(\mathbb{R})$ denotes the space of locally integrable functions.

For a sequence $(a_k)_{k\in\mathbb{Z}} \subset \mathbb{R}$ and for $\alpha \in \mathbb{R}$ we use the notation

$$a_k = \mathscr{O}(|k|^\alpha)$$

if there exist constants $M > 0$ and $N \in \mathbb{N}$ such that $|a_k| \leq M|k|^\alpha$ for all $k \in \mathbb{Z}$ with $|k| \geq N$.

Chapter 2

Generation of Reference and Disturbance Signals

In this chapter we consider ways of generating the reference and disturbance signals used in the output regulation problem. As we have already seen, this involves constructing an exosystem, which is a linear system producing the signals as its output. In output regulation of linear time-invariant systems the most commonly used exosystem is of the form

$$\dot{v}(t) = Sv(t), \qquad v(0) = v_0 \in W \tag{2.1a}$$

$$y_{ref}(t) = Fv(t), \tag{2.1b}$$

where $W = \mathbb{C}^q$, $F \in \mathscr{L}(W, Y)$ and the spectrum of the matrix S is on the imaginary axis $i\mathbb{R}$. The signals generated by this type of exosystem are of the form

$$y_{ref}(t) = y_{ref}(t) = y_n(t)t^n + \cdots + y_1(t)t + y_0(t), \tag{2.2}$$

where $y_k(\cdot)$ are linear combinations of trigonometric functions. An immediate consequence of this is that the signals generated by a finite-dimensional exosystem are always smooth functions. However, in applications it is often necessary or desirable to consider tracking of signals which are not even continuously differentiable. This is the case, for example, in the control of power supplies of proton synchrotrons, where the reference signals are periodic signals whose derivatives are not continuous [59, and references therein].

Our main goal in this thesis is to generalize the theory of output regulation of distributed parameter systems to allow more general classes of reference and disturbance signals. To this end, we will in this chapter explore two different types of generalizations of the signal generator (2.1) used in the mathematical formulation of the output regulation problem. Our main goal is to construct signal generators which are capable of generating signals of the form (2.2) where the coefficient functions $y_k(\cdot)$ are general continuous and periodic functions.

The first generalization of the exosystem (2.1) is obtained by allowing the space W to be a separable Hilbert space and letting S be an unbounded operator generating a strongly continuous group on W. In the case where the group generated by S is isometric, the output regulation problem has been studied most notably by Immonen [22], Immonen and Pohjolainen [24, 23] and by Hämäläinen and Pohjolainen [15]. The types of exosystems considered in these references are indeed very general, but they have the drawback of only being able to generate signals that are uniformly bounded. In many engineering applications it is necessary to track signals that grow polynomially in time. The simplest example of such reference signals is

$$y_{ref}(t) = t.$$

In this thesis we show how to overcome this limitation by constructing an infinite-dimensional exosystem whose system operator S is an infinite-dimensional block-diagonal operator. The resulting exosystem is very well suited to generating signals of the form (2.1) where $y_k(\cdot)$ are possibly nonsmooth functions.

The second type of generalization of the exosystem (2.1) we consider in this thesis is a time-periodic system of the form

$$\dot{v}(t) = S(t)v(t), \qquad v(0) = v_0 \in W \tag{2.3a}$$

$$y_{ref}(t) = F(t)v(t), \tag{2.3b}$$

on the space $W = \mathbb{C}^q$. In this signal generator the functions $S(\cdot) : \mathbb{R} \to \mathcal{L}(W)$ and $F(\cdot) : \mathbb{R} \to \mathcal{L}(W, Y)$ are periodic with the same period. We will see that these types of exosystems are ideal for generating the type of reference and disturbance signals we are interested in.

We will see in Chapters 3 and 7 that the output regulation problems are defined in such a way that the tracking of the reference signal and the rejection of the disturbance signals must happen for all initial states of the exosystem. Because of this, we can consider the operators S and F and the functions $S(\cdot)$ and $F(\cdot)$ as fixed *parameters* which are chosen when the signal generator is constructed. On the other hand, the initial state $v_0 \in W$ of the exosystem is a variable which can be used to generate different signals with the same exosystem. During the course of this chapter it will become clear that although the infinite-dimensional and the periodic exosystems can be used to generate very similar *individual* signals, the effect of the initial state v_0 on the signals constitutes a significant difference between the *classes* of signals these two types of exosystems generate.

The structure and the main contributions of this chapter are outlined in the following.

Section 2.1. Before the formal definitions of the signal generators, we will first discuss applications where tracking of nonsmooth signals is necessary.

Section 2.2. In this section we construct an infinite-dimensional exosystem capable of generating polynomially growing signals. We show that these types of exosystems can be used to generate reference signals of the form (2.2) where $y_j(\cdot)$ are bounded and uniformly continuous functions. Since our main interests are the cases where $y_j(\cdot)$ are periodic functions, we will analyze such reference signals in greater detail. We will show that in the periodic case there is a close connection between the smoothness of the generated signals and the choice of the inital state $v_0 \in W$ of the exosystem. In particular, we will introduce scale spaces W_α for the initial states and relate them to classes of signals in which the functions $y_j(\cdot)$ are in the Sobolev spaces of periodic functions.

Section 2.3. This section is dedicated to the study of the periodic exosystem. We show that the exosystems of this type are ideally suited to generating signals of the form (2.2) where the $y_j(\cdot)$ are continuous periodic functions. We also characterize the classes of generated reference signals and illustrate the construction of periodic exosystems with examples.

Section 2.4 In this section we compare the two types of exosystems introduced in this chapter. At this point we only give a brief account of their most apparent differences. The in-depth comparison of the signal generators is postponed until Chapter 9, where we can compare the strengths and weaknesses of the theories of output regulation corresponding to the two ways of generating the reference and disturbance signals.

2.1 Examples of Periodic Reference Signals

As was already stated, the signal generators introduced in this chapter can be used to generate reference signals that are continuous but not necessarily continuously differentiable. In this section we discuss a few applications for these types of signals. It can of course be argued — since any nonsmooth continuous signals can be approximated with any given finite accuracy using smooth signals — that a finite-dimensional signal generator is sufficiently general for any applications of output regulation we might encounter. However, in some applications where a very high relative accuracy is required, using an exosystem capable of generating nonsmooth signals can be beneficial. These types of situations are frequently encountered in — for example — the control of robot arms, control of disk drive systems and control of magnetic power supplies of proton synchrotrons [59, 16, and references therein].

The motion of a robot arm tracing a square with a constant speed is a simple example of a nonsmooth periodic path. If we want to steer the robot arm to this trajectory, we can use a reference signal consisting of the horizontal and vertical components of the motion as functions of time. Both components of the reference signal are periodic functions continuously differentiable outside a countable set of points. Figure 2.1 shows the desired motion of the robot arm along with a choice

$$y_{ref}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

for the reference function.

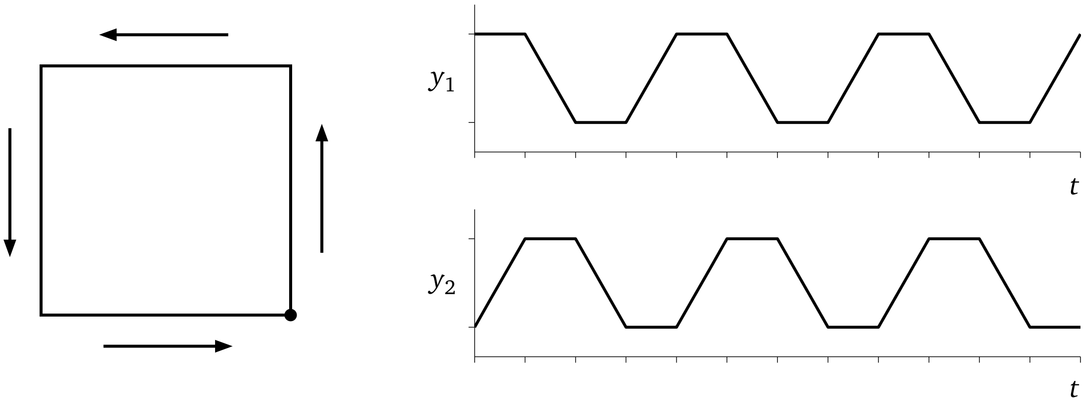

Figure 2.1: Robot arm tracing a square with a constant speed.

The reference signal resulting from the robot arm tracing a square at a constant speed is a continuous periodic $\mathbb{C}^2$-valued uniformly bounded function. These types of signals can be generated by the exosystem used in [23, 24, 15]. However, if the robot arm must move to another location between the cycles in a repetitive fashion, the appropriate reference signal must incorporate an unbounded component. This kind of situation is depicted in Figure 2.2. In the figure the dashed line denotes the translation between two starting points. In the case of horizontal translation, the resulting reference signal can be written in the form

$$y_{ref}(t) = y_0(t) + c \begin{pmatrix} 1 \\ 0 \end{pmatrix} t,$$

where $y_0(\cdot)$ is a $\mathbb{C}^2$-valued uniformly bounded periodic function and $c > 0$ is a constant.

Another common example of a nonsmooth reference signal useful in applications is the *triangle signal* shown in Figure 2.3. This is again a uniformly continuous periodic function and can be generated by a signal generator used in [23, 24, 15]. On the other hand, if $y_0(\cdot)$ is the triangle signal, then the reference signal defined as

$$y_{ref}(t) = t + y_0(t) \tag{2.4}$$

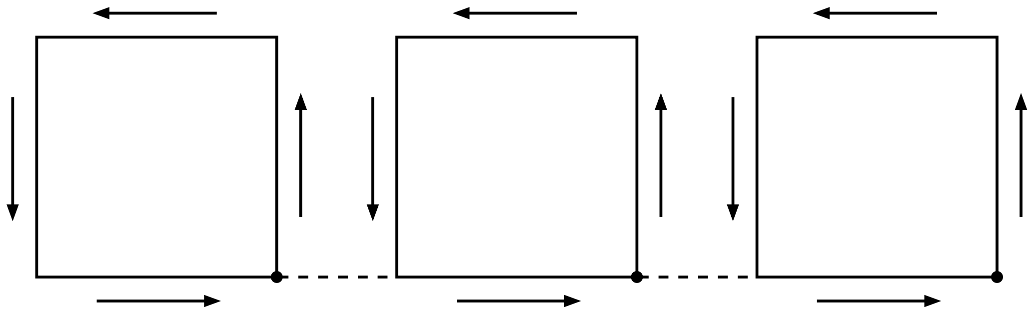

Figure 2.2: Robot arm tracing a square with translation between the cycles.

is an unbounded piecewise linear function which is uniformly continuous and differentiable almost everywhere. This signal is depicted in Figure 2.4. Both the triangle signal and the signal in (2.4) are repeatedly considered in examples throughout the rest of this thesis.

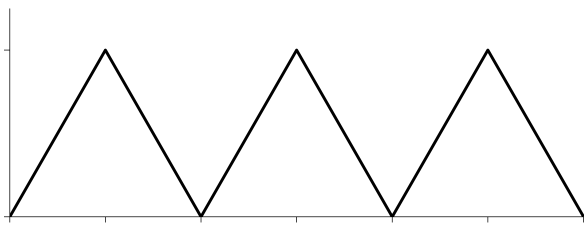

Figure 2.3: The triangle signal.

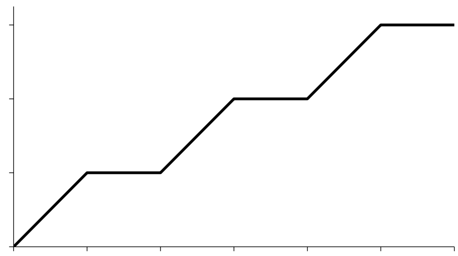

Figure 2.4: The signal $y_0(t) + t$.

2.2 The Infinite-Dimensional Exosystem

In this section we will construct an infinite-dimensional generalization of the signal generator (2.1). In this signal generator the system operator S consists of an infinite number of finite-dimensional Jordan blocks. To motivate the abstract definitions involved in the construction, we will first take a closer look at the finite-dimensional exosystem (2.1).

By possibly applying a time-invariant similarity transformation to the differential equation (2.1a), we can without loss of generality assume that the matrix S is in its Jordan canonical form, i.e.,

$$S = \operatorname{diag}\left(S_1, S_2, \ldots, S_m\right),$$

where S_k is a Jordan block of dimension $n_k \times n_k$ associated to an eigenvalue $i\omega_k \in i\mathbb{R}$. For $v \in \mathbb{C}^q$ the exponential matrix of a Jordan block S_k can be written in the form

$$e^{S_k t}v = e^{i\omega_k t}\sum_{l=1}^{n_k}\langle v, e_k^l\rangle \sum_{j=1}^{l}\frac{t^{l-j}}{(l-j)!}e_k^j,$$

where $\left\{e_k^l\right\}_{l=1}^{n_k}$ denotes the orthonormal set of generalized eigenvectors related to the eigenvalue $i\omega_k$.

In order to generalize this block-diagonal structure to an infinite-dimensional space in a mathematically sound way, we will start by defining an orthonormal basis consisting of an infinite number of finite groups of vectors. Each of these finite groups will become a sequence of generalized eigenvectors associated to a single finite-dimensional Jordan block S_k. Finally, the system operator S of the exosystem will be defined as a diagonal operator consisting of these blocks with an appropriate domain of definition.

Let the output space Y be a Hilbert space and construct the state space W of the exosystem in such a way that it is a separable Hilbert space with an orthonormal basis

$$\{\phi_k^l\}_{kl} := \left\{ \phi_k^l \in W \;\middle|\; k \in \mathbb{Z},\ l = 1,\dots,n_k \right\}.$$

By this we mean that

$$W = \overline{\operatorname{span}}\left\{\phi_k^l\right\}_{kl} \quad \text{and} \quad \langle \phi_k^l, \phi_n^m\rangle = \begin{cases} 1 & k=n,\, l=m \\ 0 & \text{otherwise.}\end{cases}$$

We assume that the lengths $n_k \in \mathbb{N}$ of the subsequences are uniformly bounded. For given frequencies $(\omega_k)_{k\in\mathbb{Z}} \subset \mathbb{R}$ the operators $S_k \in \mathscr{L}(W)$ representing the finite-dimensional Jordan blocks are defined as

$$S_k = i\omega_k\langle\cdot,\phi_k^1\rangle\phi_k^1 + \sum_{l=2}^{n_k}\langle\cdot,\phi_k^l\rangle\left(i\omega_k\phi_k^l + \phi_k^{l-1}\right).$$

Using these operators the infinite-dimensional exosystem on the space W can be defined as follows.

Definition 2.1 (The infinite-dimensional exosystem)**.** Let the separable Hilbert space W be of the form described above and let $(\omega_k)_{k\in\mathbb{Z}} \subset \mathbb{R}$. The infinite-dimensional exosystem

$$\dot{v}(t) = Sv(t), \qquad v(0) = v_0 \in W \tag{2.5a}$$

$$y_{ref}(t) = Fv(t) \tag{2.5b}$$

on the space W is constructed by choosing the system operator S of the exosystem as

$$Sv = \sum_{k\in\mathbb{Z}} S_k v, \qquad \mathscr{D}(S) = \left\{ v \in W \;\middle|\; \sum_{k\in\mathbb{Z}}\|S_k v\|^2 < \infty \right\}$$

and an output operator $F \in \mathscr{L}(W,Y)$ satisfying

$$\sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} \|F\phi_k^l\|^2 < \infty.$$

■

The operators S_k defined above satisfy

$$(i\omega_k I - S_k)\phi_k^1 = 0, \qquad (S_k - i\omega_k I)\phi_k^l = \phi_k^{l-1} \quad \forall l \in \{2,\dots,n_k\}$$

and thus they can indeed be viewed as single Jordan blocks of dimensions n_k associated to eigenvalues $i\omega_k$. Since the operator S is an infinite block diagonal operator consisting of operators the S_k, it can be seen as a generalization of a matrix in a Jordan canonical form. It is straightforward to verify that the spectrum of the operator S satisfies

$$\sigma(S) = \overline{\sigma_p(S)} = \overline{\{i\omega_k\}}_{k\in\mathbb{Z}},$$

where the line denotes the closure of the set in $\mathbb{C}$. Moreover, the operator S generates a C_0-group $T_S(t)$ on W given by

$$T_S(t)v = \sum_{k\in\mathbb{Z}} e^{i\omega_k t} \sum_{l=1}^{n_k} \langle v, \phi_k^l\rangle \sum_{j=1}^{l} \frac{t^{l-j}}{(l-j)!}\phi_k^j, \qquad v \in W,\ t \in \mathbb{R}.$$

It is easy to see that this group is polynomially bounded forward and backwards in time. More precisely, for any $n_S \in \mathbb{N}$ such that $n_S \geq n_k$ for all $k \in \mathbb{Z}$ there exists $M_S \geq 1$ such that

$$\|T_S(t)\| \leq M_S(|t|^{n_S} + 1), \qquad \forall t \in \mathbb{R}.$$

This also implies that the growth bound of the C_0-group is $\omega_0(T_S(t)) = 0$. For $k \in \mathbb{Z}$ we define

$$d_k = \max\left\{ n_l \;\middle|\; l \in \mathbb{Z},\ \omega_l = \omega_k \right\},$$

which corresponds to the dimension of the largest Jordan block associated to an eigenvalue $i\omega_k \in \sigma_p(S)$. For $k \in \mathbb{Z}$ we denote by P_k the orthogonal projection

$$P_k = \sum_{l=1}^{n_k} \langle \cdot, \phi_k^l\rangle \phi_k^l$$

onto the finite-dimensional subspace $\operatorname{span}\{\phi_k^l\}_{l=1}^{n_k}$ of W. With this notation it is easy to see that the domain of the operator S satisfies

$$\mathscr{D}(S) = \left\{ v \in W \;\middle|\; \sum_{k\in\mathbb{Z}} \omega_k^2 \|P_k v\|^2 < \infty \right\} = \left\{ v \in W \;\middle|\; \sum_{k\in\mathbb{Z}} (1+\omega_k^2)\|P_k v\|^2 < \infty \right\}.$$

Before moving on to analyze the classes of signals generated by the exosystem we will define a set of scale spaces $W_\alpha \subset W$ related to the system operator S of the exosystem. They will be used in the classification of the generated signals based on which spaces W_α the corresponding initial states belong to. Later in the section we will show that this kind of classification also has a close relationship to the regularity of the generated signals.

Definition 2.2. For $\alpha \geq 0$ we denote by $(W_\alpha, \|\cdot\|_\alpha)$ the space

$$W_\alpha = \left\{ v \in W \;\middle|\; \sum_{k\in\mathbb{Z}} (1+\omega_k^2)^\alpha \|P_k v\|^2 < \infty \right\}$$

with norm $\|\cdot\|_\alpha$ defined by

$$\|v\|_\alpha^2 = \sum_{k\in\mathbb{Z}} (1+\omega_k^2)^\alpha \|P_k v\|^2, \qquad v \in W_\alpha.$$

■

It is easy to show that for all $\alpha \geq 0$ the spaces $(W_\alpha, \|\cdot\|_\alpha)$ are Hilbert spaces and for all $0 \leq \beta \leq \alpha$ we have $W_\alpha \subset W_\beta$ and

$$\|v\|_\beta \leq \|v\|_\alpha$$

for all $v \in W_\alpha$. For nonnegative integer values $m \in \mathbb{N}_0$ the spaces W_m coincide with the domains $\mathcal{D}((S+I)^m)$ and the norms $\|\cdot\|_m$ are equivalent to the norms defined by the mappings $v \mapsto \|(S+I)^m v\|$ on W_m. Furthermore, it can be verified that the spaces W_α are invariant under the group $T_S(t)$, the restrictions $T_S(t)|_{W_\alpha}$ are strongly continuous groups on W_α and the generators of these groups are $S|_{W_\alpha} : \mathcal{D}(S|_{W_\alpha}) \subset W_\alpha \to W_\alpha$ with domains $\mathcal{D}(S|_{W_\alpha}) = W_{\alpha+1}$ [10, Sec. II.5].

The Classes of Signals

Using the formal definition and the properties of the infinite-dimensional signal generator presented in the previous section we can study the signals generated by the exosystem. We will show that the generated signals are in general of the form

$$y_{ref}(t) = y_n(t)t^n + \cdots + y_1(t)t + y_0(t) \tag{2.6}$$

where the coefficient functions $y_j(\cdot)$ are *almost periodic functions* [1, Def. 4.5.6]. These are bounded and uniformly continuous functions that can be uniformly approximated by *trigonometric polynomials*, i.e., by linear combinations of functions of the form $t \mapsto e^{i\omega t} y$, where $\omega \in \mathbb{R}$ and $y \in Y$.

In the next section we will concentrate on the case where the coefficient functions are periodic with the same period. For these types of reference signals we will in particular study the relationship between the smoothness of the coefficient signals $y_j(\cdot)$ and the choice of the initial state v_0 from the space W_α for $\alpha \geq 0$.

Theorem 2.3. *The signals generated by the infinite-dimensional exosystem are of the form* (2.6)*, where $y_j(\cdot) : \mathbb{R} \to Y$ are almost periodic functions for all $j \in \{0, \dots, n\}$ and where $n = \max_{k\in\mathbb{Z}} n_k - 1$.*

Proof. For all initial states $v_0 \in W$ the state of the exosystem is given by $v(t) = T_S(t)v_0$ and thus

$$\begin{aligned} y_{ref}(t) = Fv(t) &= \sum_{k\in\mathbb{Z}} e^{i\omega_k t} \sum_{l=1}^{n_k} \langle v_0, \phi_k^l \rangle \sum_{j=1}^{l} \frac{t^{l-j}}{(l-j)!} F\phi_k^j = \sum_{k\in\mathbb{Z}} e^{i\omega_k t} \sum_{l=1}^{n_k} \langle v_0, \phi_k^l \rangle \sum_{j=0}^{l-1} \frac{t^j}{j!} F\phi_k^{l-j} \\ &= \sum_{k\in\mathbb{Z}} e^{i\omega_k t} \sum_{j=0}^{n_k-1} t^j \cdot \frac{1}{j!} \sum_{l=j+1}^{n_k} \langle v_0, \phi_k^l \rangle F\phi_k^{l-j} = \sum_{k\in\mathbb{Z}} e^{i\omega_k t} \sum_{j=0}^{n_k-1} a_{jk} t^j, \end{aligned}$$

where we have denoted

$$a_{jk} = \frac{1}{j!} \sum_{l=j+1}^{n_k} \langle v_0, \phi_k^l \rangle F\phi_k^{l-j} \in Y. \tag{2.7}$$

Let $n = \max_{k\in\mathbb{Z}} n_k - 1$ and define $a_{jk} = 0 \in Y$ for all $k \in \mathbb{Z}$ and $j \in \{n_k + 1, \dots, n\}$. Then for any $j \in \{0, \dots, n\}$ we can use the Cauchy-Schwarz inequality twice to show that

$$\begin{aligned} j! \cdot \sum_{k\in\mathbb{Z}} \|a_{jk}\| &= \sum_{k\in\mathbb{Z}} \left\| \sum_{l=j+1}^{n_k} \langle v_0, \phi_k^l \rangle F\phi_k^{l-j} \right\| \leq \sum_{k\in\mathbb{Z}} \sum_{l=j+1}^{n_k} |\langle v_0, \phi_k^l \rangle| \cdot \|F\phi_k^{l-j}\| \\ &\leq \sum_{k\in\mathbb{Z}} \left(\sum_{l=j+1}^{n_k} |\langle v_0, \phi_k^l \rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{l=1}^{n_k-j} \|F\phi_k^l\|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} |\langle v_0, \phi_k^l \rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} \|F\phi_k^l\|^2 \right)^{\frac{1}{2}} < \infty, \end{aligned}$$

and thus $(a_{jk})_{k\in\mathbb{Z}} \in \ell^1(Y)$ for all $j \in \{0, \dots, n\}$. This implies that

$$y_{ref}(t) = \sum_{k\in\mathbb{Z}} e^{i\omega_k t} \sum_{j=0}^{n_k-1} a_{jk} t^j = \sum_{k\in\mathbb{Z}} e^{i\omega_k t} \sum_{j=0}^{n} a_{jk} t^j = \sum_{j=0}^{n} t^j \sum_{k\in\mathbb{Z}} a_{jk} e^{i\omega_k t} = \sum_{j=0}^{n} t^j y_j(t),$$

where we have in turn denoted

$$y_j(t) = \sum_{k\in\mathbb{Z}} a_{jk} e^{i\omega_k t} \tag{2.8}$$

for all $j \in \{0,\dots,n\}$. Changing the order or summation is permitted since the series are absolutely convergent. To prove the theorem it is now sufficient to show that the functions $y_j(\cdot)$ are almost periodic. This follows directly from the fact that if we have $j \in \{0,\dots,n\}$ and if $N > 0$, then the functions

$$t \mapsto \sum_{k=-N}^{N} a_{jk} e^{i\omega_k t}$$

are trigonometric polynomials and $(a_{jk})_{k\in\mathbb{Z}} \in \ell^1(Y)$ implies

$$\left\| y_j(t) - \sum_{k=-N}^{N} a_{jk} e^{i\omega_k t} \right\| = \left\| \sum_{|k|>N} a_{jk} e^{i\omega_k t} \right\| \leq \sum_{|k|>N} \|a_{jk}\| \longrightarrow 0$$

uniformly in $t \in \mathbb{R}$ as $N \to \infty$. □

The proof of Theorem 2.3 also suggests a method for constructing an exosystem to generate a given signal of the form (2.6), where the coefficient functions can be written as (2.8). This can be done in a very straightforward manner by choosing the output operator F and the initial state v_0 of the exosystem in such a way that the equations (2.7) are satisfied for all $k \in \mathbb{Z}$. The following lemma presents one possible choice for the parameters.

Lemma 2.4. *Let $(\omega_k)_{k\in\mathbb{Z}} \subset \mathbb{R}$ and assume that for all $j \in \{0,\dots,n\}$ the coefficient functions $y_j(\cdot)$ can be written in the form* (2.8) *with $(a_{jk})_{k\in\mathbb{Z}} \in \ell^1(Y)$. For $k \in \mathbb{Z}$ define*

$$c_k = \begin{cases} \max\left\{ \sqrt{\|a_{jk}\|} \right\}_{j=0}^{n} & a_{jk} \neq 0 \text{ for some } j = 0,\dots,n \\ 1 & \text{otherwise.} \end{cases}$$

The signal (2.6) *can be generated by an infinite-dimensional exosystem with $n_k = n+1$ for all $k \in \mathbb{Z}$ and $F \in \mathcal{L}(W,Y)$ satisfying*

$$F\phi_k^1 = \frac{1}{c_k}(n_k - 1)!\, a_{n_k-1,k},$$
$$F\phi_k^l = \frac{1}{c_k}\Big((n_k - l)!\, a_{n_k-l,k} - (n_k - l + 1)!\, a_{n_k-l+1,k} \Big), \quad l \in \{2,\dots,n_k\}.$$

The signal (2.6) *is generated with an initial state $v_0 \in W$ of the exosystem satisfying*

$$\langle v_0, \phi_k^l \rangle = \max\left\{ \sqrt{\|a_{jk}\|} \right\}_{j=0}^{n}$$

for all $k \in \mathbb{Z}$ and $l \in \{1,\dots,n_k\}$.

Proof. A direct computation shows that these choices of F and v_0 satisfy equations (2.7) for all $k \in \mathbb{Z}$. It is thus sufficient to show that $v_0 \in W$ and

$$\sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} \|F\phi_k^l\|^2 < \infty.$$

For the initial state we have that

$$\sum_{k\in\mathbb{Z}} \|P_k v_0\|^2 = (n+1) \sum_{k\in\mathbb{Z}} \max\left\{\|a_{jk}\|\right\}_{j=0}^n < \infty$$

since $(a_{jk})_{k\in\mathbb{Z}} \in \ell^1(Y)$ for all $j \in \{0,\dots,n\}$, and thus $v_0 \in W$. On the other hand, for the output operator

$$\begin{aligned}
\|F\phi_k^1\| &= \frac{n!}{c_k}\|a_{nk}\| \leq n! \max\left\{\sqrt{\|a_{jk}\|}\right\}_{j=0}^n \\
\|F\phi_k^l\| &\leq \frac{n!}{c_k}\left(\|a_{n+1-l,k}\| + \|a_{n+2-l,k}\|\right) \leq 2n! \max\left\{\sqrt{\|a_{jk}\|}\right\}_{j=0}^n
\end{aligned}$$

and thus

$$\sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} \|F\phi_k^l\|^2 \leq 4(n+1)(n!)^2 \sum_{k\in\mathbb{Z}} \max\left\{\|a_{jk}\|\right\}_{j=0}^n < \infty$$

again since $(a_{jk})_{k\in\mathbb{Z}} \in \ell^1(Y)$ for all $j \in \{0,\dots,n\}$. This concludes the proof. □

The formula in (2.7) shows us that the coefficients a_{jk} of the functions $y_j(\cdot)$ are determined by both the operator F of the exosystem and the choice v_0 of the initial state. Because of this, the different initial states of the same exosystem can generate very different types of signals. We will discuss this matter in greater detail in the next section.

Signals in the Sobolev Spaces of Periodic Functions

We will conclude the theoretical treatment of the infinite-dimensional signal generator by considering signals of the form (2.6) where the coefficient functions $y_j(\cdot)$ are periodic functions with the same period. To this end we assume throughout this section that

$$(\omega_k)_{k\in\mathbb{Z}} = \left(\frac{2\pi k}{\tau}\right)_{k\in\mathbb{Z}}$$

for some $\tau > 0$. It is well-known that for periodic functions the smoothness properties can be characterized via the asymptotic behavior of their Fourier coefficients. It turns out that taking an advantage of this fact we can easily relate the smoothness properties of the generated signals to the choices of the initial states of the exosystem in the spaces W_α.

To classify the generated signals we use the *Sobolev spaces* of periodic functions [27, Sec. 3.6] defined below. In the definition $\hat{f}(k) \in Y$ denote the Fourier coefficients of a continuous τ-periodic function f, i.e.,

$$f(t) = \sum_{k\in\mathbb{Z}} \hat{f}(k) e^{i\omega_k t}, \qquad \forall t \in \mathbb{R}.$$

Definition 2.5 (Sobolev spaces)**.** For $\alpha > \frac{1}{2}$ the Hilbert spaces

$$H^\alpha_{per}(0,\tau) = \left\{ f \in C_\tau(\mathbb{R}, Y) \,\middle|\, \sum_{k\in\mathbb{Z}} (1+\omega_k^2)^\alpha \|\hat{f}(k)\|^2 < \infty \right\}$$

with norms defined by

$$\|f\|^2_{per,\alpha} = \frac{1}{\tau} \sum_{k\in\mathbb{Z}} (1+\omega_k^2)^\alpha \|\hat{f}(k)\|^2, \qquad f \in H^\alpha_{per}(0,\tau)$$

are called the *Sobolev spaces* of periodic functions. ■

The order $\alpha > \frac{1}{2}$ of the space $H^\alpha_{per}(0,\tau)$ is closely related to the smoothness properties of its functions. For example, if $Y = \mathbb{C}$, then for all $m \in \mathbb{N}$ the space $H^m_{per}(0,\tau)$ contains precisely the τ-periodic functions whose distributional derivatives of orders up to m are in $L^2(0,\tau)$. In particular this implies that if $f \in C_\tau(\mathbb{R}, Y)$ is such that the derivatives $f^{(j)}$ exist and are absolutely continuous on $[0,\tau]$ for all $j \in \{0,\dots,m-1\}$, then $f \in H^m_{per}(0,\tau)$.

We will first show that for any infinite-dimensional signal generator the choice of the initial state of the exosystem in a space W_α directly translates to the smoothness of the generated signal.

Theorem 2.6. *If $v_0 \in W_\alpha$ for some $\alpha > \frac{1}{2}$, then the coefficient functions of the signal generated by the infinite-dimensional exosystem satisfy $y_j(\cdot) \in H^\alpha_{per}(0,\tau)$.*

Proof. Let $n = \max_{k\in\mathbb{Z}} n_k - 1$ and let $j \in \{0,\dots,n\}$ be arbitrary. From the proof of Theorem 2.3 we have that $(a_{jk})_{k\in\mathbb{Z}} \in \ell^1(Y)$. Together with the formula in (2.8) this implies that $(a_{jk})_{k\in\mathbb{Z}}$ are the Fourier coefficients of the function $y_j(\cdot)$. Using (2.7) we can also see that

$$\begin{aligned}\|a_{jk}\| &\le \frac{1}{j!} \sum_{l=j+1}^{n_k} |\langle v_0, \phi_k^l\rangle| \cdot \|F\phi_k^{l-j}\| \le \|F\| \sum_{l=1}^{n_k} |\langle v_0, \phi_k^l\rangle| \le \sqrt{n}\|F\| \left(\sum_{l=1}^{n_k} |\langle v_0, \phi_k^l\rangle|^2 \right)^{\frac{1}{2}} \\ &= \sqrt{n}\|F\| \cdot \|P_k v_0\|\end{aligned}$$

for all $k \in \mathbb{Z}$. The fact that $v_0 \in W_\alpha$ now implies

$$\sum_{k\in\mathbb{Z}} (1+\omega_k^2)^\alpha \|a_{jk}\|^2 \le n\|F\|^2 \cdot \sum_{k\in\mathbb{Z}} (1+\omega_k^2)^\alpha \|P_k v_0\|^2 < \infty,$$

and thus $y_j(\cdot) \in H^\alpha_{per}(0,\tau)$. Since $j \in \{0,\dots,n\}$ was arbitrary, this concludes the proof. □

The next theorem states a converse result which shows that the signal generator can be chosen in such a way that the smoothness of the reference signal is also translated to the property $v_0 \in W_\alpha$ of the corresponding initial state.

Theorem 2.7. *Let $\beta > \frac{1}{2}$ and assume $y_j(\cdot) \in H^\beta_{per}(0,\tau)$ for all $j \in \{0,\dots,n\}$. For any $0 \le \alpha < \beta - \frac{1}{2}$ the infinite-dimensional exosystem can be chosen in such a way that the reference signal* (2.6) *is generated with a choice $v_0 \in W_\alpha$ of the initial state.*

Proof. The functions $y_j(\cdot)$ are of the form

$$y_j(t) = \sum_{k\in\mathbb{Z}} a_{jk} e^{i\omega_k t}, \qquad t \in \mathbb{R}.$$

Let $0 \le \alpha < \beta - \frac{1}{2}$, define

$$c_k = \begin{cases} (1+\omega_k^2)^{\frac{\beta-\alpha}{2}} \max\left\{\|a_{jk}\|\right\}_{j=0}^n & a_{jk} \neq 0 \text{ for some } j \in \{0,\dots,n\} \\ 1 & \text{otherwise} \end{cases}$$

and choose W and $F\phi_k^l$ as in Lemma 2.4. A direct computation shows that (2.7) are satisfied if we choose

$$\langle v_0, \phi_k^l \rangle = (1+\omega_k^2)^{\frac{\beta-\alpha}{2}} \max\left\{\|a_{jk}\|\right\}_{j=0}^n.$$

It remains to show that $v_0 \in W_\alpha$ and that

$$\sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} \|F\phi_k^l\|^2 < \infty.$$

As in the proof of Lemma 2.4 it is easy to see that we have

$$\|F\phi_k^l\| \le \frac{2n!}{c_k} \max\left\{\|a_{jk}\|\right\}_{j=0}^n \le 2n!(1+\omega_k^2)^{\frac{\alpha-\beta}{2}} < \infty$$

and thus

$$\sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} \|F\phi_k^l\|^2 \le 4(n+1)(n!)^2 \sum_{k\in\mathbb{Z}} (1+\omega_k^2)^{\alpha-\beta} < \infty$$

since $\omega_k = \frac{2\pi k}{\tau}$ and $\alpha - \beta < -\frac{1}{2}$. Furthermore, we have for v_0 that

$$\sum_{k\in\mathbb{Z}} (1+\omega_k^2)^\alpha \|P_k v_0\|^2 = (n+1) \sum_{k\in\mathbb{Z}} (1+\omega_k^2)^\beta \max\left\{\|a_{jk}\|^2\right\}_{j=0}^n < \infty,$$

since $y_j(\cdot) \in H^\beta_{per}(0,\tau)$. This concludes the proof. □

As was already mentioned in the end of the previous section, the choice of the initial state of the infinite-dimensional signal generator can have a drastic effect on the generated signal. The next example illustrates this property of the infinite-dimensional exosystems.

Example 2.8. Assume $Y = \mathbb{C}$ and $\omega_k = \frac{2\pi k}{\tau}$ and choose the parameters of the infinite-dimensional exosystem in such a way that $n_k = 1$ and

$$F\phi_k = \frac{1}{|k|}$$

for all $k \in \mathbb{Z}$. As we saw before, the Fourier coefficients of the generated signals are precisely $\hat{y}_{ref}(k) = \langle v_0, \phi_k \rangle F\phi_k$. We therefore see that any function $y_{ref}(\cdot) \in H^1_{per}(0,\tau)$ can be generated with this exosystem by choosing $v_0 \in W$ such that

$$\langle v_0, \phi_k \rangle = |k| \cdot \hat{y}_{ref}(k).$$

We indeed have $v_0 \in W$, since

$$\sum_{k\in\mathbb{Z}} |\langle v_0, \phi_k \rangle|^2 = \sum_{k\in\mathbb{Z}} \frac{1+\omega_k^2}{1+\omega_k^2} \cdot k^2 |\hat{y}_{ref}(k)|^2 \le \sup_{k\in\mathbb{Z}} \frac{k^2}{1+\omega_k^2} \sum_{k\in\mathbb{Z}} (1+\omega_k^2) \cdot |\hat{y}_{ref}(k)|^2 < \infty.$$

Thus this exosystem is capable of generating any reference signal from $H^1_{per}(0,\tau)$.

This reasoning can be further extended to show that if $y_{ref}(\cdot) \in H^\gamma_{per}(0,\tau)$ for $\gamma \ge 1$, then this reference signal can be generated using this exosystem with a choice $v_0 \in W_{\gamma-1}$ of the initial state. Indeed, if we choose v_0 as above, we then have $v_0 \in W_{\gamma-1}$ because

$$\begin{aligned}\sum_{k\in\mathbb{Z}} (1+\omega_k^2)^{\gamma-1} |\langle v_0, \phi_k \rangle|^2 &= \sum_{k\in\mathbb{Z}} \frac{(1+\omega_k^2)^\gamma}{1+\omega_k^2} \cdot k^2 |\hat{y}_{ref}(k)|^2 \\ &\le \sup_{k\in\mathbb{Z}} \frac{k^2}{1+\omega_k^2} \sum_{k\in\mathbb{Z}} (1+\omega_k^2)^\gamma \cdot |\hat{y}_{ref}(k)|^2 < \infty.\end{aligned}$$

■

Composite Exosystems

Before moving on to a more elaborate example of an infinite-dimensional exosystem, we will quickly mention that any signal consisting of two (or more) independent parts, i.e., of the form

$$y_{ref}(t) = y_{ref,1}(t) + y_{ref,2}(t),$$

can be generated using a composite exosystem

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}, \qquad F = \begin{pmatrix} F_1 & F_2 \end{pmatrix}, \qquad v(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} \qquad v_0 = \begin{pmatrix} v_{01} \\ v_{02} \end{pmatrix}. \tag{2.9}$$

This knowledge can be used for example in a case where the signal can be decomposed into two periodic components $y_{ref,\tau_1}(t)$ and $y_{ref,\tau_2}(t)$ having periods τ_1 and τ_2. If the quotient τ_1/τ_2 is not rational, the signal $y_{ref}(t)$ is not a periodic function. This is the case for example with the signal $y_{ref}(t) = \sin(2t) + \sin(\sqrt{2}t)$. This function is a well-known example of an almost periodic signal, but it can in fact be generated by a finite-dimensional exosystem. In these types of situations we can also apply the results on the smoothness of the genenerated signals directly to the individual periodic subsystems of the exosystem.

Another important special case where we will use the composite exosystem is a situation where only a finite number of the Jordan blocks of the exosystem are nontrivial (i.e., $n_k > 1$ only for a finite number of indices k). In this case the generated signals are of the form (2.6) where $y_0(\cdot)$ is an almost periodic function and the rest of the coefficient functions $y_j(\cdot)$ are linear combinations of trigonometric functions. With these types of exosystems it is in fact very easy to add simple polynomially growing components such as t or t^n to a periodic or an almost periodic signal. We will see an example of this in the next section.

An example of an Infinite-Dimensional Exosystems

In this section we will consider an example demonstrating how to choose the parameters of the infinite-dimensional exosystem in order to generate a signal introduced in Section 2.1. The signal depicted in Figure 2.4 is of the form

$$y_{ref}(t) = t + y_0(t),$$

where $y_0(\cdot)$ is the triangle signal from Figure 2.3. If we consider the 2π-periodic version of the triangle signal, then on $[0, 2\pi]$ the function $y_0(\cdot)$ can be defined as

$$y_0(t) = \begin{cases} t & 0 \le t < \pi \\ -t + \pi & \pi \le t < 2\pi \end{cases}$$

and its Fourier series representation is given by

$$y_0(t) = \sum_{k\in\mathbb{Z}} a_k e^{ikt}, \qquad a_k = \frac{1}{2\pi}\int_0^{2\pi} y_0(t)e^{ikt}dt = \begin{cases} \frac{\pi}{2} & k = 0 \\ 0 & k \neq 0 \text{ and } k \text{ even} \\ -\frac{2}{\pi k^2} & k \text{ odd.} \end{cases}$$

This immediately shows that $y_0(\cdot)$ can be generated by a diagonal exosystem with frequencies $i\omega_k = ik$ for $k \in \mathbb{Z}$. Also, since $a_k = 0$ for all even $k \neq 0$, we could also decide to leave out the corresponding frequencies. We have

$$\sum_{k\in\mathbb{Z}} (1+k^2)^\beta |a_k|^2 = \frac{\pi}{2} + \sum_{\substack{k\in\mathbb{Z} \\ k \text{ odd}}} (1+k^2)^\beta \frac{4}{\pi^2 k^4} < \infty$$

if and only if $\beta < \frac{3}{2}$, and thus $y_0(\cdot) \in H^{\beta}_{per}(0,2\pi)$ for all $\beta < \frac{3}{2}$.

The signal t can be generated using a single Jordan block associated to an eigenvalue $i\omega_0 = 0$. Because of this, we choose the state space of our exosystem as

$$W = \overline{\operatorname{span}}\left\{\phi_0^1, \phi_0^2, \{\phi_k\}_{k\in\mathbb{Z}\setminus\{0\}}\right\}$$

and the system operator S as

$$S = \langle\cdot,\phi_0^2\rangle\phi_0^1 + \sum_{k\in\mathbb{Z}\setminus\{0\}} ik\langle\cdot,\phi_k\rangle\phi_k, \qquad \mathcal{D}(S) = \left\{ v\in W \;\middle|\; \sum_{k\in\mathbb{Z}\setminus\{0\}} k^2|\langle v,\phi_k\rangle|^2 < \infty \right\}.$$

A direct computation shows that the signals generated by the exosystem are given by

$$Fv(t) = \langle v_0,\phi_0^1\rangle F\phi_0^1 + \langle v_0,\phi_0^2\rangle(tF\phi_0^1 + F\phi_0^2) + \sum_{k\in\mathbb{Z}\setminus\{0\}} e^{ikt}\langle v_0,\phi_k\rangle F\phi_k$$

and that the desired reference signal can now be generated by choosing $F \in \mathcal{L}(W,\mathbb{C})$ and $v_0 \in W$ as

$$F\phi_0^1 = 1, \quad F\phi_0^2 = 0, \qquad F\phi_k = \frac{1}{k}, \quad \forall k \neq 0$$

$$\langle v_0,\phi_0^1\rangle = \frac{\pi}{2}, \quad \langle v_0,\phi_0^2\rangle = 1, \quad \langle v_0,\phi_k\rangle = \begin{cases} 0 & k\neq 0 \text{ and } k \text{ even} \\ -\frac{2}{\pi k} & k \text{ odd.} \end{cases}$$

For this choice of the initial state we clearly have $v_0 \in W_\alpha$ for all $\alpha < \frac{1}{2}$. Using Theorem 2.7 we see that since $y_0(\cdot) \in H^{\beta}_{per}(0,2\pi)$ for $\beta < \frac{3}{2}$, it would have been possible to choose the parameters of the exosystem in such a way that $v_0 \in W_\alpha$ for $\alpha < 1$. In this example it would not have been possible to achieve higher α without losing the property

$$\sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} \|F\phi_k^l\|^2 < \infty.$$

Similarly as in Example 2.8 it is easy to see that if $\gamma \geq 1$ then this particular signal generator can be used to generate all reference and disturbance signals of the form

$$y_{ref}(t) = y_1 t + y_0(t)$$

where $y_1 \in \mathbb{C}$ and $y_0(\cdot) \in H^{\gamma}_{per}(0,2\pi)$ with the appropriate choices of the initial states $v_0 \in W_{\gamma-1}$.

2.3 The Periodic Exosystem

In this section we will introduce the periodic exosystem, which will be used later in Chapters 7 and 8 to generate the reference and disturbance signals considered in the output regulation problem. It turns out that the periodic exosystem is a very natural and simple alternative of generating signals of the form (2.2) where the coefficient functions $y_j(\cdot)$ are general periodic functions with the same period. The periodic signal generator differs in many ways from the infinite-dimensional one. Apart from the obvious distinction resulting from the added dependence on time, also the classes of signals generated by the two types of exosystems turn out to be very different. We will see that in general the choice of the initial state of the periodic exosystem has a significantly smaller effect on the generated signals than it does in the case of an infinite-dimensional exosystem. This is an advantage when we are only interested in tracking a very narrow class of signals, or even only a few selected functions. In this kind of situation the periodic exosystem can be constructed in such a way that it only generates a very small class of unnecessary signals in addition to the ones we are interested in. It can be argued that this simplicity is also reflected in the controller we design to solve the associated output regulation problem.

We will start with the mathematical definition of the periodic exosystem and then move on to considering the classes of signals it can be used to generate. We will also show how to choose the parameters of the exosystem to generate given signals of the form (2.2), where the $y_j(\cdot)$ are continuous periodic functions with the same period. Finally, we will conclude the treatment of the periodic signal generators by considering two examples concerning their construction.

Definition 2.9 (The periodic exosystem)**.** The *periodic exosystem* is a linear nonautonomous system

$$\dot{v}(t) = S(t)v(t), \qquad v(0) = v_0 \in W, \tag{2.10a}$$

$$y_{ref}(t) = F(t)v(t) \tag{2.10b}$$

on the space $W = \mathbb{C}^q$. The operator-valued functions

$$S(\cdot) : \mathbb{R} \to \mathscr{L}(W), \qquad F(\cdot) : \mathbb{R} \to \mathscr{L}(W, Y)$$

are periodic with the same period $\tau > 0$ and they satisfy $S(\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathscr{L}(W))$ and $F(\cdot) \in C_\tau(\mathbb{R}, \mathscr{L}(W, Y))$. ■

We know from the theory of ordinary differential equations that the state of the periodic exosystem can be expressed using the *fundamental matrix* $U_S(t,s)$ of the differential equation (2.10a). This is a family of matrices satisfying $U_S(t,t) = I$ and

$$U_S(t,r)U_S(r,s) = U_S(t,s)$$

for all $t, s, r \in \mathbb{R}$, and our assumption $S(\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathscr{L}(W))$ guarantees that the mapping $(t,s) \mapsto U_S(t,s)$ is continuous [13, Sec. III.1]. Since the function $S(\cdot)$ is τ-periodic, we additionally have that $U_S(t+\tau, s+\tau) = U_S(t,s)$ for all $t,s \in \mathbb{R}$. In the case of a constant function $S(t) \equiv S \in \mathbb{C}^{q \times q}$, the matrices $U_S(t,s)$ are simply given by the exponential matrix of S,

$$U_S(t,s) = e^{S(t-s)}.$$

Using the fundamental matrix the state of the exosystem can be written as

$$v(t) = U_S(t,0)v_0$$

for all $t \in \mathbb{R}$. From this it is immediate that the signals generated by the exosystem are of the form

$$y_{ref}(t) = F(t)U_S(t,0)v_0, \qquad v_0 \in W, \tag{2.11}$$

and that they are continuous functions.

In this thesis we consider asymptotic tracking and disturbance rejection, and because of this we are not concerned with asymptotically decaying components of the reference and disturbance signals. We are also not interested in tracking signals which grow at exponential rates. For these reasons we make the following standing assumption that all the eigenvalues of the matrix $U_S(\tau,0)$ associated to the periodic exosystem have magnitude equal to one. This has precisely the desired effect of ruling out the types of signals we are not interested in. The requirement obviously corresponds to the assumption that the spectrum of the system operator of a time-invariant exosystem lies on the imaginary axis.

Assumption 2.10. *We have $|\lambda| = 1$ for all $\lambda \in \sigma(U_S(\tau,0))$.*

It is shown in the next section that under this assumption the signals generated by the exosystem are polynomially bounded.

The Classes of Signals

In this section we consider the signals generated by the periodic exosystem. We are again interested in signals of the form

$$y_{ref}(t) = y_n(t)t^n + \cdots + y_1(t)t + y_0(t), \tag{2.12}$$

where $y_j(\cdot)$ are continuous periodic functions with the same period. The following theorem shows that any reference signal of this type can be generated very easily with a periodic exosystem.

Theorem 2.11. *Let $y_j(\cdot)$ be continuous periodic functions with period $\tau > 0$ for all $j \in \{0,\dots,n\}$. The reference signal* (2.12) *can be generated by a periodic exosystem on the space $W = \mathbb{C}^{n+1}$ by choosing $S(t) \equiv S \in \mathscr{L}(W)$ to be a single Jordan block associated to the eigenvalue* 0 *and by choosing $F(\cdot)$ such that*

$$F(t) = \begin{pmatrix} n! \cdot y_n(t) & (n-1)! \cdot y_{n-1}(t) & \dots & 1! \cdot y_1(t) & y_0(t) \end{pmatrix} \qquad \forall t \in \mathbb{R}.$$

The reference signal (2.12) *is then generated with the choice $v_0 = \begin{pmatrix} 0 & \dots & 0 & 1 \end{pmatrix}^T \in W$ of the initial state of the exosystem.*

Proof. The chosen functions clearly satisfy the conditions imposed on the parameters of the periodic exosystem. Since $S(t)$ is independent of t, we have that the fundamental matrix $U_S(t,s)$ is the exponential matrix related to S. Furthermore, since S was chosen to be an $(n+1) \times (n+1)$ Jordan block associated to the eigenvalue 0, we have for our choices of the parameters of the exosystem and for $v_0 = \begin{pmatrix} 0 & \dots & 0 & 1 \end{pmatrix}^T \in W$ that

$$\begin{aligned} y_{ref}(t) = F(t)U_S(t,0)v_0 = F(t) \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^n}{n!} \\ & 1 & t & \cdots & \frac{t^{n-1}}{(n-1)!} \\ & & \ddots & & \vdots \\ & & & 1 & t \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = F(t) \begin{pmatrix} \frac{t^n}{n!} \\ \frac{t^{n-1}}{(n-1)!} \\ \vdots \\ t \\ 1 \end{pmatrix} \\ = y_n(t)t^n + y_{n-1}(t)t^{n-1} + \dots + y_1(t)t + y_0(t). \end{aligned}$$

This concludes the proof. □

The above theorem concludes that the types of signals we are mainly interested in can be generated using an exosystem of the form (2.10) with appropriate choices of parameters. We will now turn to determining the classes of signals generated by general exosystems of this type. As a first step in this direction we will show that a signal generator of the form (2.10) can always be written as another periodic exosystem with a constant function $\tilde{S}(\cdot) \equiv \tilde{S} \in \mathscr{L}(W)$. This follows from the so-called Floquet representation theorem for solutions of the differential equation (2.10a) [13, Sec. III.7].

Theorem 2.12. *Given a periodic exosystem of the form* (2.10)*, there exists $\tilde{S} \in \mathscr{L}(W)$ and $\tilde{F}(\cdot) \in C_\tau(\mathbb{R}, \mathscr{L}(W,Y))$ such that for all $v_0 \in W$ the signals generated by the exosystem are given by*

$$y_{ref}(t) = \tilde{F}(t)e^{\tilde{S}t}v_0.$$

Furthermore, Assumption 2.10 *is satisfied if and only if the eigenvalues of $\tilde{S}$ lie on the imaginary axis.*

Proof. The Floquet representation theorem [13, Thm. III.7.1] states that there exist a matrix $\tilde{S} \in \mathscr{L}(W)$ and a continuous τ-periodic function $U(\cdot) \in C_\tau(\mathbb{R}, \mathscr{L}(W))$ such that the fundamental matrix $U_S(t,0)$ of the differential equation (2.10a) can be written as

$$U_S(t,0) = U(t)e^{\tilde{S}t}, \qquad t \in \mathbb{R}.$$

The matrix $\tilde{S}$ is chosen in such a way that $U_S(\tau,0) = e^{\tilde{S}\tau}$. If we choose the function $\tilde{F}(\cdot)$ as $\tilde{F}(\cdot) = F(\cdot)U(\cdot) \in C_\tau(\mathbb{R}, \mathscr{L}(W,Y))$, then the signals generated by the exosystem (2.10) are given by

$$y_{ref}(t) = F(t)U_S(t,0)v_0 = F(t)U(t)e^{\tilde{S}t}v_0 = \tilde{F}(t)e^{\tilde{S}t}v_0.$$

This proves the first claim of the theorem.

Let $\tilde{S} = RJR^{-1}$, where J is the Jordan canonical form of $\tilde{S}$ and $R \in \mathscr{L}(W)$. We then have

$$U_S(\tau,0) = e^{\tilde{S}\tau} = Re^{J\tau}R^{-1}.$$

If $\sigma(\tilde{S}) = \{\lambda_k\}_{k=1}^m$, then $e^{J\tau}$ is an upper triangular matrix with eigenvalues $\{e^{\lambda_k\tau}\}_{k=1}^m$. The property that Assumption 2.10 is satisfied if and only if $\sigma(\tilde{S}) \subset i\mathbb{R}$ now follows directly from the fact that

$$1 = |e^{\lambda_k\tau}| = e^{\tau\cdot \operatorname{Re}\lambda_k} \qquad \Leftrightarrow \qquad \operatorname{Re}\lambda_k = 0$$

for all $k \in \{1,\ldots,m\}$. □

As stated above, the previous theorem shows that any exosystem of the form (2.10) can be written as a system

$$\dot{w}(t) = \tilde{S}w(t), \qquad w(0) = v_0 \in W, \tag{2.13a}$$

$$y_{ref}(t) = \tilde{F}(t)w(t), \tag{2.13b}$$

where $\tilde{S}$ and $\tilde{F}(\cdot)$ are as in Theorem 2.12. In particular this means that there is no added generality in allowing the matrices $S(t)$ to depend on time. We nevertheless choose to work with exosystems of the form (2.10) in this thesis. We do this mainly because in some cases it is more illustrative or more natural for a given application to have time-dependence in $S(\cdot)$. We will encounter this type of situation in the next section when generating signals containing periodically modulated frequencies.

We will, however, use the form (2.13) to prove the following theorem describing the classes of signals generated by general periodic exosystems. The result states that the generated signals are polynomially bounded and of the form (2.12), but the coefficient functions $y_j(\cdot)$ might not be periodic functions.

Theorem 2.13. *If Assumption* 2.10 *is satisfied, the signals generated by the periodic exosystem are of the form* (2.12)*, where* $y_j(\cdot) \in C(\mathbb{R}, Y)$ *are bounded and uniformly continuous functions. If the corresponding system* (2.13) *is such that* $\sigma(\tilde{S}) = \{i\omega_k\}_{k=1}^m$ *and* $\tilde{n} \in \mathbb{N}$ *is the dimension of the largest Jordan block of* $\tilde{S}$*, then* $n = \tilde{n} - 1$ *and the coefficient functions* $y_j(\cdot)$ *are of the form*

$$y_j(t) = \sum_{k=1}^m a_{jk}(t) e^{i\omega_k t}, \qquad t \in \mathbb{R} \tag{2.14}$$

with $a_{jk}(\cdot) \in C_\tau(\mathbb{R}, Y)$*. The functions* $y_j(\cdot)$ *are* τ*-periodic for all initial states* $v_0 \in W$ *and for all* $j \in \{0, \ldots, n\}$ *if and only if* $\sigma(\tilde{S}) \subset \{i\frac{2\pi k}{\tau}\}_{k \in \mathbb{Z}}$*.*

Proof. By possibly applying a time-invariant similarity transformation to the exosystem (2.13) we can assume that the matrix $\tilde{S}$ is in its Jordan canonical form, i.e.,

$$\tilde{S} = \operatorname{diag}\left(S_1, S_2, \ldots, S_m\right),$$

where S_k is a Jordan block of dimension $n_k \times n_k$ associated to an eigenvalue $i\omega_k \in i\mathbb{R}$. For $v \in W$ the exponential matrix of the matrix $\tilde{S}$ is given by

$$e^{\tilde{S}t} v = \sum_{k=1}^m e^{i\omega_k t} \sum_{l=1}^{n_k} \langle v, e_k^l \rangle \sum_{j=1}^{l} \frac{t^{l-j}}{(l-j)!} e_k^j,$$

where $\{e_k^l\}_{l=1}^{n_k}$ denotes the orthonormal set of generalized eigenvectors related to the eigenvalue $i\omega_k$.

Similarly as in the proof of Theorem 2.3, we can see that the signals generated by the system (2.13) are given by

$$\begin{aligned}
y_{ref}(t) = \tilde{F}(t) e^{\tilde{S}t} v_0 &= \sum_{k=1}^m e^{i\omega_k t} \sum_{l=1}^{n_k} \langle v_0, e_k^l \rangle \sum_{j=1}^{l} \frac{t^{l-j}}{(l-j)!} \tilde{F}(t) e_k^j \\
&= \sum_{k=1}^m e^{i\omega_k t} \sum_{l=1}^{n_k} \langle v_0, e_k^l \rangle \sum_{j=0}^{l-1} \frac{t^j}{j!} \tilde{F}(t) e_k^{l-j} = \sum_{k=1}^m e^{i\omega_k t} \sum_{j=0}^{n_k-1} t^j \cdot \frac{1}{j!} \sum_{l=j+1}^{n_k} \langle v_0, e_k^l \rangle \tilde{F}(t) e_k^{l-j} \\
&= \sum_{k=1}^m e^{i\omega_k t} \sum_{j=0}^{n_k-1} a_{jk}(t) t^j,
\end{aligned}$$

where we have denoted by $a_{jk}(\cdot) \in C_\tau(\mathbb{R}, Y)$ the functions satisfying

$$a_{jk}(t) = \frac{1}{j!} \sum_{l=j+1}^{n_k} \langle v_0, e_k^l \rangle \tilde{F}(t) e_k^{l-j}, \qquad t \in \mathbb{R}.$$

Let $n = \max_{k=1,\dots,m} n_k - 1$ and define $a_{jk}(\cdot) \equiv 0 \in C_\tau(\mathbb{R}, Y)$ for all $k \in \{1,\dots,m\}$ and $j \in \{n_k+1,\dots,n\}$. Using the notation in (2.14) we can now write

$$y_{ref}(t) = \sum_{k=1}^{m} e^{i\omega_k t} \sum_{j=0}^{n_k-1} a_{jk}(t)t^j = \sum_{k=1}^{m} e^{i\omega_k t} \sum_{j=0}^{n} a_{jk}(t)t^j = \sum_{j=0}^{n} t^j \sum_{k=1}^{m} a_{jk}(t)e^{i\omega_k t} = \sum_{j=0}^{n} t^j y_j(t).$$

The boundedness and uniform continuity of $y_j(\cdot)$ follow directly from (2.14) and the fact that $a_{jk}(\cdot)$ and $t \mapsto e^{i\omega_k t}$ are bounded and uniformly continuous functions. Since the functions $t \mapsto e^{i\omega_k t}$ are τ-periodic if and only if ω_k are multiples of $\frac{2\pi}{\tau}$, it is easy to see that $y_j(\cdot) \in C_\tau(\mathbb{R}, Y)$ for all initial states $v_0 \in W$ and for all $j \in \{0,\dots,n\}$ precisely if we have $\sigma(\tilde{S}) \subset \left\{ i\frac{2\pi k}{\tau} \right\}_{k\in\mathbb{Z}}$. □

The Floquet representation theorem can also be used to derive the following polynomial bound for the growth of the fundamental matrix $U_S(t,s)$. This estimate will be used later in Chapter 7.

Lemma 2.14. *Under Assumption* 2.10 *there exist constants* $M_S \geq 0$ *and* $n_S \in \mathbb{N}_0$ *such that the fundamental matrix* $U_S(t,s)$ *of the periodic exosystem* (2.10) *satisfies*

$$\|U_S(t,s)\| \leq M_S(|t-s|^{n_S} + 1)$$

for all $t,s \in \mathbb{R}$.

Proof. As we saw in the proof of Theorem 2.12, there exist a matrix $\tilde{S} \in \mathcal{L}(W)$ and a continuous τ-periodic function $U(\cdot) \in C_\tau(\mathbb{R}, \mathcal{L}(W))$ such that $U_S(t,0) = U(t)e^{\tilde{S}t}$ for all $t \in \mathbb{R}$. Moreover, the matrices $U(t)$ are invertible for all $t \in \mathbb{R}$ [13, Thm. III.7.1]. Since Assumption 2.10 is satisfied, we have $\sigma(\tilde{S}) \subset i\mathbb{R}$ by Theorem 2.12. Therefore there exists $\tilde{M}_S \geq 0$ and $n_S \in \mathbb{N}_0$ such that

$$\|e^{\tilde{S}t}\| \leq \tilde{M}_S(|t|^{n_S} + 1), \qquad t \in \mathbb{R}.$$

Using this and the properties of the fundamental matrix $U_S(t,s)$ we see that

$$\begin{aligned}\|U_S(t,s)\| &= \|U_S(t,0)U_S(0,s)\| = \|U_S(t,0)U_S(s,0)^{-1}\| = \|U(t)e^{\tilde{S}t}e^{-\tilde{S}s}U(s)^{-1}\| \\ &\leq \|U(t)\| \cdot \|e^{\tilde{S}(t-s)}\| \cdot \|U(s)^{-1}\| \leq \|U(\cdot)\|_\infty \|U(\cdot)^{-1}\|_\infty \cdot \tilde{M}_S(|t-s|^{n_S} + 1)\end{aligned}$$

for all $t,s \in \mathbb{R}$. This concludes the proof. □

Since the signals generated by the periodic exosystem are of the form (2.11), the effect of the choice of the initial state $v_0 \in W$ on the generated reference signals can be easily seen if we write

$$y_{ref}(t) = F(t)U_S(t,0)v_0 = \begin{pmatrix} g_1(t) & \dots & g_q(t) \end{pmatrix} v_0 = v_0^1 g_1(t) + \dots + v_0^q g_q(t),$$

where $g_j(\cdot)$ are continuous functions. This shows us that essentially the reference signals generated by a periodic exosystem with fixed parameters are linear combinations of at most q fixed continuous signals. Thus, as we already mentioned, in the case of the periodic exosystem the effect of the choice of the initial state has a significantly smaller and more easily predictable effect on the generated signals than it does in the case of the infinite-dimensional exosystem. To further illustrate this we will in the next example construct a periodic signal generator capable of generating a given signal $y_0(\cdot) \in H^1_{per}(0,\tau)$. The situation can be compared to the one in Example 2.8, where we completed the same task using an infinite-dimensional exosystem.

Example 2.15. Let $y_0(\cdot) \in H^1_{per}(0,\tau)$. This signal can be generated by choosing the parameters of the periodic exosystem as $W = \mathbb{C}$, $S(t) \equiv 0$ and $F(\cdot) = y_0(\cdot)$. The desired reference signal corresponds to the initial state $v_0 = 1$ and all generated signals are of the form

$$y_{ref}(t) = F(t)U_S(t,0)v_0 = y_0(t)v_0, \qquad v_0 \in \mathbb{C}.$$

This shows us that the class of signals generated by this exosystem consists precisely of the scalar multiples of the function $y_0(\cdot)$. ■

Examples of Periodic Exosystems

In this section we will consider two examples demonstrating more concretely how to choose the parameters of a periodic exosystem. In particular, we have seen in Theorem 2.11 that any reference signal of the form (2.12) can be generated using a periodic exosystem with a constant function $S(\cdot) = S$. In Example 2.17 we show that it is — nevertheless — sometimes useful to be able to choose a periodically time-dependent function $S(\cdot)$. This example also further illustrates the fact that the coefficient functions $y_j(\cdot)$ of the generated signals are not necessarily periodic.

Example 2.16. In this example we will construct a periodic exosystem capable of generating the signal depicted in Figure 2.4. For $\tau = 2\pi$ this signal is of the form

$$y_{ref}(t) = t + y_0(t),$$

where $y_0(\cdot) \in C_\tau(\mathbb{R},\mathbb{C})$ is given by

$$y_0(t) = \begin{cases} t & 0 \le t < \pi \\ -t + \pi & \pi \le t < 2\pi. \end{cases}$$

Theorem 2.11 shows that we can choose $W = \mathbb{C}^2$,

$$S(t) \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and $F(t) = (1 \;\; y_0(t))$. The reference signal in Figure 2.4 can be generated with the choice $v_0 = (0 \;\; 1)^T$ of the initial state, and the complete class of signals generated by this exosystem consists of the signals of the form

$$y_{ref}(t) = F(t)U_S(t,0)v_0 = \begin{pmatrix}1 & y_0(t)\end{pmatrix}\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\begin{pmatrix}v_0^1\\ v_0^2\end{pmatrix} = v_0^1 + v_0^2(t + y_0(t)),$$

where $v_0^1, v_0^2 \in \mathbb{C}$. ■

In the second example we construct a periodic exosystem generating trigonometric signals whose frequencies are modulated periodically.

Example 2.17. In this example we will construct a periodic exosystem to generate signals of the form

$$y_{ref}(t) = a_1\cos(\omega_0 t + \omega_m(t)) + a_2\sin(\omega_0 t + \omega_m(t)), \tag{2.15}$$

where $a_1, a_2 \in \mathbb{C}$ are the amplitudes of the components, $\omega_0 \in \mathbb{R}$ is the base frequency and $\omega_m(\cdot) : \mathbb{R} \to \mathbb{R}$ is an absolutely continuous τ-periodic function. Signals of this type are usually encountered in frequency modulation. An example of a signal of this form is depicted in Figure 2.5 together with the modulating signal.

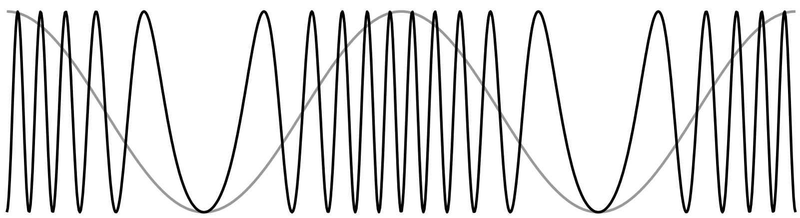

Figure 2.5: A signal with a periodically modulated frequency.

It is important to note that the signals of the form (2.15) are not necessarily periodic, or they can have period different from τ. Indeed, these signals are τ-periodic precisely if for some $n \in \mathbb{Z}$ we have

$$\omega_0(t+\tau) + \omega_m(t+\tau) = \omega_0 t + \omega_m(t) + 2\pi n \qquad \Leftrightarrow \qquad \omega_0 = \frac{2\pi n}{\tau}.$$

Consequently, this example also illustrates the fact that in general the periodic exosystems can be capable of generating signals other than those of the form (2.12) with periodic coefficient functions $y_j(\cdot) \in C_\tau(\mathbb{R}, Y)$.

Since $\omega_m(\cdot)$ is absolutely continuous, it has a derivative in the sense of Lebesgue integration satisfying $\omega_m'(\cdot) \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{C})$. We choose the parameters of the periodic exosystem on $W=\mathbb{C}^2$ as $F(\cdot)\equiv F=(1\ \ 0)$ and

$$S(t)=\begin{pmatrix}0 & \omega_0+\omega_m'(t)\\ -\omega_0-\omega_m'(t) & 0\end{pmatrix}$$

for all $t\in\mathbb{R}$. Since the matrices $S(t)$ commute, i.e., $S(t)S(s)=S(s)S(t)$ for all $t,s\in\mathbb{R}$, the evolution family $U_S(t,s)$ can be expressed using an exponential formula

$$U_S(t,s)=e^{\int_s^t S(r)dr} \qquad \forall s,t\in\mathbb{R}.$$

A direct computation now shows that the signal corresponding to the initial state

$$v_0=\begin{pmatrix}\cos(\omega_m(0)) & \sin(\omega_m(0))\\ -\sin(\omega_m(0)) & \cos(\omega_m(0))\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix}$$

of the exosystem is given by

$$\begin{aligned}
y_{ref}(t)&=F(t)U_S(t,0)v_0=Fe^{\int_0^t S(s)ds}v_0=Fe^{\int_0^t(\omega_0+\omega_m'(s))ds\cdot\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)}v_0\\
&=Fe^{\left(\omega_0 t+\omega_m(t)-\omega_m(0)\right)\cdot\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)}v_0\\
&=F\begin{pmatrix}\cos(\omega_0 t+\omega_m(t)) & \sin(\omega_0 t+\omega_m(t))\\ -\sin(\omega_0 t+\omega_m(t)) & \cos(\omega_0 t+\omega_m(t))\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix}\\
&=a_1\cos(\omega_0 t+\omega_m(t))+a_2\sin(\omega_0 t+\omega_m(t)).
\end{aligned}$$

■

2.4 Comparison of the Two Types of Exosystems

In Example 2.8 we constructed an infinite-dimensional exosystem having the property that any function from the space $H^1_{per}(0,\tau)$ could be generated with a proper choice of the initial state $v_0\in W$. On the other hand, it was also evident that even if we were originally only interested in generating a single reference signal containing all the frequencies ω_k, the infinite-dimensional exosystem couldn't still have been chosen to be any smaller. Even without going into details it is easy to see that this is also true in a more general sense: The infinite-dimensional exosystems are usually capable of generating large classes of signals, like $H^1_{per}(0,\tau)$ in Example 2.8.

As we saw in Example 2.15 the situation is very different in the case of periodic exosystems. It was shown that given a periodic reference signal, the exosystem can be chosen to only generate scalar multiples of this given signal. In the case of multiple periodic functions, we can also choose the exosystem to generate a class of signals consisting precisely of the linear combinations of the original functions. Because of this, the periodic exosystem can be easily designed to generate a smaller predefined group of signals.

As we will see in the next chapter, by definition a controller solving the output regulation problem is able to steer the output of the system to any reference signal generated by the exosystem. Even if we are only interested in tracking a few selected signals, a controller designed based on the appropriate infinite-dimensional exosystem will also be able to achieve asymptotic tracking of a large class of extraneous signals. It can be argued that this capability is also reflected in the controller as unnecessary complexity and that — in the light of the above comparison — this complexity can be reduced by instead designing the controller based on a periodic exosystem generating the relevant signals.

The theories of output regulation related to the infinite-dimensional and the periodic exosystems can be treated with similar methods, as we will see in the following chapters, but while studying these problems we will also encounter the most dramatic differences between the two types of exosystems. The key elements include the implementability of the controllers, the theory for the stabilization of the closed-loop system and in general the strength of the results on the regulation problems. These differences will be further discussed in Chapter 9.

Chapter 3

Output Regulation with Infinite-Dimensional Exosystems

In this chapter we consider the output regulation and robust output regulation of distributed parameter systems with the infinite-dimensional exosystem introduced in Chapter 2. These problems consist of choosing a controller in such a way that the output of the original system can be steered to any of the reference signals generated by the exosystem despite disturbance signals originating from the same exosystem. In robust output regulation it is additionally required that the same controller achieves the tracking of the reference signals and disturbance rejection even if the parameters of the system are perturbed in such a way that the stability of the closed-loop system consisting of the system and the controller is preserved.

As already mentioned in Chapter 1, the main tools in the analysis of the output regulation problems are the infinite-dimensional Sylvester equations of the form

$$\Sigma S = A_e \Sigma + B_e, \tag{3.1}$$

where S is the system operator of the exosystem and A_e and B_e are operators of the closed-loop system. In particular the solution of the Sylvester equation can be used to study the asymptotic behaviors of the state of the closed-loop system and the regulation error. This close connection between the closed-loop system and the Sylvester equation allows us to characterize the solvability of the output regulation problem using the solvability of the Sylvester equation (3.1) and an additional *regulation constraint*.

One of the unique features of this characterization, resulting from the use of the Sylvester equation (3.1), is that it only involves the parameters of the closed-loop system. Because of this, the result is independent of the form of the controller in the sense that it remains valid for any controllers that can be written in a similar closed-loop form with the plant.

Another benefit resulting from the use of the Sylvester equations is that we will find a very strong connection between the conditions for the solvability of the output regulation problem and the level of smoothness of the reference and disturbance signals. We will see that this can be accomplished very naturally by allowing the solution Σ of the Sylvester equation (3.1) to be an unbounded operator. The scale spaces W_α of the exosystem play a crucial part in the analysis. In particular, we allow the solution of the Sylvester equation (3.1) to be an operator belonging to $\mathscr{L}(W_\alpha, X_e)$. If considered as an operator $\Sigma : W \to X_e$ with domain $\mathscr{D}(\Sigma) = W_\alpha$, the value $\alpha \geq 0$ can be seen as the level of unboundedness of the operator Σ. The choice of this scale space has a direct effect on the definition of the output regulation problem. Roughly stated, we will see that if we allow the solution of (3.1) to be in $\mathscr{L}(W_\alpha, X_e)$, we can then consider tracking of signals of the infinite-dimensional exosystem generated by initial states in W_α. This establishes the minimal level of smoothness for the reference and disturbance signals considered. On the other hand, the higher values of $\alpha \geq 0$ immediately result in relaxed conditions for the solvability of the output regulation problem.

We will use the following terminology when considering the unbounded solutions of the Sylvester equation (3.1).

Definition 3.1. Let $\alpha \geq 0$. We say that an operator $\Sigma \in \mathscr{L}(W_\alpha, X_e)$ is a *solution of the Sylvester equation* (3.1) *on* $W_{\alpha+1}$ if the operator satisfies $\Sigma(W_{\alpha+1}) \subset \mathscr{D}(A_e)$ and if

$$\Sigma S v = A_e \Sigma v + B_e v$$

for all $v \in W_{\alpha+1}$. ■

Before moving on to the theory we will briefly outline the organization of the chapter and the main contributions presented in each of the sections.

Section 3.1 In the first section we introduce the system and the controller considered in the output regulation problem and state the basic assumptions on their parameters. The main emphasis in this part of the thesis is on the problem of robust output regulation and we therefore only consider the dynamic error feedback controller.

Section 3.2 In this section we formulate the output regulation problem mathematically. We also state our first main result, which shows that the solvability of the output regulation problem can be characterized using the solvability of certain constrained Sylvester equation.

Section 3.3 The topic of this section is the connection between the state of the closed-loop system and the associated Sylvester equation. The main result of the section states that the solvability of the Sylvester equation is equivalent to the state of the closed-loop system having a special form. We will see that the obtained formula can also be used to study the asymptotic behavior of the regulation error, and ultimately to prove the main result of the preceding section.

Section 3.4 The statement of the output regulation problem and the result on the characterization of the controllers solving it involve assumptions on the solvability of the associated Sylvester equations. In this section we derive sufficient conditions for the existence of unique solutions to these equations.

Section 3.5 In this section we formulate the robust output regulation problem for distributed parameters with infinite-dimensional exosystems. We also relate the solvability of this problem to a condition involving properties of certain Sylvester equations. At the end of the section we show that this condition is independent of the smoothness properties of the reference and disturbance signals. This condition is used as a starting point in the further study of the solvability of the robust output regulation problem carried out in Chapter 4.

The theory of output regulation presented in this chapter generalizes the treatments in [24, 15], where the problem was considered for infinite-dimensional exosystems capable of generating uniformly bounded reference and disturbance signals. Our main generalization is to consider an exosystem whose system operator S consist of a possibly infinite number of nontrivial Jordan blocks, allowing us to track and reject polynomially bounded exogeneous signals. In addition, the theory is generalized to allow unbounded solutions of the Sylvester equations.

3.1 The System and the Controller

We consider a linear system

$$\dot{x}(t) = Ax(t) + Bu(t) + w_s(t), \qquad x(0) = x_0 \in X, \tag{3.2a}$$

$$y(t) = Cx(t) + Du(t) + w_m(t) \tag{3.2b}$$

on a Banach space X. Here $x(t) \in X$ is the state of the system, $y(t) \in Y$ is the output, and $u(t) \in U$ is the input for $t \geq 0$. The input space U and the output space Y are possibly infinite-dimensional Hilbert spaces. The terms $w_s(t)$ and $w_m(t)$ are the disturbances to the state and the measurement, respectively. We assume that the operator $A : \mathscr{D}(A) \subset X \to X$ generates a C_0-semigroup $T_A(t)$ on X and that the other operators are bounded, $B \in \mathscr{L}(U,X)$, $C \in \mathscr{L}(X,Y)$, and $D \in \mathscr{L}(U,Y)$. For $\lambda \in \rho(A)$ the transfer function of the plant is denoted by $P(\lambda) = CR(\lambda,A)B + D \in \mathscr{L}(U,Y)$.

In this chapter we consider output regulation in a situation where the reference signal $y_{ref}(\cdot)$ to be tracked and the disturbance signals $w_s(\cdot)$ and $w_m(\cdot)$ are generated by an infinite-dimensional exosystem

$$\dot{v}(t) = Sv(t), \qquad v(0) = v_0 \in W. \tag{3.3}$$

The reference and disturbance signals are obtained as outputs of this system,

$$w_s(t) = Ev(t), \qquad w_m(t) = F_m v(t), \qquad y_{ref}(t) = F_r v(t),$$

and the operators $S : \mathcal{D}(S) \subset W \to W$, $F_r, F_m \in \mathcal{L}(W,Y)$, and $E \in \mathcal{L}(W,X)$ satisfy the conditions of Definition 2.1. We further assume that the operator A of the system (3.2) is such that $\sigma(A) \cap \sigma_p(S) = \varnothing$, and that the values $P(i\omega_k)$ of the transfer function are boundedly invertible operators for all $k \in \mathbb{Z}$.

Denoting the regulation error by $e(t) = y(t) - y_{ref}(t)$ and defining the operator $F = F_m - F_r \in \mathcal{L}(W,Y)$ the system can be written in a standard form as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ev(t), \qquad x(0) = x_0 \in X, \\ e(t) &= Cx(t) + Du(t) + Fv(t). \end{aligned}$$

The control signal u of the system is obtained as an output of a controller. As our main interest in this part of the thesis is on robust output regulation, we consider a dynamic error feedback controller of the form

$$\begin{aligned} \dot{z}(t) &= \mathcal{G}_1 z(t) + \mathcal{G}_2 e(t), \qquad z(0) = z_0 \in Z, \\ u(t) &= Kz(t) \end{aligned}$$

on a Banach space Z. Here the operator $\mathcal{G}_1 : \mathcal{D}(\mathcal{G}_1) \subset Z \to Z$ generates a C_0-semigroup on Z and the other operators are bounded, $\mathcal{G}_2 \in \mathcal{L}(Y,Z)$ and $K \in \mathcal{L}(Z,U)$.

The system and the controller can be written together as a closed-loop system on the Banach space $X_e = X \times Z$. This composite system with state $x_e(t) = (x(t) \;\; z(t))^T$ is given by

$$\dot{x}_e(t) = A_e x_e(t) + B_e v(t), \qquad x_e(0) = x_{e0} = (x_0 \;\; z_0)^T, \tag{3.4a}$$

$$e(t) = C_e x_e(t) + D_e v(t), \tag{3.4b}$$

where $C_e = \begin{pmatrix} C & DK \end{pmatrix}$, $D_e = F$,

$$A_e = \begin{pmatrix} A & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 DK \end{pmatrix}, \quad \text{and} \quad B_e = \begin{pmatrix} E \\ \mathcal{G}_2 F \end{pmatrix}.$$

The C_0-semigroup generated by the operator $A_e : \mathcal{D}(A) \times \mathcal{D}(\mathcal{G}_1) \subset X_e \to X_e$ is denoted by $T_e(t)$. Since the operators E, F_r and F_m satisfy the conditions of Definition 2.1, the operator B_e also satisfies

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} \|B_e \phi_k^l\|^2 &= \sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} \left(\|E\phi_k^l\| + \|\mathcal{G}_2 (F_m - F_r)\phi_k^l\| \right)^2 \\ &= \sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} 2 \left(\|E\phi_k^l\|^2 + 2\|\mathcal{G}_2\|^2 \left(\|F_m \phi_k^l\|^2 + \|F_r \phi_k^l\|^2 \right) \right) < \infty. \end{aligned} \tag{3.5}$$

Even though in this part of the thesis we only consider the dynamic error feedback controller, it should be noted that the results presented in this chapter apply to all controllers for which the closed-loop system can be written in the form (3.4). In particular this is true for static state feedback laws and for feedforward-error feedback controllers [21]. The latter type are dynamic error feedback controllers incorporating an additional feedthrough term to improve the performance of the control law.

3.2 The Output Regulation Problem

The output regulation problem on W_α consists of choosing the controller parameters in such a way that the controlled system can track the reference signals and reject the disturbance signals originating from the initial states $v_0 \in W_\alpha$ of the infinite-dimensional exosystem (3.3). As we saw in Chapter 2, in the case of the periodic reference and disturbance signals the choices of the initial states of the exosystem are directly related to the level of smoothness of the signals to be tracked and rejected. Because of this, the chosen value of $\alpha \geq 0$ in the statement of the output regulation problem establishes the *minimal* level of smoothness that the considered reference and disturbance signals have. The formal definition of the problem is presented in the following.

The Output Regulation Problem on W_α. Let $\alpha \geq 0$. Find $(\mathcal{G}_1, \mathcal{G}_2, K)$ such that the following are satisfied:

1. The closed-loop system operator A_e generates a strongly stable C_0-semigroup on X_e.
2. For all initial states $v_0 \in W_\alpha$ and $x_{e0} \in X_e$ the regulation error goes to zero asymptotically, i.e.,

$$\lim_{t\to\infty} e(t) = 0.$$

■

The statement of the problem contains two parts. We will first consider the second one, which requires that the regulation error decays to zero asymptotically provided that the closed-loop system is strongly stable. The treatement of the first part, the stabilization of the closed-loop system, is presented later in Chapter 5. There we will design an observer-based controller solving the robust output regulation problem, which will be introduced in Section 3.5.

The main result of this chapter, the characterization of the solvability of the output regulation problem with the aid of the Sylvester equation (3.1), is stated in the next theorem. This result shows that the solvability of the output regulation problem is equivalent to the solution of the Sylvester equation (3.1) satisfying an additional *regulation constraint* (3.7). Together the Sylvester equation and the regulation constraint form the well-known *regulator equations*. The theorem is proved in the next section, where we study in greater detail the backbone of this result: The connection between the behavior of the state of the closed-loop system and the associated Sylvester equation.

Theorem 3.2. *Assume the controller* $(\mathscr{G}_1, \mathscr{G}_2, K)$ *is such that* A_e *generates a strongly stable* C_0*-semigroup on* X_e *and that the Sylvester equation*

$$\Sigma S = A_e \Sigma + B_e \tag{3.6}$$

on $W_{\alpha+1}$ *has a solution* $\Sigma \in \mathscr{L}(W_\alpha, X_e)$ $\big($*satisfying* $\Sigma(W_{\alpha+1}) \subset \mathscr{D}(A_e)\big)$. *Then the following are equivalent:*

(a) *The controller* $(\mathscr{G}_1, \mathscr{G}_2, K)$ *solves the output regulation problem on* W_α.

(b) *The solution* Σ *of the Sylvester equation* (3.6) *satisfies*

$$C_e \Sigma + D_e = 0 \tag{3.7}$$

on W_α.

Theorem 3.2 also completes the relationship between the smoothness of the considered reference and disturbance signals and the conditions required for the solvability of the output regulation problem. As already noted at the beginning of this section, the value $\alpha \geq 0$ in the statement of the output regulation problem determines the minimal level of smoothness the signals have. The above theorem states that to solve the output regulation problem on W_α, it is sufficient that the Sylvester equation (3.6) has a solution Σ in $\mathscr{L}(W_\alpha, X_e)$ satisfying the regulation constraint (3.7). The fact that the value of α can be seen as the level of unboundedness of the operator Σ immediately implies that the higher the value of α, the less strict the requirement of the solvability of (3.6) is. The relationship will become even more concrete in Section 3.4, where the effect of the value of α on the solvability of the Sylvester equation is investigated further.

The characterization in Theorem 3.2 can be used to determine whether a given controller solves the output regulation problem. However, the condition of the solvability of the Sylvester equation (3.6) can be difficult to verify on an abstract level. For this reason we will provide sufficient conditions for the existence of a solution to (3.6) later in Section 3.4. For the reader's convenience the main conclusion of this consideration is stated in advance in the next theorem.

Theorem 3.3. *Assume the controller* $(\mathcal{G}_1, \mathcal{G}_2, K)$ *is such that* A_e *generates a strongly stable* C_0*-semigroup on* X_e*, its spectrum satisfies* $\sigma(A_e) \cap \sigma_p(S) = \varnothing$ *and*

$$\sup_{k\in\mathbb{Z}} \frac{\|R(i\omega_k, A_e)\|^{2n_k}}{(1+\omega_k^2)^\alpha} < \infty.$$

The Sylvester equation (3.6) *has a unique solution* $\Sigma \in \mathcal{L}(W_\alpha, X_e)$ *given by*

$$\Sigma v = \sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} \langle v, \phi_k^l \rangle \sum_{j=1}^{l} (-1)^{l-j} R(i\omega_k, A_e)^{l+1-j} B_e \phi_k^j, \qquad v \in W_\alpha$$

and the following are equivalent:

(a) *The controller* $(\mathcal{G}_1, \mathcal{G}_2, K)$ *solves the output regulation problem on* W_α*.*

(b) *The solution* Σ *of the Sylvester equation* (3.6) *satisfies* $C_e\Sigma + D_e = 0$ *on* W_α*.*

To prove Theorem 3.2 we will first need to study the connection between the state of the closed-loop system and the Sylvester equation (3.6). This is the topic of the next section.

3.3 The Dynamic Steady-State of the Closed-Loop System

In this section we will show that the Sylvester equation (3.6) is closely connected to the behavior of the state of the closed-loop system. Our main result presented in the next theorem states that any solution Σ of the Sylvester equation (3.6) can be used to express the state of the closed-loop system and, conversely, if the state of the closed-loop system has this particular form for a given operator Σ, then Σ is necessarily a solution of the equation (3.6). This connection also immediately allows us to express the regulation error using the solution of the Sylvester equation (3.6), and ultimately to prove Theorem 3.2 at the end of this section.

Theorem 3.4. *Let* $\Sigma \in \mathcal{L}(W_\alpha, X_e)$*. Then the following are equivalent.*

(a) *The operator* Σ *is a solution of the Sylvester equation* (3.6) *on* $W_{\alpha+1}$*.*

(b) *For all initial states* $x_{e0} \in X_e$ *and* $v_0 \in W_\alpha$ *of the closed-loop system and the exosystem the state of the closed-loop system can be written as*

$$x_e(t) = T_e(t)(x_{e0} - \Sigma v_0) + \Sigma v(t). \tag{3.8}$$

If (b) *is satisfied, then the regulation error is given by*

$$e(t) = C_e T_e(t)(x_{e0} - \Sigma v_0) + (C_e\Sigma + D_e)v(t), \qquad t \geq 0 \tag{3.9}$$

for all initial states $x_{e0} \in X_e$ and $v_0 \in W_\alpha$ of the closed-loop system and the exosystem.

The fact that the state of the closed-loop system can be represented in the form (3.8) also allows us to describe the asymptotic behavior of a stable closed-loop system. This is due to the fact that if the closed-loop system is strongly stable, then the first term on the right-hand side of (3.8) goes to zero for all initial states of the closed-loop system and the signal generator. This shows that the state of a strongly stable closed-loop system behaves asymptotically as

$$x_e(t) \sim \Sigma v(t).$$

Because of this, the mapping $t \mapsto \Sigma v(t)$ can be seen as a sort of *dynamic steady state* of the closed-loop system. Furthermore, for a strongly stable closed-loop system the formula (3.9) analogously shows that the asymptotic behavior of the regulation error is given by

$$e(t) \sim (C_e\Sigma + D_e)v(t).$$

This last expression also makes the role of the regulation constraint (3.7) in characterizing the controllers solving the output regulation problem very clear. Indeed, to prove Theorem 3.2 it remains to verify that this asymptotic behavior of the regulation error is equal to zero if and only if the regulation constraint is satisfied.

The properties of the dynamic steady state of the closed-loop system are summarized in the following corollary.

Corollary 3.5. *Assume that the closed-loop system is strongly stable and $\Sigma \in \mathscr{L}(W_\alpha, X_e)$ is a solution of the Sylvester equation* (3.6) *on $W_{\alpha+1}$. Then for all initial states $x_{e0} \in X_e$ and $v_0 \in W_\alpha$ of the closed-loop system and the signal generator the state of the closed-loop system and the regulation error satisfy*

$$\lim_{t\to\infty} \|x_e(t) - \Sigma v(t)\| = 0 \qquad \textit{and} \qquad \lim_{t\to\infty} \|e(t) - (C_e\Sigma + D_e)v(t)\| = 0. \tag{3.10}$$

To prove Theorem 3.4 we will need the mild form of the Sylvester equation given in the next lemma. The Sylvester operator equation has a very nice property that this mild form is in fact equivalent to the strong form. This is no longer true for the time-dependent Sylvester differential equation we will study later in Chapter 6. It should be noted that in the case $\alpha = 0$ we can replace $W_{\alpha+1}$ with $\mathscr{D}(S)$ and the proof of this lemma is independent of the form of the operator S. The conclusion of the lemma is then in fact satisfied for any S generating a semigroup on a Banach space W.

Lemma 3.6. *Assume $\Sigma \in \mathscr{L}(W_\alpha, X_e)$. Then the operator Σ satisfies $\Sigma(W_{\alpha+1}) \subset \mathscr{D}(A_e)$ and*

$$\Sigma S = A_e \Sigma + B_e \tag{3.11}$$

on $W_{\alpha+1}$ if and only if

$$\int_0^t T_e(t-s) B_e T_S(s) v \, ds = \Sigma T_S(t) v - T_e(t) \Sigma v, \qquad \forall v \in W_\alpha \tag{3.12}$$

for all $t \in [0, \tau]$ for one/all $\tau > 0$.

Proof. If $\Sigma(W_{\alpha+1}) \subset \mathscr{D}(A_e)$ and if the Sylvester equation (3.11) is satisfied, we have for any $v \in W_{\alpha+1} \subset \mathscr{D}(S)$ and for all $t > s$

$$\begin{aligned} T_e(t-s) B_e T_S(s) v &= T_e(t-s)(\Sigma S - A_e \Sigma) T_S(s) v \\ &= -T_e(t-s) A_e \Sigma T_S(s) v + T_e(t-s) \Sigma S T_S(s) v \\ &= \frac{d}{ds} \left(T_e(t-s) \Sigma T_S(s) v \right). \end{aligned}$$

Integrating both sides of this equation from 0 to an arbitrary $t > 0$ gives

$$\int_0^t T_e(t-s) B_e T_S(s) v \, ds = \Sigma T_S(t) v - T_e(t) \Sigma v.$$

Since the operators on both sides of this equation are in $\mathscr{L}(W_\alpha, X_e)$ and since $W_{\alpha+1}$ is dense in W_α, we have that (3.12) holds for all $v \in W_\alpha$ and $t > 0$.

Assume (3.12) is satisfied on $[0, \tau]$ for some $\tau > 0$ and let $v_0 \in W_{\alpha+1}$. We have that

$$x_e(t) = \int_0^t T_e(t-s) B_e T_S(s) v_0 \, ds$$

is the mild state of the closed-loop system corresponding to the initial states $x_e(0) = 0$ and $v(0) = v_0$ of the closed-loop system and the exosystem, respectively. Since the mapping $t \mapsto T_S(t) v_0$ is continuously differentiable due to the fact that $v \in W_{\alpha+1} \subset \mathscr{D}(S)$, and since $x_e(0) = 0 \in \mathscr{D}(A_e)$, we have from [7, Thm. 3.1.3] that $x_e(\cdot)$ is in fact the classical state of the closed-loop system. This implies that we have $x_e(\cdot) \in C^1([0, \tau], X_e)$ and that

$$\frac{1}{t} \int_0^t T_e(t-s) B_e T_S(s) v_0 = \frac{x_e(t) - x_e(0)}{t} \longrightarrow B_e v_0$$

as $t \to 0+$. Using this and the fact that the restriction $T_S(t)|_{W_\alpha}$ is a strongly continuous semigroup on W_α generated by $S|_{W_\alpha} : W_{\alpha+1} \to W_\alpha$, we can use equation (3.12) to show that for any $t \in (0,\tau]$ we have

$$\begin{aligned}
&\left\| \frac{1}{t}(T_e(t)\Sigma v_0 - \Sigma v_0) - \Sigma S v_0 + B_e v_0 \right\|_{X_e} \\
=\ &\left\| \frac{1}{t}\left(\Sigma T_S(t) v_0 - \int_0^t T_e(t-s) B_e T_S(s) v_0 ds - \Sigma v_0 \right) - \Sigma S v_0 + B_e v_0 \right\|_{X_e} \\
\leq\ &\left\| \Sigma \left(\frac{T_S(t)v_0 - v_0}{t} - S v_0 \right) \right\|_{X_e} + \left\| \frac{1}{t} \int_0^t T_e(t-s) B_e T_S(s) v_0 ds - B_e v_0 \right\|_{X_e} \\
\leq\ &\|\Sigma\|_{\mathscr{L}(W_\alpha, X_e)} \left\| \frac{1}{t}(T_S(t)v_0 - v_0) - S v_0 \right\|_\alpha + \left\| \frac{1}{t} \int_0^t T_e(t-s) B_e T_S(s) v_0 ds - B_e v_0 \right\|_{X_e},
\end{aligned}$$

which converges to 0 as $t \to 0+$. By definition this shows that $\Sigma v_0 \in \mathscr{D}(A_e)$ and

$$A_e \Sigma v_0 = \Sigma S v_0 + B_e v_0.$$

Since $v_0 \in W_{\alpha+1}$ was arbitrary, this concludes the proof. □

Theorem 3.4 can now be proved in a very straightforward manner using the previous lemma.

Proof of Theorem 3.4. For all $x_{e0} \in X_e$ and $v_0 \in W_\alpha$ the mild state of the closed-loop system is given by

$$x_e(t) = T_e(t) x_{e0} + \int_0^t T_e(t-s) B_e T_S(s) v_0 ds.$$

Because of this, the state of the closed-loop system having representation (3.8) for all $x_{e0} \in X_e$ and $v_0 \in W_\alpha$ is equivalent to

$$\begin{aligned}
& T_e(t) x_{e0} + \int_0^t T_e(t-s) B_e T_S(s) v_0 ds = T_e(t)(x_{e0} - \Sigma v_0) + \Sigma v(t) \qquad \forall x_{e0} \in X_e,\ v_0 \in W_\alpha \\
\Leftrightarrow \quad & \int_0^t T_e(t-s) B_e T_S(s) v_0 ds = -T_e(t)\Sigma v_0 + \Sigma T_S(t) v_0, \qquad \forall v_0 \in W_\alpha.
\end{aligned}$$

Since $\Sigma \in \mathscr{L}(W_\alpha, X_e)$, this is by Lemma 3.6 in turn equivalent to the fact that Σ is a solution of the Sylvester equation (3.6) on $W_{\alpha+1}$.

If the state of the closed-loop system can be written as (3.8), the regulation error is then given by

$$e(t) = C_e x_e(t) + D_e v(t) = C_e T_e(t)(x_{e0} - \Sigma v_0) + (C_e \Sigma + D_e) v(t).$$

□

The final auxiliary result we need for the proof of Theorem 3.2 is given in the following lemma. It shows that the asymptotic part of the regulation error decays to zero precisely if the regulation constraint (3.7) is satisfied.

Lemma 3.7. *Let $\Sigma \in \mathscr{L}(W_\alpha, X_e)$. We have*

$$\lim_{t\to\infty} (C_e\Sigma + D_e)T_S(t)v_0 = 0, \qquad \forall v_0 \in W_\alpha$$

if and only if $C_e\Sigma + D_e = 0$ on W_α.

Proof. It is clearly sufficient to show that the limit implies $C_e\Sigma + D_e = 0$ on W_α. Let $k \in \mathbb{Z}$ be arbitrary. For any $l \in \{1, \dots, n_k\}$ we have

$$T_S(t)\phi_k^l = e^{i\omega_k t} \sum_{j=1}^{l} \frac{t^{l-j}}{(l-j)!}\phi_k^j \tag{3.13}$$

for all $t \in \mathbb{R}$. Using the above formula we can see that for $l = 1$

$$0 = \lim_{t\to\infty} \left\|(C_e\Sigma + D_e)T_S(t)\phi_k^1\right\| = \lim_{t\to\infty} \left\|e^{i\omega_k t}(C_e\Sigma + D_e)\phi_k^1\right\| = \left\|(C_e\Sigma + D_e)\phi_k^1\right\|,$$

and thus $(C_e\Sigma + D_e)\phi_k^1 = 0$. This and another application of the formula (3.13) show that for $l = 2$

$$\begin{aligned} 0 &= \lim_{t\to\infty} \left\|(C_e\Sigma + D_e)T_S(t)\phi_k^2\right\| = \lim_{t\to\infty} \left\|e^{i\omega_k t}\left(t(C_e\Sigma + D_e)\phi_k^1 + (C_e\Sigma + D_e)\phi_k^2\right)\right\| \\ &= \left\|(C_e\Sigma + D_e)\phi_k^2\right\|, \end{aligned}$$

which in turn implies $(C_e\Sigma + D_e)\phi_k^2 = 0$. Repeating these steps for $l \in \{3, \dots, n_k - 1\}$ shows that $(C_e\Sigma + D_e)\phi_k^l = 0$ for every $l \in \{1, \dots, n_k - 1\}$. Finally for $l = n_k$

$$\begin{aligned} 0 &= \lim_{t\to\infty} \left\|(C_e\Sigma + D_e)T_S(t)\phi_k^{n_k}\right\| = \lim_{t\to\infty} \left\|\sum_{j=1}^{n_k-1} \frac{t^{n_k-j}}{(n_k-j)!}(C_e\Sigma + D_e)\phi_k^j + (C_e\Sigma + D_e)\phi_k^{n_k}\right\| \\ &= \left\|(C_e\Sigma + D_e)\phi_k^{n_k}\right\|, \end{aligned}$$

and thus $(C_e\Sigma + D_e)\phi_k^{n_k} = 0$. Since $k \in \mathbb{Z}$ was arbitrary and since the set $\{\phi_k^l\}_{kl}$ is a basis of W_α, we must have $C_e\Sigma + D_e = 0$ on W_α. □

We can now conclude this section by collecting the above results to prove Theorem 3.2.

Proof of Theorem 3.2. We will first show that (b) implies (a). Assume the regulation constraint (3.7) is satisfied. Since $T_e(t)$ is strongly stable, we have from Corollary 3.5 that for all initial states $x_{e0} \in X_e$ and $v_0 \in W_\alpha$

$$\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \left\|e(t) - (C_e\Sigma + D_e)v(t)\right\| = 0,$$

since $C_e\Sigma + D_e = 0$. Thus the controller solves the output regulation problem on W_α.

It remains to prove that (a) implies (b). Assume the controller solves the output regulation problem on W_α and $\Sigma \in \mathscr{L}(W_\alpha, X_e)$ is a solution of the Sylvester equation (3.6) on $W_{\alpha+1}$. Since the regulation error decays to zero asymptotically for all initial states of the closed-loop system and the exosystem, Corollary 3.5 implies that for all $x_{e0} \in X_e$ and $v_0 \in W_\alpha$ we must have

$$\|(C_e\Sigma + D_e)T_S(t)v_0\| \leq \|(C_e\Sigma + D_e)T_S(t)v_0 - e(t)\| + \|e(t)\| \overset{t\to\infty}{\longrightarrow} 0,$$

and thus $\lim_{t\to\infty}(C_e\Sigma + D_e)T_S(t)v_0 = 0$ for every $v_0 \in W_\alpha$. This together with Lemma 3.7 concludes that the regulation constraint (3.7) is satisfied. □

3.4 Sufficient Conditions for the Solvability of the Sylvester Equation

In this section we will present sufficient conditions for the solvability of the Sylvester equation (3.6) in Theorem 3.2. A fair amount of theory exists on the solvability of general Sylvester equations with unbounded operators [53, 43, 2], but unfortunately the lack of exponential stability of the closed-loop system prevents us from using the majority of these results. However, it turns out that the structure of the signal generator allows us to use more straightforward methods to derive conditions for the solvability of the equation.

As already discussed in Section 3.2, the conditions presented here will further illustrate the effect of the space W_α on the conditions for the solvability of the Sylvester equation. This follows immediately from the fact that the parameter $\alpha \geq 0$ plays a key role in determining the strictness of these conditions. The connection between the parameter α and the solvability of the Sylvester equation (3.6) will, in turn, make the relationship between the regularity of the reference and disturbance signals and the assumptions required for the solvability of the output regulation problem more concrete.

The first sufficient condition for the solvability of the Sylvester equation is presented in the next theorem. Due to the fact that this condition can be complicated to check, we will also present a simpler sufficient condition later in the section. This second condition is considerably stronger but has the advatages of being easier to verify and being independent of the operators E and F. We will see in the next section that this becomes especially useful in the context of robust output regulation.

Theorem 3.8. *Let $\alpha \geq 0$. Assume that $\sigma_p(A_e) \cap \sigma_p(S) = \varnothing$ and $B_e\phi_k^l \in \mathscr{R}(i\omega_k I - A_e)^{n_k - l + 1}$ for every $k \in \mathbb{Z}$ and $l \in \{1, \ldots, n_k\}$, and that*

$$\sup_{\|x_e^*\|\leq 1} \sum_{k\in\mathbb{Z}} \frac{1}{(1+\omega_k^2)^\alpha} \sum_{l=1}^{n_k} \left| \sum_{j=1}^{l} (-1)^{l-j} \langle R(i\omega_k, A_e)^{l+1-j} B_e \phi_k^j, x_e^* \rangle \right|^2 < \infty, \qquad (3.14)$$

where $x_e^* \in X_e^*$, the dual space of X_e. Then the Sylvester equation $\Sigma S = A_e\Sigma + B_e$ on $W_{\alpha+1}$ has a unique solution $\Sigma \in \mathcal{L}(W_\alpha, X_e)$ given by

$$\Sigma v = \sum_{k\in\mathbb{Z}}\sum_{l=1}^{n_k}\langle v, \phi_k^l\rangle \sum_{j=1}^{l}(-1)^{l-j}R(i\omega_k, A_e)^{l+1-j}B_e\phi_k^j, \qquad v \in W_\alpha. \tag{3.15}$$

Proof. For brevity we denote $R_k = R(i\omega_k, A_e) : \mathcal{R}(i\omega_k I - A_e) \subset X_e \to X_e$ for $k \in \mathbb{Z}$. The operators R_k are well-defined, since $\sigma_p(A_e) \cap \sigma_p(S) = \varnothing$ implies that $i\omega_k I - A_e$ is injective for all $k \in \mathbb{Z}$. We will first show that $\Sigma \in \mathcal{L}(W_\alpha, X_e)$. Using the Cauchy–Schwarz inequality twice we can see that for all $v \in W_\alpha$

$$\begin{aligned}
\|\Sigma v\| &= \sup_{\|x_e^*\|\le 1}\left|\langle \Sigma v, x_e^*\rangle\right| \le \sup_{\|x_e^*\|\le 1}\sum_{k\in\mathbb{Z}}\sum_{l=1}^{n_k}\left|\langle v, \phi_k^l\rangle\right|\left|\sum_{j=1}^{l}(-1)^{l-j}\langle R_k^{l+1-j}B_e\phi_k^j, x_e^*\rangle\right| \\
&\le \sup_{\|x_e^*\|\le 1}\sum_{k\in\mathbb{Z}}\left\|P_k v\right\|\frac{(1+\omega_k^2)^{\frac{\alpha}{2}}}{(1+\omega_k^2)^{\frac{\alpha}{2}}}\cdot\left(\sum_{l=1}^{n_k}\left|\sum_{j=1}^{l}(-1)^{l-j}\langle R_k^{l+1-j}B_e\phi_k^j, x_e^*\rangle\right|^2\right)^{\frac{1}{2}} \\
&\le \|v\|_\alpha\cdot\left(\sup_{\|x_e^*\|\le 1}\sum_{k\in\mathbb{Z}}\frac{1}{(1+\omega_k^2)^\alpha}\sum_{l=1}^{n_k}\left|\sum_{j=1}^{l}(-1)^{l-j}\langle R_k^{l+1-j}B_e\phi_k^j, x_e^*\rangle\right|^2\right)^{\frac{1}{2}}
\end{aligned}$$

and thus $\Sigma \in \mathcal{L}(W_\alpha, X_e)$. Next we will show that Σ is a solution of the Sylvester equation on W_α. Let $s \in \rho(A_e)$ and $v \in W_{\alpha+1}$, and denote $R_s = R(s, A_e)$. Now

$$R(s, A_e)\Sigma(S - sI)v = \sum_{k\in\mathbb{Z}}\sum_{l=1}^{n_k}\langle (S - sI)v, \phi_k^l\rangle \sum_{j=1}^{l}(-1)^{l-j}R_sR_k^{l+1-j}B_e\phi_k^j. \tag{3.16}$$

Using the definition of S we see that each of the terms in the sum over $k \in \mathbb{Z}$ satisfies

$$\begin{aligned}
&\sum_{l=1}^{n_k}\langle (S - sI)v, \phi_k^l\rangle \sum_{j=1}^{l}(-1)^{l-j}R_sR_k^{l+1-j}B_e\phi_k^j \\
=\;& \sum_{l=1}^{n_k-1}\sum_{j=1}^{l}\left(\langle v, \phi_k^l\rangle(-1)^{l-j}(i\omega_k - s)R_sR_k^{l+1-j}B_e\phi_k^j + \langle v, \phi_k^{l+1}\rangle(-1)^{l-j}R_sR_k^{l+1-j}B_e\phi_k^j\right) \\
&+ \langle v, \phi_{n_k}^{n_k}\rangle \sum_{j=1}^{n_k}(-1)^{n_k-j}(i\omega_k - s)R_sR_k^{n_k+1-j}B_e\phi_k^j.
\end{aligned}$$

Using the resolvent equation we see that these, in turn, are equal to

$$\begin{aligned}
&\sum_{j=1}^{n_k-1}\left(\sum_{l=j}^{n_k-1}\langle v,\phi_k^l\rangle(-1)^{l-j}R_sR_k^{l-j}B_e\phi_k^j+\sum_{l=j+1}^{n_k}\langle v,\phi_k^l\rangle(-1)^{l-j+1}R_sR_k^{l-j}B_e\phi_k^j\right)\\
&-\sum_{j=1}^{n_k-1}\sum_{l=j}^{n_k-1}\langle v,\phi_k^l\rangle(-1)^{l-j}R_k^{l+1-j}B_e\phi_k^j-\langle v,\phi_k^{n_k}\rangle\sum_{j=1}^{n_k}(-1)^{n_k-j}R_k^{n_k+1-j}B_e\phi_k^j\\
&+\langle v,\phi_k^{n_k}\rangle\sum_{j=1}^{n_k}(-1)^{n_k-j}R_sR_k^{n_k-j}B_e\phi_k^j\\
=\ &-\sum_{l=1}^{n_k}\sum_{j=1}^{l}\langle v,\phi_k^l\rangle(-1)^{l-j}R_k^{l+1-j}B_e\phi_k^j+\sum_{j=1}^{n_k}\langle v,\phi_k^j\rangle R_sB_e\phi_k^j.
\end{aligned}$$

Substituting the last expression into the original formula (3.16) yields

$$\begin{aligned}
R(s,A_e)\Sigma(S-sI)v&=\sum_{k\in\mathbb{Z}}\left(-\sum_{l=1}^{n_k}\sum_{j=1}^{l}\langle v,\phi_k^l\rangle(-1)^{l-j}R_k^{l+1-j}B_e\phi_k^j+\sum_{j=1}^{n_k}\langle v,\phi_k^j\rangle R_sB_e\phi_k^j\right)\\
&=-\Sigma v+R(s,A_e)B_ev,
\end{aligned}$$

or $\Sigma v=R(s,A_e)B_ev-R(s,A_e)\Sigma(S-sI)v$. This implies that $\Sigma v\in\mathcal{D}(A_e)$ and

$$(sI-A_e)\Sigma v=B_ev-\Sigma(S-sI)v.$$

Since $v\in W_{\alpha+1}$ was arbitrary, this concludes that Σ is a solution of the Sylvester equation $\Sigma S=A_e\Sigma+B_e$ on $W_{\alpha+1}$.

Finally, we will show that Σ is unique. Assume that the operator $\Sigma_1\in\mathcal{L}(W_\alpha,X_e)$ is such that $\Sigma_1(W_{\alpha+1})\subset\mathcal{D}(A_e)$ and $\Sigma_1S=A_e\Sigma_1+B_e$ on $W_{\alpha+1}$. For all $k\in\mathbb{Z}$ we have

$$S\phi_k^1=i\omega_k\phi_k^1\qquad\text{and}\qquad S\phi_k^l=i\omega_k\phi_k^l+\phi_k^{l-1}\quad\forall l\in\{2,\dots,n_k\}.$$

Using this we see that for any $k\in\mathbb{Z}$

$$\begin{aligned}
&B_e\phi_k^1=(i\omega_kI-A_e)\Sigma_1\phi_k^1,\qquad B_e\phi_k^2=(i\omega_kI-A_e)\Sigma_1\phi_k^2+\Sigma_1\phi_k^1,\quad\dots\\
&B_e\phi_k^{n_k}=(i\omega_kI-A_e)\Sigma_1\phi_k^{n_k}+\Sigma_1\phi_k^{n_k-1}
\end{aligned}$$

A direct computation shows that this implies $\Sigma_1\phi_k^l=\sum_{j=1}^l(-1)^{l-j}R(i\omega_k,A_e)^{l+1-j}B_e\phi_k^j$ for all $l\in\{1,\dots,n_k\}$, and thus for any $v\in W_\alpha$ we have

$$\Sigma_1v=\sum_{k\in\mathbb{Z}}\sum_{l=1}^{n_k}\langle v,\phi_k^l\rangle\Sigma_1\phi_k^l=\sum_{k\in\mathbb{Z}}\sum_{l=1}^{n_k}\langle v,\phi_k^l\rangle\sum_{j=1}^{l}(-1)^{l-j}R(i\omega_k,A_e)^{l+1-j}B_e\phi_k^j=\Sigma v.$$

This concludes that $\Sigma_1=\Sigma$. □

The following theorem shows that the result of Theorem 3.8 is in a particular sense very sharp: The stated conditions are in fact necessary for the operator Σ in the theorem to be in $\mathscr{L}(W_\alpha, X_e)$. However, the conditions are still purely sufficient, as they guarantee the existence of a *unique* solution to the Sylvester equation. In particular, if the point spectra of A_e and S are not disjoint, then there exists a $k \in \mathbb{Z}$ for which the operator $i\omega_k I - A_e$ is not injective. If the condition $B_e \phi_k^l \in \mathscr{R}(i\omega_k I - A_e)^{n_k - l + 1}$ for all $l \in \{1, \ldots, n_k\}$ is still satisfied, then the Sylvester equation (3.6) has an infinite number of solutions.

Theorem 3.9. *If the operator* Σ *defined by* (3.15) *is in* $\mathscr{L}(W_\alpha, X_e)$*, then the condition* (3.14) *is satisfied.*

Proof. Assume $\Sigma \in \mathscr{L}(W_\alpha, X_e)$ and denote $R_k = R(i\omega_k, A_e)$ for brevity. There exists a constant $M \geq 0$ such that for all $v \in W_\alpha$ we have

$$\sup_{\|x_e^*\| \leq 1} \left| \sum_{k \in \mathbb{Z}} \sum_{l=1}^{n_k} \langle v, \phi_k^l \rangle \sum_{j=1}^{l} (-1)^{l-j} \langle R_k^{l+1-j} B_e \phi_k^j, x_e^* \rangle \right| = \|\Sigma v\| \leq M \|v\|_\alpha. \tag{3.17}$$

Let $x_e^* \in X_e^*$ be such that $\|x_e^*\| \leq 1$ and let $N_1, N_2 \in \mathbb{N}$. Choose $v \in W_\alpha$ in such a way that if $-N_1 \leq k \leq N_2$, then

$$\langle v, \phi_k^l \rangle = \frac{1}{(1+\omega_k^2)^\alpha} \sum_{j=1}^{l} (-1)^{l-j} \overline{\langle R_k^{l+1-j} B_e \phi_k^j, x_e^* \rangle}$$

and $\langle v, \phi_k^l \rangle = 0$ otherwise. Substituting v into (3.17) shows that

$$\begin{aligned}
&\sum_{k=-N_1}^{N_2} \frac{1}{(1+\omega_k^2)^\alpha} \sum_{l=1}^{n_k} \left| \sum_{j=1}^{l} (-1)^{l-j} \langle R_k^{l+1-j} B_e \phi_k^j, x_e^* \rangle \right|^2 \leq \|\Sigma v\| \leq M \|v\|_\alpha \\
&= M \left(\sum_{k=-N_1}^{N_2} \frac{(1+\omega_k^2)^\alpha}{(1+\omega_k^2)^{2\alpha}} \sum_{l=1}^{n_k} \left| \sum_{j=1}^{l} (-1)^{l-j} \langle R_k^{l+1-j} B_e \phi_k^j, x_e^* \rangle \right|^2 \right)^{\frac{1}{2}},
\end{aligned}$$

and thus

$$\sum_{k=-N_1}^{N_2} \frac{1}{(1+\omega_k^2)^\alpha} \sum_{l=1}^{n_k} \left| \sum_{j=1}^{l} (-1)^{l-j} \langle R_k^{l+1-j} B_e \phi_k^j, x_e^* \rangle \right|^2 \leq M^2.$$

This holds for all $N_1, N_2 \in \mathbb{N}$, and letting $N_1, N_2 \to \infty$ we see that

$$\sum_{k \in \mathbb{Z}} \frac{1}{(1+\omega_k^2)^\alpha} \sum_{l=1}^{n_k} \left| \sum_{j=1}^{l} (-1)^{l-j} \langle R_k^{l+1-j} B_e \phi_k^j, x_e^* \rangle \right|^2 \leq M^2.$$

Since $x_e^* \in X_e^*$ with $\|x_e^*\| \leq 1$ was arbitrary, also the supremum over these elements must be bounded from above by M^2. This concludes the proof. □

We conclude this section by considering a simpler sufficient condition for the solvability of the Sylvester equation already encountered in the statement of Theorem 3.3. This condition has a nice additional property that it is independent of the operator B_e of the closed-loop system and, consequently, of the operators E and F containing the output operators of the infinite-dimensional exosystem.

Lemma 3.10. *Let $\alpha \geq 0$. If $\sigma(A_e) \cap \sigma_p(S) = \varnothing$ and if*

$$\sup_{k\in\mathbb{Z}} \frac{\|R(i\omega_k, A_e)\|^{2n_k}}{(1+\omega_k^2)^\alpha} < \infty, \tag{3.18}$$

then the Sylvester equation $\Sigma S = A_e \Sigma + B_e$ on $W_{\alpha+1}$ has a unique solution given by (3.15).

Proof. We will verify that under our assumptions the conditions of Theorem 3.8 are satisfied. Since $\sigma(A_e) \cap \sigma_p(S) = \varnothing$, we have $\mathscr{R}(i\omega_k I - A_e)^l = X_e$ for all $l \in \{1, \dots, n_k\}$, and $R(i\omega_k, A_e)$ are bounded operators for all $k \in \mathbb{Z}$. Choose $n \in \mathbb{N}$ such that $n_k \leq n$ for all $k \in \mathbb{Z}$. Then for any $x_e^* \in X_e^*$ with $\|x_e^*\| \leq 1$ we have

$$\left| \sum_{j=1}^{l} (-1)^{l-j} \langle R(i\omega_k, A_e)^{l+1-j} B_e \phi_k^j, x_e^* \rangle \right| \leq \sum_{j=1}^{l} \left\| R(i\omega_k, A_e) \right\|^{l+1-j} \left\| B_e \phi_k^j \right\| \cdot \left\| x_e^* \right\|$$

$$\leq \max\left\{ \left\| R(i\omega_k, A_e) \right\|, \left\| R(i\omega_k, A_e) \right\|^{n_k} \right\} \cdot \sum_{j=1}^{n_k} \left\| B_e \phi_k^j \right\|$$

$$\leq \max\left\{ 1, \left\| R(i\omega_k, A_e) \right\|^{n_k} \right\} \cdot \sqrt{n_k} \cdot \left(\sum_{j=1}^{n_k} \left\| B_e \phi_k^j \right\|^2 \right)^{\frac{1}{2}}$$

for all $k \in \mathbb{Z}$. The condition (3.18) implies that there exists $M \geq 0$ such that

$$\frac{\max\left\{1, \|R(i\omega_k, A_e)\|^{2n_k}\right\}}{(1+\omega_k^2)^\alpha} \leq M$$

for all $k \in \mathbb{Z}$. Since by (3.5) the operator B_e satisfies $\left(B_e \phi_k^l\right)_{kl} \in \ell^2(X_e)$, we have

$$\sup_{\|x_e^*\|\leq 1} \sum_{k\in\mathbb{Z}} \frac{1}{(1+\omega_k^2)^\alpha} \sum_{l=1}^{n_k} \left| \sum_{j=1}^{l} (-1)^{l-j} \langle R(i\omega_k, A_e)^{l+1-j} B_e \phi_k^j, x_e^* \rangle \right|^2$$

$$\leq \sum_{k\in\mathbb{Z}} \frac{\max\left\{1, \|R(i\omega_k, A_e)\|^{2n_k}\right\}}{(1+\omega_k^2)^\alpha} \sum_{l=1}^{n_k} n_k \sum_{j=1}^{n_k} \left\| B_e \phi_k^j \right\|^2 \leq n^2 M \sum_{k\in\mathbb{Z}} \sum_{j=1}^{n_k} \left\| B_e \phi_k^j \right\|^2 < \infty.$$

The conclusion of the lemma now follows from Theorem 3.8. □

With the aid of the above lemma the proof of Theorem 3.3 becomes very simple.

Proof of Theorem 3.3. Lemma 3.10 implies that the given operator Σ is in $\mathscr{L}(W_\alpha, X_e)$ and that it is the unique solution of the Sylvester equation (3.6). The rest of the theorem follows directly from Theorem 3.2. □

3.5 The Robust Output Regulation Problem

In this section we will define the problem of robust output regulation. This problem consists of choosing the controller in such a way that the tracking of the reference signals and disturbance rejection are achieved despite perturbations to the parameters of the system, provided that the closed-loop system remains strongly stable. We continue the study in the next chapter, where we characterize the controllers solving the robust output regulation problem for a given infinite-dimensional exosystem.

In this thesis we are interested in the control structure's tolerance to uncertainties in the parameters of the plant. This, however, is only one of many different types of robustness desirable in practical applications. One particularly important question is whether the control structure is robust with respect to addition of small delays in the feedback loop. These types of delays are an inevitable consequence of finite computation times and time lags in signal transmissions. For infinite-dimensional signal generators this problem turns out to be extremely difficult. We will see in the next chapter that our main result concerning robust output regulation problem, the internal model principle, together with the results presented in [30] imply a negative result concerning this type of robustness. More precisely, we will see that if we want to design a controller that is robust with respect to perturbations to the parameters of the system and if the exosystem has an infinite number of eigenvalues on the imaginary axis, then the controller can not be robust with respect to small delays.

The formal statement of our main problem is given below.

The Robust Output Regulation Problem on W_α. Let $\alpha \geq 0$. Find $(\mathcal{G}_1, \mathcal{G}_2, K)$ such that the following are satisfied:

1. The closed-loop system operator A_e generates a strongly stable C_0-semigroup on X_e.
2. For all initial states $v_0 \in W_\alpha$ and $x_{e0} \in X_e$ the regulation error goes to zero asymptotically, i.e., $\lim_{t\to\infty} e(t) = 0$.
3. If the operators (A, B, C, D, E, F) are perturbed to (A', B', C', D', E', F') in such a way that the new closed-loop system (A'_e, B'_e, C'_e, D'_e) is strongly stable and the Sylvester equation
$$\Sigma' S = A'_e \Sigma' + B'_e \tag{3.19}$$
on $W_{\alpha+1}$ has a solution $\Sigma' \in \mathcal{L}(W_\alpha, X_e)$, then $\lim_{t\to\infty} e(t) = 0$ for all initial states $v_0 \in W_\alpha$ and $x_{e0} \in X_e$.

■

The formula

$$e(t) = C_e T_e(t)(x_{e0} - \Sigma v_0) + (C_e \Sigma + D_e)v(t) \tag{3.20}$$

in Theorem 3.4 is essential in studying the solvability of the robust output regulation problem and in characterizing the controllers solving it. It shows that the regulation error consists of two terms which are fairly independent of each other. The behavior of the first term with respect to time only depends on the semigroup $T_e(t)$ of the closed-loop system. This part of the regulation error decays to zero for all initial states x_{e0} and v_0 whenever the closed-loop system is strongly stable. On the other hand, the second term in the formula depends mainly on the behavior of the exosystem, and the closed-loop system only affects it through the solution Σ of the Sylvester equation. We have seen in Lemma 3.7 that this term goes to zero asymptotically if and only if the regulation constraint

$$C_e \Sigma + D_e = 0$$

in Theorem 3.2 is satisfied. Using these observations the robust output regulation problem can be divided into two somewhat independent parts.

The first part of the problem can be seen as the problem of *robust stabilization* of the closed-loop system. When the operators of the system are perturbed in such a way that the strong stability of the closed-loop is preserved and the Sylvester equation (3.19) with perturbed parameters has a solution, then by Theorem 3.4 the regulation error for this perturbed system is again described by a formula similar to the one in (3.20). Since the perturbed closed-loop system is strongly stable, the first term of the regulation error decays again to zero asymptotically.

The second part of the problem consists of choosing the controller parameters in such a way that also the second term of (3.20) approaches zero for all perturbations of the system's parameters. Since the requirement is that the controller solves the output regulation problem for all perturbations for which the closed-loop system is stable and the Sylvester equation (3.19) has a solution, we have from Theorem 3.2 that the requirement for the decay of the second term is equivalent to the requirement that the regulation constraint

$$C_e' \Sigma' + D_e' = 0$$

is satisfied for all such perturbations. This part of the problem can be seen as a problem of choosing a controller for which the conditional property

> "given the closed-loop stability, the controller tracks and rejects the signals generated by the exosystem"

is robust with respect to perturbations to the operators of the system. Using the above reasoning this property of the controller can be formulated mathematically using Sylvester equations.

Definition 3.11. A controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ is said to be *conditionally robust* if the implication

$$\Sigma' S = A_e' \Sigma' + B_e' \quad \Rightarrow \quad C_e' \Sigma' + D_e' = 0 \tag{3.21}$$

is satisfied for all operators A_e', B_e', C_e', D_e' of the closed-loop system and $\Sigma' \in \mathcal{L}(W_\alpha, X_e)$ such that $\Sigma'(W_{\alpha+1}) \subset \mathcal{D}(A_e')$. ■

Even though we call the property in Definition 3.11 *conditional* robustness, we can see from the definition that it is in fact independent of the problem of stabilizing the closed-loop system. Using this concept we indeed achieve division of the robust output regulation problem into two independent parts that can be solved separately. The name of the property comes from our general approach to solving the robust output regulation problem and, on a very fundamental level, from the fact that we consider strong stability of the closed-loop system. The robustness properties of this type of stability are considerably weaker than the ones of exponential stability, which is usually used when the exosystem is finite-dimensional. In the case of exponentially stability it is easy to determine classes of perturbations preserving the stability of the closed-loop system, which in turn also guarantees the solvability of the perturbed Sylvester equation (3.19) [43]. However, in the case of strong stability we need to state the robust output regulation problem in a conditional form where we — being unable to characterize suitable classes of perturbations doing so — assume the perturbations preserve the stability of the closed-loop system and the solvability of the Sylvester equation (3.19). The concept of conditional robustness allows us to temporarily disregard this complication and concentrate on choosing the parameters of the controller in such a way that the regulation constraint is satisfied robustly. This latter problem, in fact, turns out to be the more interesting part of the robust output regulation problem.

The following theorem verifies that the conditional robustness of the controller as defined above is indeed the appropriate concept to use in dividing the robust output regulation problem into parts. It shows that provided the controller stabilizes the closed-loop system, the controller solves the robust output regulation problem if and only if it is conditionally robust. We will prove Theorem 3.12 later in Section 4.1, where we have a more convenient characterization for the conditionally robust controllers at our disposal.

Theorem 3.12. *If the controller* $(\mathscr{G}_1, \mathscr{G}_2, K)$ *stabilizes the closed-loop strongly and the Sylvester equation*

$$\Sigma S = A_e \Sigma + B_e \tag{3.22}$$

on $W_{\alpha+1}$ *has a solution, then the controller solves the robust output regulation problem on* W_α *if and only if it is conditionally robust.*

As mentioned above, characterizing classes of perturbations preserving strong stability of a semigroup is a difficult task, and no results on this problem exist for general semigroups. In a special case where the generator of the semigroup is a Riesz-spectral operator [7, Sec. 2.3] this question has been studied in [38], where classes of perturbations preserving the strong and polynomial stability types of a semigroup are presented.

We conclude this section by presenting two properties of conditionally robust controllers. The first one shows that conditional robustness is in fact independent of the chosen space W_α. Because of this, we can without loss of generality always consider the condition (3.21) for $\alpha = 0$. Moreover, we will show that we can state the robust output regulation problem for general operators $E' \in \mathscr{L}(W,X)$ and $F' \in \mathscr{L}(W,Y)$, whereas technically we should require that they, consisting of output operators of the infinite-dimensional exosystem, also satisfy

$$\sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} \|E'\phi_k^l\|^2 < \infty \qquad \text{and} \qquad \sum_{k\in\mathbb{Z}} \sum_{l=1}^{n_k} \|F'\phi_k^l\|^2 < \infty$$

in accordance with Definition 2.1. The second part of the next theorem shows that we can safely omit this additional requirement.

Theorem 3.13. *The following are true.*

1. *A controller* $(\mathscr{G}_1, \mathscr{G}_2, K)$ *is conditionally robust on* W_α *for some* $\alpha \geq 0$ *if and only if it is conditionally robust on* W.
2. *The implication in* (3.21) *is satisfied for all operators* (A', B', C', D', E', F') *if and only if it is satisfied for all such operators with* E' *and* F' *satisfying* $\left(E'\phi_k^l\right)_{kl} \in \ell^2(X)$ *and* $\left(F'\phi_k^l\right)_{kl} \in \ell^2(Y)$.

Proof. We will begin by considering the first claim. Assume that the controller is conditionally robust on W_α for some $\alpha \geq 0$ and let $\Sigma \in \mathscr{L}(W, X_e)$ be a solution of the Sylvester equation

$$\Sigma S = A_e \Sigma + B_e$$

on W_1. The operator Σ satisfies $\Sigma \in \mathscr{L}(W_\alpha, X_e)$ and $\Sigma(W_{\alpha+1}) \subset \mathscr{D}(A_e)$, and thus it is also a solution of this Sylvester equation on $W_{\alpha+1}$. Condition (3.21) implies that $C_e\Sigma + D_e = 0$

on W_α. However, this is also satisfied on W, since $\Sigma \in \mathscr{L}(W,X_e)$ and since W_α is dense in W. This concludes that the controller is conditionally robust on W.

On the other hand, assume that the controller is conditionally robust on W. Let $\alpha \geq 0$ be arbitrary and let $\Sigma \in \mathscr{L}(W_\alpha,X_e)$ be a solution of the Sylvester equation

$$\Sigma S = A_e \Sigma + B_e$$

on $W_{\alpha+1}$. To show that the controller is conditionally robust on W_α we will need to show that $C_e\Sigma + D_e = 0$ on W_α. For this we will use the projections $P_k \in \mathscr{L}(W)$ onto the finite-dimensional subspaces $P_k W = \operatorname{span}\left\{\phi_k^l\right\}_{l=1}^{n_k}$ of W.

Let $k \in \mathbb{Z}$. Since for any $v \in \mathscr{D}(S)$ we have $P_k v \in W_{\alpha+1}$, the Sylvester equation and the commutativity relation $SP_k v = P_k S v$ imply

$$\Sigma P_k S v = A_e \Sigma P_k v + B_e P_k v \qquad \Leftrightarrow \qquad \tilde{\Sigma} S v = \tilde{A}_e \tilde{\Sigma} v + \tilde{B}_e v$$

on $\mathscr{D}(S)$ where $\tilde{\Sigma} = \Sigma P_k \in \mathscr{L}(W,X_e)$ satisfies $\mathscr{R}(\tilde{\Sigma}) \subset \mathscr{D}(A_e)$, and we have $\tilde{A}_e = A_e$ and

$$\tilde{B}_e = B_e P_k = \begin{pmatrix} EP_k \\ \mathscr{G}_2 F P_k \end{pmatrix} = \begin{pmatrix} \tilde{E} \\ \mathscr{G}_2 \tilde{F} \end{pmatrix} \in \mathscr{L}(W,X_e).$$

For the corresponding operators $\tilde{C}_e$ and $\tilde{D}_e$ we have

$$\tilde{C}_e = \begin{pmatrix} C & DK \end{pmatrix} = C_e, \qquad \tilde{D}_e = \tilde{F} = F P_k = D_e P_k,$$

and the fact that the controller is conditionally robust on W implies

$$0 = \tilde{C}_e \tilde{\Sigma} + \tilde{D}_e = C_e \Sigma P_k + D_e P_k = (C_e \Sigma + D_e) P_k.$$

Since $k \in \mathbb{Z}$ was arbitrary, we must have $C_e\Sigma + D_e = 0$. This shows that the controller is conditionally robust on W_α and concludes that the first claim of the theorem is true.

To prove the second part of the theorem it is clearly sufficient to show that the "if"-part of the claim is true. This can be done in exactly the same way as above using the projections P_k. This follows from the fact that if $E \in \mathscr{L}(W,X)$ and $F \in \mathscr{L}(W,Y)$, then for any $k \in \mathbb{Z}$ the operators $\tilde{E} = EP_k$ and $\tilde{F} = FP_k$ satisfy

$$\sum_{k' \in \mathbb{Z}} \sum_{l=1}^{n_{k'}} \left\|\tilde{E}\phi_{k'}^l\right\|^2 = \sum_{l=1}^{n_k} \left\|E\phi_k^l\right\|^2 < \infty \qquad \text{and} \qquad \sum_{k' \in \mathbb{Z}} \sum_{l=1}^{n_{k'}} \left\|\tilde{F}\phi_{k'}^l\right\|^2 = \sum_{l=1}^{n_k} \left\|F\phi_k^l\right\|^2 < \infty.$$

Using these operators we can conlude that the implication in (3.21) is satisfied in exactly the same way as above. □

The sufficient conditions for the solvability of the Sylvester equation given in Theorem 3.8 and Lemma 3.10 can also be used to determine classes of perturbations admissible in the sense of the robust output regulation problem. This can be done simply by requiring that the perturbations considered in the problem preserve the strong stability of the closed-loop system and the conditions of either Theorem 3.8 or Lemma 3.10. In particular the classes of perturbations obtained by using the latter set of conditions contains all perturbations of the operators E and F because they do not appear in the operator A_e. For this same reason, they also do not affect the stability of the closed-loop system. The perturbations of these two operators correspond to the perturbation of the output operators of the infinite-dimensional exosystem. For this reason it is easy to see that if the closed-loop system is stabilized in such a way that the conditions of Lemma 3.10 are satisfied for some $\alpha \geq 0$, then the controller solving the robust output regulation problem is in fact capable of tracking any reference signal corresponding to an initial state $v_0 \in W_\alpha$ of *any* infinite-dimensional exosystem with the given spectrum and Jordan block structure.

Chapter 4

The Internal Model Principle for Infinite-Dimensional Exosystems

In this chapter we continue the study of robust output regulation. Our main interest is the characterization of the controllers solving the robust output regulation problem via the so-called *internal model principle*. In the finite-dimensional linear control theory this classical result by Francis and Wonham [11] states that a stabilizing controller solves the robust output regulation problem if and only if

> whenever $s \in \sigma(S)$ is an eigenvalue of S and $d(s)$ is the dimension of the largest Jordan block associated to s, then $s \in \sigma(\mathcal{G}_1)$ and $\mathcal{G}_1$ has at least p Jordan blocks of dimension greater than or equal to $d(s)$ associated to s,

where p refers to the dimension of the output space. If a controller has this property, it is said to *incorporate a p-copy internal model* of the exosystem. This elegant result shows that in order for the dynamic error feedback controller to solve the robust output regulation problem, its dynamics must be able to reproduce the behavior of the exosystem.

It is clear that since the Jordan canonical form is not available for operators on infinite-dimensional spaces, the generalization of the internal model principle to distributed parameter systems with infinite-dimensional exosystem first of all requires generalizing the concept of the internal model in an appropriate way. However, in our situation we can derive a very natural generalization for the above concept. Due to our way of constructing the infinite-dimensional exosystem, we already have access to all the necessary information concerning the operator S, namely the knowledge of its spectrum and the dimensions of the largest Jordan blocks associated to its eigenvalues. Moreover, although the operator $\mathcal{G}_1$ does not have a complete canonical form, there are no difficulties in defining individual Jordan chains even for unbounded operators. A straightforward generalization of the finite-dimensional internal model based on these observations leads to the following definition.

Definition 4.1 (The p-copy internal model)**.** Assume $\dim Y < \infty$. A controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ is said to *incorporate a p-copy internal model* of the exosystem S if for all $k \in \mathbb{Z}$ we have

$$\dim \mathcal{N}(i\omega_k I - \mathcal{G}_1) \geq \dim Y$$

and $\mathcal{G}_1$ has at least $\dim Y$ independent Jordan chains of length greater than or equal to d_k associated to the eigenvalue $i\omega_k$. ■

It turns out that this definition is ideally suited to our purposes. First of all, it is clear that this is indeed a very direct generalization of the finite-dimensional p-copy internal model. Moreover, this concept can also be used to characterize the controllers solving the robust output regulation problem precisely as its finite-dimensional counterpart. In fact, the sole purpose of this chapter is to show that this is true by proving the following theorem, which states that the conditionally robust controllers — and hence also the controllers solving the robust output regulation problem — are exactly those incorporating a p-copy internal model of the exosystem. In other words, the theorem generalizes the p-copy internal model principle of Francis and Wonham for distributed parameter systems with infinite-dimensional exosystems.

Theorem 4.2. *Assume Y is finite-dimensional and $\sigma(A_e) \cap \sigma_p(S) = \varnothing$. The controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ is conditionally robust if and only if it incorporates a p-copy internal model of the exosystem.*

The assumption on the disjointness of the spectra of the closed-loop system and the exosystem in Theorem 4.2 is again a consequence of considering strongly stabilizable closed-loop systems. Since $\sigma(S) \subset i\mathbb{R}$, this condition is always satisfied for finite-dimensional systems and for distributed parameter systems for which the closed-loop can be stabilized exponentially. Assumptions under which this condition can be satisfied in the case of an infinite-dimensional exosystem are studied in greater detail in Chapter 5.

During the process of proving Theorem 4.2 we will encounter two other ways of defining an 'internal model' found in the literature. The first one is the *internal model structure* defined by Immonen [23, 22]. This property of the controller is expressed using a Sylvester equation involving the system operator S of the exosystem and the parameters of the error feedback controller. It was shown in [22] that this property can be used to characterize the controllers solving the robust output regulation problem in very much the same way as the p-copy internal model can be used in the finite-dimensional theory. However, the applicability of this definition and the associated result are severely limited due to the fact that the definition of the internal model structure is very abstract and particularly difficult to verify for actual controllers.

The second alternative definition for the internal model consists of the conditions introduced by Pohjolainen and Hämäläinen [46, 15] imposed on the ranges and kernels of the operators of the error feedback controller. To distinguish it from the other concepts we call this set of constraints the *$\mathscr{G}$-conditions*. They were first introduced as purely sufficient conditions for the controller to be conditionally robust. However, it turns out that these conditions are, in fact, equivalent to the conditional robustness of the controller under minimal assumptions. The $\mathscr{G}$-conditions have an advantage over the internal model structure in being more concrete and easier to verify. On the other hand, they also turn out to be applicable under more general assumptions than our p-copy internal model. In particular, they continue to be meaningful even in the case of an infinite-dimensional output space, a situation where Definition 4.1 becomes ambiguous.

As we already mentioned in Section 3.5, in the case of infinite-dimensional exosystems the internal model principle in Theorem 4.2 also implies a negative result concerning the robustness of the associated control structures with respect to small delays. It has been shown in [30] that if a system has an infinite number of eigenvalues on the imaginary axis, then there exists no stabilizing controller for which the controlled system is robust with respect to delays. We encounter this situation when designing a controller to solve the robust output regulation problem for an exosystem with an infinite spectrum. The internal model principle implies that any such controller must itself have an infinite number of imaginary eigenvalues. Therefore the results in [30] conclude that it is not possible to choose the parameters of the controller in such a way that the resulting closed-loop system would be robust with respect to small delays.

The structure of the proof of the internal model principle is sketched in Figure 4.1. We proceed from left to right and consider the three relationships between the concepts in their own dedicated sections. In the proofs of these interrelations we also use weaker assumptions whenever possible.

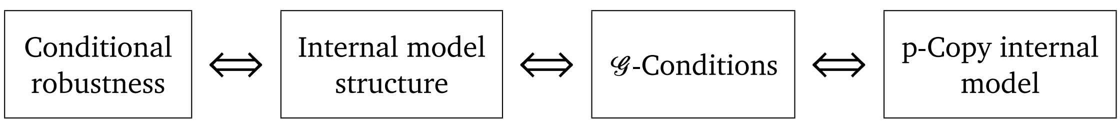

Figure 4.1: Outline of the proof of the internal model principle

The organization and the main contributions of this chapter are outlined in the following.

Section 4.1 In this section we study the internal model structure by Immonen. We show that this concept is equivalent to the conditional robustness of the controller. With the aid of this result we also prove Theorem 3.12.

Section 4.2 The $\mathscr{G}$-conditions introduced by Pohjolainen and Hämäläinen are studied in this section. These conditions are first generalized for infinite-dimensional exosystems having nontrivial Jordan block structrure. We consider the relationship between the $\mathscr{G}$-conditions and the internal model structure and show that under suitable assumptions these properties are equivalent. The proof of the equivalence is split into parts in such a way that the precise conditions for each of the implications are clearly visible.

Section 4.3 In this section we show that the p-copy internal model given in Definition 4.1 is under suitable assumptions equivalent to the $\mathscr{G}$-conditions. The proof of this equivalence is again split into a series of lemmas. These individual results show that parts of the interrelation hold even under more general conditions. Finally, combining the results presented in the three sections we can prove Theorem 4.2, the main result of the chapter.

The results in this chapter generalize mainly those presented in [24, 15]. The relationship between the internal model structure and the controller solving the robust output regulation problem has been established in [24], but our use of conditional robustness provides a logical midway between these two properties. In particular, conditional robustness is independent of the form of the controller. The fact that the $\mathscr{G}$-conditions imply conditional robustness was proved in [46] for finite-dimensional and in [15] for diagonal exosystems.

4.1 The Internal Model Structure

In this section we concentrate on the internal model structure of Immonen [23] and study its relationship to conditional robustness of the controller. We will see that with the aid of the obtained results we will also be able to prove Theorem 3.12 fairly effortlessly. We start by stating the formal definition of this property of the controller.

Definition 4.3 (Internal model structure)**.** A controller $(\mathscr{G}_1, \mathscr{G}_2, K)$ is said to have *internal model structure* if

$$\forall\ \Gamma, \Delta : \qquad \Gamma S = \mathscr{G}_1 \Gamma + \mathscr{G}_2 \Delta \quad \Rightarrow \quad \Delta = 0, \tag{4.1}$$

where $\Gamma \in \mathscr{L}(W, Z)$ is such that $\Gamma(\mathscr{D}(S)) \subset \mathscr{D}(\mathscr{G}_1)$ and $\Delta \in \mathscr{L}(W, Y)$. ■

At the end of the previous chapter we saw that conditional robustness of the controller is independent of the choice of the space W_α. Because of this, we can without loss of generality always consider Definition 3.11 for $\alpha = 0$. Accordingly, it is not necessary to extend the definition of the internal model structure for unbounded solutions of the Sylvester equations.

The following theorem shows that conditional robustness of a controller is equivalent to the internal model structure without any additional assumptions. Because of this, the internal model structure of a controller can indeed be seen as a way of defining an internal model for distributed parameter systems. However, it is also clear that the usefulness of this concept is limited due to the fact that condition (4.1) is not very easy to verify in practice.

Theorem 4.4. *A controller $(\mathscr{G}_1, \mathscr{G}_2, K)$ is conditionally robust if and only if it has internal model structure.*

Proof. Assume that the controller $(\mathscr{G}_1, \mathscr{G}_2, K)$ is conditionally robust. Let the operators $\Gamma \in \mathscr{L}(W,Z)$ and $\Delta \in \mathscr{L}(W,Y)$ be such that $\Gamma(\mathscr{D}(S)) \subset \mathscr{D}(\mathscr{G}_1)$ and $\Gamma S = \mathscr{G}_1\Gamma + \mathscr{G}_2\Delta$ on is satisfied $\mathscr{D}(S)$. We need to show that $\Delta = 0$. Let A, B, C, and D be the operators of an arbitrary plant and choose

$$E = -BK\Gamma \in \mathscr{L}(W,X), \qquad F = \Delta - DK\Gamma \in \mathscr{L}(W,Y).$$

Now $\Sigma = (0 \quad \Gamma)^T \in \mathscr{L}(W,X_e)$ is an operator satisfying $\Sigma(\mathscr{D}(S)) \subset \mathscr{D}(A) \times \mathscr{D}(\mathscr{G}_1) = \mathscr{D}(A_e)$. For any $v \in \mathscr{D}(S)$ we can see using $\Gamma S v = \mathscr{G}_1\Gamma v + \mathscr{G}_2\Delta v$ that

$$\Sigma S v = \begin{pmatrix} 0Sv \\ \Gamma S v \end{pmatrix} = \begin{pmatrix} A0v + BK\Gamma v \\ \mathscr{G}_2 C0v + \mathscr{G}_1\Gamma v + \mathscr{G}_2 DK\Gamma v \end{pmatrix} + \begin{pmatrix} -BK\Gamma v \\ \mathscr{G}_2(\Delta v - DK\Gamma v) \end{pmatrix} = A_e\Sigma v + B_e v,$$

and thus $\Sigma S = A_e\Sigma + B_e$ on $\mathscr{D}(S)$. The fact that the controller is conditionally robust now implies that for all $v \in W$

$$0 = C_e\Sigma v + D_e v = C0v + DK\Gamma v + (\Delta - DK\Gamma)v = \Delta v.$$

This concludes that $\Delta = 0$ and thus the controller $(\mathscr{G}_1, \mathscr{G}_2, K)$ has internal model structure.

Assume now that the controller $(\mathscr{G}_1, \mathscr{G}_2, K)$ has internal model structure and that the operators A_e and B_e of the closed-loop system and $\Sigma \in \mathscr{L}(W,X_e)$ satisfy $\Sigma(\mathscr{D}(S)) \subset \mathscr{D}(A_e)$ and $\Sigma S = A_e\Sigma + B_e$. We will need to show that $C_e\Sigma + D_e = 0$. Since $X_e = X \times Z$ and $\mathscr{D}(A_e) = \mathscr{D}(A) \times \mathscr{D}(\mathscr{G}_1)$, the operator Σ is of the form

$$\Sigma = \begin{pmatrix} \Pi \\ \Gamma \end{pmatrix}, \qquad \Pi \in \mathscr{L}(W,X), \;\; \Gamma \in \mathscr{L}(W,Z),$$

where $\Pi(\mathscr{D}(S)) \subset \mathscr{D}(A)$ and $\Gamma(\mathscr{D}(S)) \subset \mathscr{D}(\mathscr{G}_1)$. The Sylvester equation $\Sigma S = A_e\Sigma + B_e$ implies that for all $v \in \mathscr{D}(S)$ we have

$$\begin{pmatrix} \Pi S v \\ \Gamma S v \end{pmatrix} = \begin{pmatrix} A & BK \\ \mathscr{G}_2 C & \mathscr{G}_1 + \mathscr{G}_2 DK \end{pmatrix} \begin{pmatrix} \Pi v \\ \Gamma v \end{pmatrix} + \begin{pmatrix} Ev \\ \mathscr{G}_2 F v \end{pmatrix} = \begin{pmatrix} A\Pi v + BK\Gamma v + Ev \\ \mathscr{G}_1\Gamma v + \mathscr{G}_2(C\Pi + DK\Gamma + F)v \end{pmatrix}.$$

The lower line of this equation states that

$$\Gamma S = \mathscr{G}_1\Gamma + \mathscr{G}_2(C\Pi + DK\Gamma + F)$$

on $\mathscr{D}(S)$, where $C\Pi + DK\Gamma + F \in \mathscr{L}(W,Y)$. Since the controller has internal model structure, we have from (4.1) that

$$0 = C\Pi + DK\Gamma + F = \begin{pmatrix} C & DK \end{pmatrix} \begin{pmatrix} \Pi \\ \Gamma \end{pmatrix} + F = C_e\Sigma + D_e,$$

and thus the controller is conditionally robust. This concludes the proof. □

Since the definition of the internal model structure is independent of the parameter K of the controller, by Theorem 4.4 the same applies to conditional robustness. Indeed, this operator is only used in stabilizing the closed-loop system in order to solve the robust output regulation problem, and can be chosen freely without the risk of affecting conditional robustness of the controller. However, the choice of K does affect the spectrum of the system operator A_e of the closed-loop system, and thus in particular the condition on the distjointness of the spectra of A_e and S present in Theorem 4.2.

In Section 3.5 we saw that condition (3.21) in the definition of conditional robustness was chosen to ensure that the regulation error would decay asymptotically for all perturbations preserving the strong stability of the closed-loop system and for which the Sylvester equation (3.19) has a solution. Because of this, it would seem sufficient to require condition (3.21) to be satisfied for all operators for which the closed-loop system is strongly stable. However, by slightly modifying the proof of Theorem 4.4 it is easy to show that provided there exist operators for which the closed-loop system is stable, condition (3.21) is satisfied for arbitrary operators if and only if it is satisfied for all operators for which the closed-loop system is stable.

Corollary 4.5. *Assume there exist operators (A,B,C,D,E,F) such that the closed-loop system is strongly stable and let $\alpha \geq 0$. A controller $(\mathscr{G}_1,\mathscr{G}_2,K)$ is conditionally robust if and only if the implication*

$$\Sigma' S = A_e'\Sigma' + B_e' \quad \Rightarrow \quad C_e'\Sigma' + D_e' = 0$$

is satisfied for all A_e', B_e', C_e', D_e' for which the closed-loop system is strongly stable, and for all $\Sigma' \in \mathscr{L}(W_\alpha, X_e)$ satisfying $\Sigma'(W_{\alpha+1}) \subset \mathscr{D}(A_e')$.

Proof. It is clearly enough to show the "if"-part of the statement. We will first show that if the implication is satisfied, then the controller has internal model structure. This can be seen directly from the first part of the proof of Theorem 4.4, if we choose the operators A, B, C, and D in such a way that the closed-loop system is strongly stable. Since the operators E and F do not appear in the operator A_e, they do not affect the stability of the closed-loop system. Because of this, they can be chosen as before. The modified proof now implies that the controller has internal model structure. Finally, applying Theorem 4.4 concludes that the controller is conditionally robust. □

Using the above corollary we can prove Theorem 3.12 presented at the end of the previous chapter. The statement of this theorem is that a controller stabilizing the closed-loop system solves the robust output regulation problem on W_α if and only if it is conditionally robust.

Proof of Theorem 3.12. Assume that the controller is conditionally robust. This implies that the solution Σ of the Sylvester equation (3.22) satisfies $C_e\Sigma + D_e = 0$ on W_α. Since the closed-loop system is strongly stable, we have from Theorem 3.2 that the controller solves the output regulation problem on W_α and thus the regulation error $e(t)$ goes to zero as $t \to \infty$ for all initial states $x_{e0} \in X_e$ and $v_0 \in W_\alpha$.

Moreover, if the parameters (A,B,C,D,E,F) are perturbed in such a way that the strong stability of the closed-loop system is preserved and the perturbed Sylvester equation (3.19) has a solution, for the same reasons as above the controller solves the output regulation problem for these perturbed parameters. Because of this, the corresponding regulation error $e(t)$ decays to zero asymptotically for all initial states $x_{e0} \in X_e$ and $v_0 \in W_\alpha$. This concludes that the controller solves the robust output regulation problem.

Assume now that the controller solves the robust output regulation problem. If the operators (A',B',C',D',E',F') are such that the corresponding closed-loop system is strongly stable and the Sylvester equation

$$\Sigma' S = A'_e\Sigma' + B'_e$$

on $W_{\alpha+1}$ has a solution $\Sigma' \in \mathscr{L}(W_\alpha, X_e)$, then the regulation error $e(t)$ goes to zero as $t \to \infty$ for all initial states $x_{e0} \in X_e$ and $v_0 \in W_\alpha$. In other words, the controller solves the output regulation problem on W_α for these parameters. We thus have from Theorem 3.2 that $C'_e\Sigma' + D'_e = 0$ on W_α. Since the perturbations of the parameters were arbitrary, this concludes that the condition (3.21) is satisfied for all operators A'_e, B'_e, C'_e, D'_e for which the closed-loop system is stable. This and Corollary 4.5 conclude that the controller is conditionally robust. □

4.2 The $\mathscr{G}$-Conditions

In this section we study the relationship between the internal model structure considered in the previous section and the $\mathscr{G}$-conditions introduced by Hämäläinen and Pohjolainen [15, 46]. As discussed in the beginning of this chapter, the results presented here show that this concept can again be considered as an alternative way of defining an internal model for infinite-dimensional controllers.

We begin by stating the formal definition of the $\mathscr{G}$-conditions. They have been studied earlier only in the cases where the exosystem has been either finite-dimensional, or the system operator S has been diagonal. Using again the special structure of our exosystem, we can easily generalize the definition to the infinite-dimensional exosystem considered in this thesis.

Definition 4.6 (The $\mathscr{G}$-conditions)**.** A controller $(\mathscr{G}_1, \mathscr{G}_2, K)$ is said to satisfy the *$\mathscr{G}$-conditions* related to the infinite-dimensional exosystem in Definition 2.1 if

$$\mathscr{R}(i\omega_k I - \mathscr{G}_1) \cap \mathscr{R}(\mathscr{G}_2) = \{0\} \qquad \forall k \in \mathbb{Z}, \tag{4.2a}$$

$$\mathscr{N}(\mathscr{G}_2) = \{0\}, \tag{4.2b}$$

and

$$\mathscr{N}(i\omega_k I - \mathscr{G}_1)^{d_k - 1} \subset \mathscr{R}(i\omega_k I - \mathscr{G}_1) \qquad \forall k \in \mathbb{Z}. \tag{4.2c}$$

■

This variant of the internal model is expressed as a set of conditions involving the ranges and kernels of the operators $\mathscr{G}_1$ and $\mathscr{G}_2$ of the controller. The contribution of the exosystem to the definition are the eigenvalues $i\omega_k$ of its system operator S and the dimensions d_k of the largest Jordan blocks associated to them. It is immediately clear that these conditions are much more concrete than the concept of the internal model structure, and that they are also easier to verify for actual controllers. The reason behind this contrast is that in defining the $\mathscr{G}$-conditions we have used the particular form of our signal generator, whereas the internal model structure remains valid even for a general operator S generating a C_0-group on a Hilbert space W. For us, however, this generality of the internal model structure is of no use, since we are only interested in tracking the types of reference signals considered in Chapter 2.

On the other hand, the $\mathscr{G}$-conditions also differ greatly from the definition of the p-copy internal model. If we again compare the different aspects of these two concepts, we can see that the definition of the p-copy internal model is still much easier to grasp than the $\mathscr{G}$-conditions. In particular, the fundamental property of the internal model — the fact that the controller is able to reproduce the dynamics of the exosystem — is not as clearly visible from the statement of Definition 4.6 as it is from the definition of the p-copy internal model. However, when it comes to the ease of verification of the conditions, finding the eigenvalues and the associated generalized eigenvectors of an unbounded operator can be a complicated task. Therefore, the conditions in Definition 4.6 can sometimes be easier to verify than the seemingly simpler p-copy internal model. In fact, we will see this kind of situation in the next chapter when designing a controller to solve the robust output regulation problem. Furthermore, the definition of the p-copy internal model uses the dimension of the output space and becomes ambiguous if this space is infinite-dimensional. An infinite-dimensional output space, however, poses no problem for using the $\mathscr{G}$-conditions. It is demonstrated in [15] that this definition can be successfully applied even if Y is a general Hilbert space.

The following theorem is the main result of this section. It states that under suitable assumptions the controller $(\mathscr{G}_1, \mathscr{G}_2, K)$ has internal model structure if and only if it satisfies the $\mathscr{G}$-conditions. We will also show in Lemma 4.12 at the end of the section that the condition appearing in the theorem can be substituted with an assumption on the disjointness of the spectra of A_e and S.

Theorem 4.7. *Assume the controller $(\mathscr{G}_1, \mathscr{G}_2, K)$ satisfies $Z = \mathscr{R}(i\omega_k I - \mathscr{G}_1) + \mathscr{R}(\mathscr{G}_2)$ for all $k \in \mathbb{Z}$. Then it has internal model structure if and only if it satisfies the $\mathscr{G}$-conditions.*

The theorem is proved in a series of lemmas. Lemmas 4.8, 4.9, and 4.10 show that the internal model structure of the controller implies that it satisfies the $\mathscr{G}$-conditions. Lemma 4.11 concludes that also the converse holds. It is also worthwhile to note that the assumption of the theorem is required only in showing that the internal model structure implies the condition (4.2c). This means that even without this assumption the $\mathscr{G}$-conditions still imply that the controller has internal model structure, and that the condition can be omitted completely if the exosystem is diagonal.

Lemma 4.8. *If the controller $(\mathscr{G}_1, \mathscr{G}_2, K)$ has internal model structure, then* (4.2a) *is satisfied.*

Proof. Let $k \in \mathbb{Z}$ and $w \in \mathscr{R}(i\omega_k I - \mathscr{G}_1) \cap \mathscr{R}(\mathscr{G}_2)$. Then there exist $z \in \mathscr{D}(\mathscr{G}_1)$ and $y \in Y$ such that

$$w = (i\omega_k I - \mathscr{G}_1)z = \mathscr{G}_2 y.$$

Choose

$$\Gamma = \langle \cdot, \phi_k^{n_k} \rangle z \in \mathscr{L}(W, Z) \qquad \text{and} \qquad \Delta = \langle \cdot, \phi_k^{n_k} \rangle y \in \mathscr{L}(W, Y).$$

Now $\mathscr{R}(\Gamma) \subset \mathscr{D}(\mathscr{G}_1)$ and for any $v \in \mathscr{D}(S)$ we have

$$\begin{aligned}(\Gamma S - \mathscr{G}_1 \Gamma)v &= \langle Sv, \phi_k^{n_k} \rangle z - \langle v, \phi_k^{n_k} \rangle \mathscr{G}_1 z = \langle S_k v, \phi_k^{n_k} \rangle z - \langle v, \phi_k^{n_k} \rangle \mathscr{G}_1 z \\ &= \left\langle i\omega_k \langle v, \phi_k^1 \rangle \phi_k^1 + \sum_{l=2}^{n_k} \langle v, \phi_k^l \rangle \left(i\omega_k \phi_k^l + \phi_k^{l-1} \right), \phi_k^{n_k} \right\rangle z - \langle v, \phi_k^{n_k} \rangle \mathscr{G}_1 z \\ &= \langle v, \phi_k^{n_k} \rangle (i\omega_k I - \mathscr{G}_1) z = \langle v, \phi_k^{n_k} \rangle \mathscr{G}_2 y = \mathscr{G}_2 \left(\langle v, \phi_k^{n_k} \rangle y \right) = \mathscr{G}_2 \Delta v.\end{aligned}$$

Thus we have $\Gamma S = \mathscr{G}_1 \Gamma + \mathscr{G}_2 \Delta$ on $\mathscr{D}(S)$, and the fact that the controller has internal model structure implies $\Delta = 0$. This immediately concludes that

$$0 = \Delta \phi_k^{n_k} = \langle \phi_k^{n_k}, \phi_k^{n_k} \rangle y = y$$

and $w = \mathscr{G}_2 y = 0$. $\square$

Lemma 4.9. *If the controller* $(\mathcal{G}_1, \mathcal{G}_2, K)$ *has internal model structure, then* (4.2b) *is satisfied.*

Proof. Let $y \in \mathcal{N}(\mathcal{G}_2)$ and let $\phi \in \mathcal{D}(S)$ be such that $\|\phi\| = 1$. Choose

$$\Gamma = 0 \in \mathcal{L}(W, Z) \qquad \text{and} \qquad \Delta = \langle \cdot, \phi \rangle y.$$

Now $\mathcal{R}(\Gamma) = \{0\} \subset \mathcal{D}(\mathcal{G}_1)$, and for all $v \in \mathcal{D}(S)$ we have $\Gamma S v = 0$ and

$$\mathcal{G}_1 \Gamma v + \mathcal{G}_2 \Delta v = 0 + \langle v, \phi \rangle \mathcal{G}_2 y = 0.$$

Thus we have $\Gamma S = \mathcal{G}_1 \Gamma + \mathcal{G}_2 \Delta$ on $\mathcal{D}(S)$ and the fact that the controller has internal model structure implies $\Delta = 0$. This further concludes that $0 = \Delta \phi = \langle \phi, \phi \rangle y = y$. □

The assumption that $Z = \mathcal{R}(i\omega_k I - \mathcal{G}_1) + \mathcal{R}(\mathcal{G}_2)$ for all $k \in \mathbb{Z}$ is required to show that the internal model structure implies the last one of the $\mathcal{G}$-conditions.

Lemma 4.10. *If* $Z = \mathcal{R}(i\omega_k I - \mathcal{G}_1) + \mathcal{R}(\mathcal{G}_2)$ *for all* $k \in \mathbb{Z}$*, and if the controller* $(\mathcal{G}_1, \mathcal{G}_2, K)$ *has internal model structure, then* (4.2c) *is satisfied.*

Proof. Since

$$d_k = \max \left\{ n_j \;\middle|\; j \in \mathbb{Z},\ \omega_j = \omega_k \right\},$$

it is sufficient to show that for all $k \in \mathbb{Z}$ we have

$$\mathcal{N}(i\omega_k I - \mathcal{G}_1)^{n_k - 1} \subset \mathcal{R}(i\omega_k I - \mathcal{G}_1).$$

Let $k \in \mathbb{Z}$ and $z \in \mathcal{N}(i\omega_k I - \mathcal{G}_1)^{n_k - 1}$. Since $Z = \mathcal{R}(i\omega_k I - \mathcal{G}_1) + \mathcal{R}(\mathcal{G}_2)$, there exist $z_1 \in \mathcal{D}(\mathcal{G}_1)$ and $y \in Y$ such that

$$z = (i\omega_k I - \mathcal{G}_1) z_1 + \mathcal{G}_2 y. \tag{4.3}$$

To prove the claim it is now sufficient to show that $y = 0$. Choose $\Gamma \in \mathcal{L}(W, Z)$ and $\Delta \in \mathcal{L}(W, Y)$ such that $\Delta = (-1)^{n_k} \langle \cdot, \phi_k^{n_k} \rangle y$ and

$$\Gamma = \left(\sum_{l=1}^{n_k - 1} (-1)^{l-1} \langle \cdot, \phi_k^l \rangle (i\omega_k I - \mathcal{G}_1)^{n_k - 1 - l} z \right) + (-1)^{n_k - 1} \langle \cdot, \phi_k^{n_k} \rangle z_1.$$

Since $z_1 \in \mathcal{D}(\mathcal{G}_1)$ and since $(i\omega_k I - \mathcal{G}_1)^l z \in \mathcal{D}(\mathcal{G}_1)$ for all $l \in \{0, \dots, n_k - 2\}$, we have $\mathcal{R}(\Gamma) \subset \mathcal{D}(\mathcal{G}_1)$. Since $\left\{ \phi_k^l \right\}_{l=1}^{n_k}$ are generalized eigenvectors of the operator S associated to the eigenvalue $i\omega_k$, they satisfy

$$S\phi_k^1 = i\omega_k \phi_k^1, \qquad S\phi_k^l = i\omega_k \phi_k^l + \phi_k^{l-1} \quad \forall l \in \{2, \dots, n_k\}.$$

Using this we see that for our choices of the operators Γ and Δ we have

$$(\Gamma S - \mathscr{G}_1\Gamma)\phi_k^1 = (i\omega_k I - \mathscr{G}_1)\Gamma\phi_k^1 = (i\omega_k I - \mathscr{G}_1)^{n_k-1}z = 0 = \mathscr{G}_2\Delta\phi_k^1,$$

$$\begin{aligned}(\Gamma S - \mathscr{G}_1\Gamma)\phi_k^l = (i\omega_k I - \mathscr{G}_1)\Gamma\phi_k^l + \Gamma\phi_k^{l-1} &= (-1)^{l-1}(i\omega_k I - \mathscr{G}_1)(i\omega_k I - \mathscr{G}_1)^{n_k-1-l}z \\ &+ (-1)^{l-2}(i\omega_k I - \mathscr{G}_1)^{n_k-1-(l-1)}z = 0 = \mathscr{G}_2\Delta\phi_k^l,\end{aligned}$$

for $l \in \{2, \ldots, n_k - 1\}$, and finally using (4.3) we obtain

$$\begin{aligned}(\Gamma S - \mathscr{G}_1\Gamma)\phi_k^{n_k} &= (i\omega_k I - \mathscr{G}_1)\Gamma\phi_k^{n_k} + \Gamma\phi_k^{n_k-1} = (-1)^{n_k-1}(i\omega_k I - \mathscr{G}_1)z_1 + (-1)^{n_k-2}z \\ &= (-1)^{n_k-1}\left((i\omega_k I - \mathscr{G}_1)z_1 - z\right) = (-1)^{n_k-1}\left(-\mathscr{G}_2 y\right) \\ &= \mathscr{G}_2\left((-1)^{n_k}\langle\phi_k^{n_k}, \phi_k^{n_k}\rangle y\right) = \mathscr{G}_2\Delta\phi_k^{n_k}.\end{aligned}$$

This concludes that $\Gamma S v = \mathscr{G}_1\Gamma v + \mathscr{G}_2\Delta v$ for all $v \in \operatorname{span}\left\{\phi_k^l\right\}_{l=1}^{n_k}$. However, since clearly $\Gamma\phi_j^l = 0$ and $\Delta\phi_j^l = 0$ for all $j \neq k$ and $l \in \{1, \ldots, n_j\}$, we have that for all $v \in \mathscr{D}(S)$

$$\Gamma S v = \Gamma P_k S v = \Gamma S P_k v = \mathscr{G}_1\Gamma P_k v + \mathscr{G}_2\Delta P_k v = \mathscr{G}_1\Gamma v + \mathscr{G}_2\Delta v,$$

and thus $\Gamma S = \mathscr{G}_1\Gamma + \mathscr{G}_2\Delta$ on $\mathscr{D}(S)$. The fact that the controller has internal model structure implies $\Delta = 0$, and further that

$$0 = (-1)^{n_k}\Delta\phi_k^{n_k} = \langle\phi_k^{n_k}, \phi_k^{n_k}\rangle y = y.$$

Substituting this into equation (4.3) we obtain $z = (i\omega_k I - \mathscr{G}_1)z_1$, which concludes $z \in \mathscr{R}(i\omega_k I - \mathscr{G}_1)$. □

Finally, Lemma 4.11 shows that the $\mathscr{G}$-conditions imply that the controller has internal model structure.

Lemma 4.11. *If the controller $(\mathscr{G}_1, \mathscr{G}_2, K)$ satisfies the $\mathscr{G}$-conditions, then it has internal model structure.*

Proof. Let $\Gamma \in \mathscr{L}(W, Z)$ and $\Delta \in \mathscr{L}(W, Y)$ be such that $\Gamma(\mathscr{D}(S)) \subset \mathscr{D}(\mathscr{G}_1)$ and

$$\Gamma S = \mathscr{G}_1\Gamma + \mathscr{G}_2\Delta. \tag{4.4}$$

Let $k \in \mathbb{Z}$ be arbitrary. We will show that $\Delta\phi_k^l = 0$ for all $l \in \{1, \ldots, n_k\}$. Applying both sides of the Sylvester equation (4.4) to ϕ_k^1 we obtain

$$(i\omega_k I - \mathscr{G}_1)\Gamma\phi_k^1 = \mathscr{G}_2\Delta\phi_k^1.$$

Now conditions (4.2a) and (4.2b) imply that $\Delta\phi_k^1 = 0$ and $(i\omega_k I - \mathscr{G}_1)\Gamma\phi_k^1 = 0$.

If $n_k \geq 2$, we also need to consider $\Delta\phi_k^2$. Using condition (4.2c) we see that

$$\Gamma\phi_k^1 \in \mathcal{N}(i\omega_k I - \mathcal{G}_1) \subset \mathcal{N}(i\omega_k I - \mathcal{G}_1)^{d_k-1} \subset \mathcal{R}(i\omega_k I - \mathcal{G}_1).$$

Applying both sides of (4.4) to ϕ_k^2 we obtain

$$(i\omega_k I - \mathcal{G}_1)\Gamma\phi_k^2 + \Gamma\phi_k^1 = \mathcal{G}_2\Delta\phi_k^2.$$

Since $\Gamma\phi_k^1 \in \mathcal{R}(i\omega_k I - \mathcal{G}_1)$, conditions (4.2a) and (4.2b) imply $\Delta\phi_k^2 = 0$ and

$$(i\omega_k I - \mathcal{G}_1)\Gamma\phi_k^2 + \Gamma\phi_k^1 = 0.$$

If $n_k \geq 3$, we proceed to considering $\Delta\phi_k^3$. Since $\Gamma\phi_k^1 \in \mathcal{D}(\mathcal{G}_1)$, the above equation implies $\Gamma\phi_k^2 \in \mathcal{D}(i\omega_k I - \mathcal{G}_1)^2$. Applying $(i\omega_k I - \mathcal{G}_1)$ to both sides of this equation and using $\Gamma\phi_k^1 \in \mathcal{N}(i\omega_k I - \mathcal{G}_1)$ also shows us that $(i\omega_k I - \mathcal{G}_1)^2\Gamma\phi_k^2 = 0$. Condition (4.2c) further implies

$$\Gamma\phi_k^2 \in \mathcal{N}(i\omega_k I - \mathcal{G}_1)^2 \subset \mathcal{N}(i\omega_k I - \mathcal{G}_1)^{d_k-1} \subset \mathcal{R}(i\omega_k I - \mathcal{G}_1).$$

Again, applying both sides of (4.4) to ϕ_k^3 we obtain

$$(i\omega_k I - \mathcal{G}_1)\Gamma\phi_k^3 + \Gamma\phi_k^2 = \mathcal{G}_2\Delta\phi_k^3.$$

Since $\Gamma\phi_k^2 \in \mathcal{R}(i\omega_k I - \mathcal{G}_1)$, conditions (4.2a) and (4.2b) imply that $\Delta\phi_k^3 = 0$ and $(i\omega_k I - \mathcal{G}_1)\Gamma\phi_k^3 + \Gamma\phi_k^2 = 0$.

By repeating the above steps as many times as necessary we can show that $\Delta\phi_k^l = 0$ and

$$\Gamma\phi_k^l \in \mathcal{N}(i\omega_k I - \mathcal{G}_1)^l \subset \mathcal{N}(i\omega_k I - \mathcal{G}_1)^{d_k-1} \subset \mathcal{R}(i\omega_k I - \mathcal{G}_1)$$

for all $l \in \{1, \dots, n_k - 1\}$. Finally, applying both sides of (4.4) to $\phi_k^{n_k}$ we obtain

$$(i\omega_k I - \mathcal{G}_1)\Gamma\phi_k^{n_k} + \Gamma\phi_k^{n_k-1} = \mathcal{G}_2\Delta\phi_k^{n_k},$$

and conditions (4.2a) and (4.2b) imply $\Delta\phi_k^{n_k} = 0$. Since $k \in \mathbb{Z}$ was arbitrary, we have shown that $\Delta\phi_k^l = 0$ for all $k \in \mathbb{Z}$ and $l \in \{1, \dots, n_k\}$. Since $\{\phi_k^l\}$ is a basis of W, we must have $\Delta = 0$. This concludes that the controller has internal model structure. □

The above lemma completes the proof of Theorem 4.7. We conclude this section by presenting a sufficient condition for the assumption appearing in the theorem. In finite-dimensional control theory the condition

$$\mathcal{R}(i\omega_k I - \mathcal{G}_1) + \mathcal{R}(\mathcal{G}_2) = Z, \qquad \forall k$$

means that all the modes of the exosystem in the system operator $\mathcal{G}_1$ of the controller are controllable by $\mathcal{G}_2$. Since these modes are unstable, it can be shown that in the finite-dimensional case this condition must necessarily be satisfied if the the closed-loop system can be stabilized. For infinite-dimensional systems the situation is in general more complicated, but it is shown below that if the closed-loop system can be stabilized in such a way that the spectra of the closed-loop system and the exosystem are disjoint, then this condition is satisfied.

Lemma 4.12. *If $\sigma(A_e) \cap \sigma_p(S) = \varnothing$, then $\mathcal{R}(i\omega_k I - \mathcal{G}_1) + \mathcal{R}(\mathcal{G}_2) = Z$ for all $k \in \mathbb{Z}$.*

Proof. Let $k \in \mathbb{Z}$ and $z \in Z$. We need to show that there exist $z_0 \in \mathcal{D}(\mathcal{G}_1)$ and $y \in Y$ such that

$$z = (i\omega_k I - \mathcal{G}_1)z_0 + \mathcal{G}_2 y.$$

Since $\sigma(A_e) \cap \sigma_p(S) = \varnothing$, we have $i\omega_k \in \rho(A_e)$ and the operator $i\omega_k I - A_e$ is surjective. Thus there exist $x_1 \in \mathcal{D}(A)$ and $z_1 \in \mathcal{D}(\mathcal{G}_1)$ such that

$$\begin{pmatrix} 0 \\ z \end{pmatrix} = (i\omega_k I - A_e) \begin{pmatrix} x_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} (i\omega_k I - A)x_1 - BKz_1 \\ -\mathcal{G}_2 C x_1 + (i\omega_k I - \mathcal{G}_1)z_1 - \mathcal{G}_2 DKz_1 \end{pmatrix}.$$

The second equation shows that $z = (i\omega_k I - \mathcal{G}_1)z_1 + \mathcal{G}_2(-Cx_1 - DKz_1)$, and thus we can choose $z_0 = z_1 \in \mathcal{D}(\mathcal{G}_1)$ and $y = -Cx_1 - DKz_1 \in Y$. □

4.3 The p-Copy Internal Model

In this section we will finally complete the proof of Theorem 4.2 generalizing the p-copy internal model principle to distributed parameter systems with infinite-dimensional exosystems. We will do this by first showing that the p-copy internal model is equivalent to the $\mathcal{G}$-conditions considered in the previous section. We can then collect the results presented earlier in this chapter to prove Theorem 4.2.

The following theorem establishing the equivalence of the $\mathcal{G}$-conditions and the p-copy internal model is again proved using a series of lemmas. These independent results also show that parts of the theorem hold even under weaker conditions. In particular the $\mathcal{G}$-conditions — and thus also conditional robustness of the controller — imply that the controller incorporates a p-copy internal model even if the output space Y is infinite-dimensional. On the other hand, for infinite-dimensional output spaces the lack of the converse implication again suggests what we already discussed in the previous section: The $\mathcal{G}$-conditions are a more suitable choice for the definition of an internal model when Y is infinite-dimensional.

Theorem 4.13. *Let $\sigma(A_e) \cap \sigma_p(S) = \varnothing$ and $\dim Y < \infty$. A controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ incorporates a p-copy internal model of the exosystem if and only if it satisfies the $\mathcal{G}$-conditions.*

The proof of this theorem repeatedly uses an interesting result stating that if either the $\mathcal{G}$-conditions are satisfied or if the controller incorporates a p-copy internal model of the exosystem, then under our assumption for any $k \in \mathbb{Z}$ the operator $P(i\omega_k)K$ restricted to the eigenspace $\mathcal{N}(i\omega_k I - \mathcal{G}_1)$ is an isomorphism between the eigenspace and the output space. This property is first of all used to establishes the fact that if the controller satisfies the $\mathcal{G}$-conditions, then every eigenvalue $i\omega_k$ of the exosystem is an eigenvalue of $\mathcal{G}_1$ with a geometric multiplicity equal to the dimension of the output space. Furthermore, it is also an essential part of the proof of the converse implication. The following lemma shows that under the assumption on the spectra of A_e and S, this operator is injective. The result is used in the proofs of Lemmas 4.15–4.18.

Lemma 4.14. *If $\sigma_p(A_e) \cap \sigma_p(S) = \varnothing$, then the operator $(P(i\omega_k)K)|_{\mathcal{N}(i\omega_k I - \mathcal{G}_1)}$ is injective for every $k \in \mathbb{Z}$.*

Proof. Let $k \in \mathbb{Z}$, and let $z \in \mathcal{N}(i\omega_k I - \mathcal{G}_1)$ be such that $P(i\omega_k)Kz = 0$. Since A is the system operator of the plant, the standing assumptions made in Section 3.1 require that it satisfies $\sigma(A) \cap \sigma_p(S) = \varnothing$. We can therefore choose $x = R(i\omega_k, A)BKz \in \mathcal{D}(A)$. Now

$$(i\omega_k I - A_e)\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} (i\omega_k I - A)x - BKz \\ -\mathcal{G}_2 Cx + (i\omega_k I - \mathcal{G}_1)z - \mathcal{G}_2 DKz \end{pmatrix}$$

$$= \begin{pmatrix} BKz - BKz \\ -\mathcal{G}_2(CR(i\omega_k, A)B + D)Kz + (i\omega_k I - \mathcal{G}_1)z \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathcal{G}_2 P(i\omega_k)Kz \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $i\omega_k \in \sigma_p(S)$, we know that $i\omega_k \notin \sigma_p(A_e)$ and thus $i\omega_k I - A_e$ is injective. This implies that $z = 0$, which further concludes that the restriction of $P(i\omega_k)K$ to $\mathcal{N}(i\omega_k I - \mathcal{G}_1)$ is an injection. □

The following lemma states that if the $\mathcal{G}$-conditions are satisfied, then for all $k \in \mathbb{Z}$ the space $\mathcal{N}(i\omega_k I - \mathcal{G}_1)$ is isomorphic to Y, and $\mathcal{G}_1$ has $\dim Y$ independent Jordan chains of length greater than or equal to d_k associated to the eigenvalue $i\omega_k$. This proves one of the implications in Theorem 4.13, but the result also holds for infinite-dimensional output space Y.

Lemma 4.15. *If $\sigma(A_e) \cap \sigma_p(S) = \varnothing$ and the controller satisfies the $\mathcal{G}$-conditions, then for all $k \in \mathbb{Z}$ the operator $(P(i\omega_k)K)|_{\mathcal{N}(i\omega_k I - \mathcal{G}_1)}$ is an isomorphism between $\mathcal{N}(i\omega_k I - \mathcal{G}_1)$ and Y, and $\mathcal{G}_1$ has $\dim Y$ independent Jordan chains of length greater than or equal to d_k associated to the eigenvalue $i\omega_k$.*

Proof. Let $k \in \mathbb{Z}$. We have from Lemma 4.14 that the operator $(P(i\omega_k)K)|_{\mathcal{N}(i\omega_k I - \mathcal{G}_1)}$ is injective, and thus to prove the first claim it is sufficient to show that it is also surjective.

Since $\sigma(A_e) \cap \sigma_p(S) = \varnothing$, we have $i\omega_k \in \rho(A_e)$ and the operator $i\omega_k I - A_e$ is surjective. This implies that for any $z \in Z$ there exist $x_1 \in \mathcal{D}(A)$ and $z_1 \in \mathcal{D}(\mathcal{G}_1)$ such that

$$\begin{pmatrix} 0 \\ z \end{pmatrix} = (i\omega_k I - A_e) \begin{pmatrix} x_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} (i\omega_k I - A)x_1 - BKz_1 \\ -\mathcal{G}_2 C x_1 + (i\omega_k I - \mathcal{G}_1) z_1 - \mathcal{G}_2 DK z_1 \end{pmatrix}.$$

Since A is the system operator of the plant, we have $\sigma(A) \cap \sigma_p(S) = \varnothing$ by the standing assumptions made in Section 3.1. We therefore have $i\omega_k \in \rho(A)$, and the first line of the above equation implies $x_1 = R(i\omega_k, A)BKz_1$. Substituting x_1 into the second equation we obtain

$$z = -\mathcal{G}_2 CR(i\omega_k, A)BKz_1 + (i\omega_k I - \mathcal{G}_1)z_1 - \mathcal{G}_2 DKz_1 = (i\omega_k I - \mathcal{G}_1)z_1 - \mathcal{G}_2 P(i\omega_k)Kz_1. \quad (4.5)$$

Let $y \in Y$ be arbitrary. Then $z = -\mathcal{G}_2 y \in \mathcal{R}(\mathcal{G}_2) \subset Z$ and we can choose $z_1 \in \mathcal{D}(\mathcal{G}_1)$ in such a way that (4.5) is satisfied. The first two $\mathcal{G}$-conditions,

$$\mathcal{R}(i\omega_k I - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\} \qquad \text{and} \qquad \mathcal{N}(\mathcal{G}_2) = \{0\},$$

can now be used to show that

$$\begin{aligned} & -\mathcal{G}_2 y = (i\omega_k I - \mathcal{G}_1)z_1 - \mathcal{G}_2 P(i\omega_k)Kz_1 \\ \Leftrightarrow \quad & \underbrace{-\mathcal{G}_2 y + \mathcal{G}_2 P(i\omega_k)Kz_1}_{\in \mathcal{R}(\mathcal{G}_2)} = \underbrace{(i\omega_k I - \mathcal{G}_1)z_1}_{\in \mathcal{R}(i\omega_k I - \mathcal{G}_1)} \\ \Leftrightarrow \quad & \begin{cases} \mathcal{G}_2 y = \mathcal{G}_2 P(i\omega_k)Kz_1 \\ 0 = (i\omega_k I - \mathcal{G}_1)z_1 \end{cases} \\ \Leftrightarrow \quad & \begin{cases} y = P(i\omega_k)Kz_1 \\ 0 = (i\omega_k I - \mathcal{G}_1)z_1 \end{cases} \end{aligned}$$

Since $y \in Y$ was arbitrary, the above derivation shows that for any such element there exists $z_1 \in \mathcal{N}(i\omega_k I - \mathcal{G}_1)$ such that $y = P(i\omega_k)Kz_1$. This concludes that the restriction of the operator $P(i\omega_k)K$ to $\mathcal{N}(i\omega_k I - \mathcal{G}_1)$ is surjective.

The fact that the operator $(P(i\omega_k)K)|_{\mathcal{N}(i\omega_k I - \mathcal{G}_1)}$ is an isomorphism also establishes $\dim \mathcal{N}(i\omega_k I - \mathcal{G}_1) = \dim Y$. Since the Jordan chains related to linearly independent eigenvectors are independent, it remains to show that there exists a Jordan chain of length greater than or equal to d_k related to every element of $\mathcal{N}(i\omega_k I - \mathcal{G}_1)$.

We can assume $d_k \geq 2$, since otherwise the proof is complete. Since for all $l \in \mathbb{N}$ we have $\mathcal{N}(i\omega_k I - \mathcal{G}_1)^l \subset \mathcal{N}(i\omega_k I - \mathcal{G}_1)^{l+1}$, condition (4.2c) implies

$$\mathcal{N}(i\omega_k I - \mathcal{G}_1) \subset \mathcal{N}(i\omega_k I - \mathcal{G}_1)^2 \subset \cdots \subset \mathcal{N}(i\omega_k I - \mathcal{G}_1)^{d_k - 1} \subset \mathcal{R}(i\omega_k I - \mathcal{G}_1). \quad (4.6)$$

Let $\psi_1 \in \mathcal{N}(i\omega_k I - \mathcal{G}_1)$ be arbitrary and define a sequence $(\psi_l)_{l=2}^{d_k}$ recursively as follows:

Let $l \in \{1, \dots, d_k - 1\}$. Assume $\psi_l \in \mathcal{N}(i\omega_k I - \mathcal{G}_1)^l$. We have from (4.6) that there exists $\psi_{l+1} \in \mathcal{D}(\mathcal{G}_1)$ such that

$$(\mathcal{G}_1 - i\omega_k I)\psi_{l+1} = \psi_l \in \mathcal{N}(i\omega_k I - \mathcal{G}_1)^l \subset \mathcal{D}(i\omega_k I - \mathcal{G}_1)^l.$$

We thus have $\psi_{l+1} \in \mathcal{D}(i\omega_k I - \mathcal{G}_1)^{l+1}$ and

$$(\mathcal{G}_1 - i\omega_k I)^{l+1}\psi_{l+1} = (\mathcal{G}_1 - i\omega_k I)^l \psi_l = 0.$$

This implies that $\psi_{l+1} \in \mathcal{N}(i\omega_k I - \mathcal{G}_1)^{l+1}$.

The sequence $(\psi_l)_{l=1}^{d_k}$ obtained this way has the properties that $(i\omega_k I - \mathcal{G}_1)\psi_1 = 0$ and $(\mathcal{G}_1 - i\omega_k I)\psi_l = \psi_{l-1}$ for every $l \in \{2, \dots, d_k\}$. Thus by possibly adding elements to this set we obtain a Jordan chain $(\psi_l)_{l=1}^m$ with length $m \geq d_k$. □

In the previous lemma we saw that the $\mathcal{G}$-conditions actually imply that the operator $\mathcal{G}_1$ has *precisely* $\dim Y$ independent Jordan chains. The fact that a larger number is not possible follows from our assumption $\sigma_p(A_e) \cap \sigma_p(S) = \emptyset$, as is shown in the next lemma.

Lemma 4.16. *If* $\sigma_p(A_e) \cap \sigma_p(S) = \emptyset$, *then* $\dim \mathcal{N}(i\omega_k I - \mathcal{G}_1) \leq \dim Y$ *for all* $k \in \mathbb{Z}$.

Proof. Let $k \in \mathbb{Z}$. We have from Lemma 4.14 that the operator

$$(P(i\omega_k)K)|_{\mathcal{N}(i\omega_k I - \mathcal{G}_1)} \in \mathcal{L}(\mathcal{N}(i\omega_k I - \mathcal{G}_1), Y)$$

is injective. Using the rank-nullity theorem [33, Thm. 4.7.7] we can conclude that

$$\begin{aligned}\dim \mathcal{N}(i\omega_k I - \mathcal{G}_1) &= \dim \mathcal{R}\left((P(i\omega_k)K)|_{\mathcal{N}(i\omega_k I - \mathcal{G}_1)}\right) + \dim \mathcal{N}\left((P(i\omega_k)K)|_{\mathcal{N}(i\omega_k I - \mathcal{G}_1)}\right) \\ &= \dim \mathcal{R}\left((P(i\omega_k)K)|_{\mathcal{N}(i\omega_k I - \mathcal{G}_1)}\right) \leq \dim Y.\end{aligned}$$

□

The following three lemmas show that if $\dim Y < \infty$ and if the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ incorporates a p-copy internal model of the exosystem, then this controller satisfies the $\mathcal{G}$-conditions. For this it is sufficient to assume $\sigma_p(A_e) \cap \sigma_p(S) = \emptyset$. This condition is satisfied whenever the operator A_e generates a strongly stable C_0-semigroup, since in this case we necessarily have $\sigma_p(A_e) \subset \mathbb{C}^-$ [18].

Lemma 4.17. *If* $\sigma_p(A_e) \cap \sigma_p(S) = \emptyset$, *if* $\dim Y < \infty$, *and if the controller* $(\mathcal{G}_1, \mathcal{G}_2, K)$ *incorporates a p-copy internal model of the exosystem, then* $\mathcal{R}(i\omega_k I - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2) = \{0\}$ *for all* $k \in \mathbb{Z}$.

Proof. Let $k \in \mathbb{Z}$. The p-copy internal model and Lemma 4.16 imply that we have $\dim \mathcal{N}(i\omega_k I - \mathcal{G}_1) = \dim Y$. If we take $v \in \mathcal{R}(i\omega_k I - \mathcal{G}_1) \cap \mathcal{R}(\mathcal{G}_2)$, then there exist $y \in Y$ and $z \in \mathcal{D}(\mathcal{G}_1)$ such that $v = \mathcal{G}_2 y = (i\omega_k I - \mathcal{G}_1)z$. We will first show that there also exists $z_1 \in \mathcal{D}(\mathcal{G}_1)$ satisfying

$$v = \mathcal{G}_2 P(i\omega_k)Kz_1 = (i\omega_k I - \mathcal{G}_1)z_1.$$

We have from Lemma 4.14 that the operator $(P(i\omega_k)K)|_{\mathcal{N}(i\omega_k I - \mathcal{G}_1)}$ is injective, and $\dim \mathcal{N}(i\omega_k I - \mathcal{G}_1) = \dim Y$ further implies that it is invertible. Because of this we can choose $z_0 \in \mathcal{N}(i\omega_k I - \mathcal{G}_1)$ in such a way that

$$P(i\omega_k)Kz_0 = y - P(i\omega_k)Kz \in Y \qquad \Leftrightarrow \qquad y = P(i\omega_k)K(z + z_0).$$

We then have

$$\mathcal{G}_2 P(i\omega_k)K(z+z_0) = \mathcal{G}_2 y = v = (i\omega_k I - \mathcal{G}_1)z = (i\omega_k I - \mathcal{G}_1)(z+z_0),$$

and we can thus choose $z_1 = z + z_0$.

Since A is the system operator of the plant, we have $\sigma(A) \cap \sigma_p(S) = \varnothing$ by our standing assumptions. We can therefore choose $x_1 = R(i\omega_k, A)BKz_1 \in \mathcal{D}(A)$ and, similarly as in the proof of Lemma 4.14, we can see that

$$(i\omega_k I - A_e)\begin{pmatrix} x_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathcal{G}_2 P(i\omega_k)Kz_1 + (i\omega_k I - \mathcal{G}_1)z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $i\omega_k \in \sigma_p(S)$ and $\sigma_p(A_e) \cap \sigma_p(S) = \varnothing$, the operator $i\omega_k I - A_e$ is injective and we must have $z_1 = 0$. This concludes that $v = (i\omega_k I - \mathcal{G}_1)z_1 = 0$. □

Lemma 4.18. *If $\sigma_p(A_e) \cap \sigma_p(S) = \varnothing$, if $\dim Y < \infty$, and if the controller $(\mathcal{G}_1, \mathcal{G}_2, K)$ incorporates a p-copy internal model of the exosystem, then $\mathcal{N}(\mathcal{G}_2) = \{0\}$.*

Proof. Let $y \in \mathcal{N}(\mathcal{G}_2)$ and $k \in \mathbb{Z}$. The p-copy internal model and Lemma 4.16 imply that $\dim \mathcal{N}(i\omega_k I - \mathcal{G}_1) = \dim Y$. From Lemma 4.14 we have that $(P(i\omega_k)K)|_{\mathcal{N}(i\omega_k I - \mathcal{G}_1)}$ is injective, and the fact that $\dim \mathcal{N}(i\omega_k I - \mathcal{G}_1) = \dim Y$ further implies that it is invertible. This implies that there exists $z_1 \in \mathcal{N}(i\omega_k I - \mathcal{G}_1)$ such that $y = P(i\omega_k)Kz_1$, and further $\mathcal{G}_2 P(i\omega_k)Kz_1 = 0$. Since $\sigma(A) \cap \sigma_p(S) = \varnothing$ by our standing assumptions, we can choose $x_1 = R(i\omega_k, A)BKz_1 \in \mathcal{D}(A)$. As in the proof of Lemma 4.14 we see that

$$(i\omega_k I - A_e)\begin{pmatrix} x_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathcal{G}_2 P(i\omega_k)Kz_1 + (i\omega_k I - \mathcal{G}_1)z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $i\omega_k \in \sigma_p(S)$ and $\sigma_p(A_e) \cap \sigma_p(S) = \varnothing$, the operator $i\omega_k I - A_e$ is injective and we must have $z_1 = 0$. This also implies $y = P(i\omega_k)Kz_1 = 0$, and thus $\mathcal{N}(\mathcal{G}_2) = \{0\}$. □

Lemma 4.19. *If $\sigma_p(A_e)\cap\sigma_p(S)=\varnothing$, if $\dim Y<\infty$, and if the controller $(\mathcal{G}_1,\mathcal{G}_2,K)$ incorporates a p-copy internal model of the exosystem, then $\mathcal{N}(i\omega_k I-\mathcal{G}_1)^{d_k-1}\subset\mathcal{R}(i\omega_k I-\mathcal{G}_1)$ for all $k\in\mathbb{Z}$.*

Proof. Let $k\in\mathbb{Z}$ and denote $N=\dim Y$. The p-copy internal model and Lemma 4.16 imply that $\dim\mathcal{N}(sI-\mathcal{G}_1)=N$.

Since the controller incorporates a p-copy internal model of the exosystem, the operator $\mathcal{G}_1$ has N independent Jordan chains $\left(\psi_n^l\right)_{l=1}^{m_n}$ with $m_n\geq d_k$ associated to $i\omega_k$. Since the definition of a Jordan chain implies $\psi_n^k\in\mathcal{R}(i\omega_k I-\mathcal{G}_1)$ for all $n\in\{1,\dots,N\}$ and $k\in\{1,\dots,d_k-1\}$, it is sufficient to show that

$$\mathcal{N}(i\omega_k I-\mathcal{G}_1)^m\subset\operatorname{span}\left\{\,\psi_n^l\;\middle|\;n=1,\dots,N,\ l=1,\dots,m\,\right\}\tag{4.7}$$

for all $m\in\{1,\dots,d_k-1\}$. We will do this using induction. Since the set $\left\{\psi_n^1\right\}_{n=1}^N$ is linearly independent and since $\psi_n^1\in\mathcal{N}(i\omega_k I-\mathcal{G}_1)$ for all $n\in\{1,\dots,N\}$, we have

$$\mathcal{N}(i\omega_k I-\mathcal{G}_1)=\operatorname{span}\left\{\psi_n^1\right\}_{n=1}^N.\tag{4.8}$$

This concludes that (4.7) is satisfied for $m=1$.

Assume (4.7) is satisfied for $m=j\in\{1,\dots,d_k-2\}$ and let $z\in\mathcal{N}(i\omega_k I-\mathcal{G}_1)^{j+1}$. Then $z\in\mathcal{D}(\mathcal{G}_1)$ and $(i\omega_k I-\mathcal{G}_1)z\in\mathcal{N}(i\omega_k I-\mathcal{G}_1)^j$. Since we assumed that (4.7) is satisfied for $m=j$, there exist constants $\left\{\,\alpha_n^l\;\middle|\;n=1,\dots,N,\,l=1,\dots,j\,\right\}$ such that

$$(i\omega_k I-\mathcal{G}_1)z=\sum_{n=1}^N\sum_{l=1}^j\alpha_n^l\psi_n^l=\sum_{n=1}^N\sum_{l=1}^j\alpha_n^l(\mathcal{G}_1-i\omega_k I)\psi_n^{l+1},$$

where the second equality follows from the fact that $\left(\psi_n^l\right)_{l=1}^{m_n}$ are Jordan chains of $\mathcal{G}_1$. The above equation implies

$$(i\omega_k I-\mathcal{G}_1)\left(z+\sum_{n=1}^N\sum_{l=1}^j\alpha_n^l\psi_n^{l+1}\right)=0\qquad\Rightarrow\qquad z+\sum_{n=1}^N\sum_{l=1}^j\alpha_n^l\psi_n^{l+1}\in\mathcal{N}(i\omega_k I-\mathcal{G}_1).$$

We now have from (4.8) that there exist constants $\left\{\alpha_n^0\right\}_{n=1}^N$ such that

$$z+\sum_{n=1}^N\sum_{l=1}^j\alpha_n^l\psi_n^{l+1}=\sum_{n=1}^N\alpha_n^0\psi_n^1.$$

However, this immediately implies that (4.7) holds for $m=j+1$. By induction this concludes that (4.7) holds for all $m\in\{1,\dots,d_k-1\}$, and thus completes the proof. □

We are now in a position to use the previous lemmas to prove Theorem 4.7.

Proof of Theorem 4.7. Lemma 4.15 shows that if the controller $(\mathscr{G}_1, \mathscr{G}_2, K)$ satisfies the $\mathscr{G}$-conditions, then it incorporates a p-copy internal model of the exosystem. Lemmas 4.17, 4.18, and 4.19 conclude that also the converse is true. □

We will conclude the section by presenting the proof of the main result of this chapter, the generalization of the internal model principle for distributed parameter systems with infinite-dimensional exosystems.

Proof of Theorem 4.2. Theorem 4.4 states that a controller is conditionally robust if and only if it has internal model structure. Theorem 4.7 and Lemma 4.12 together imply that under our assumptions the controller has internal model structure if and only if it satisfies the $\mathscr{G}$-conditions. Finally, Theorem 4.13 states that under our assumptions the controller satisfies the $\mathscr{G}$-conditions if and only if it incorporates a p-copy internal model of the exosystem. □

This completes our study of the internal model principle. In the next chapter we turn to the design of controllers solving the robust output regulation problem. In particular this requires stabilizing the closed-loop system in such a way that the associated Sylvester equation has a solution.

Chapter 5

Controller Design for Robust Output Regulation

In this chapter we consider designing an error feedback controller to solve the robust output regulation problem on W_α for some $\alpha \geq 0$. Combining the results presented in Chapters 3 and 4 we can see that it suffices to choose a controller which

1. is conditionally robust,
2. strongly stabilizes the closed-loop system,
3. satisfies conditions of Lemma 3.10 for α.

We restrict our attention to a situation that is a special case in two regards. First of all, we assume our exosystem has at most finite number of nontrivial Jordan blocks and that the infinite part of its spectrum consists of simple and uniformly separated eigenvalues. As we saw in Section 2.2, this type of signal generator can be viewed as a composite exosystem consisting of a finite-dimensional part and an infinite-dimensional diagonal part. Therefore, the signals we can under our assumptions consider are in general of the form

$$y_{ref}(t) = y_{ap}(t) + y_n(t)t^n + \cdots + y_1(t)t,$$

where $y_{ap}(\cdot)$ is an almost periodic function and the functions $y_j(\cdot)$ for $j \in \{1, \ldots, n\}$ are linear combinations of trigonometric functions. This assumption on the structure of the signal generator is restrictive, but the generated signals still include the most important polynomially bounded functions considered in applications, and for example all the signals discussed in Section 2.1 can be generated with exosystems of this type. Moreover, we can also consider any signal of the above form where $y_{ap}(\cdot)$ is a continuous periodic function. Indeed, we saw in Section 2.2 that in order to generate a continuous τ-periodic function it is sufficient that the exosystem contains the simple and uniformly separated eigenvalues

$$i\omega_k = i\frac{2\pi k}{\tau}.$$

In this chapter we also only consider the single-input single-output case. This restriction is not essential to the existence of a controller solving the robust output regulation problem. In fact, similar methods are also applicable in the case of a finite-dimensional output space, and even for an inifinite-dimensional output space provided that we replace the strong stability type of the closed-loop system with weak stability [15, Sec. 7]. However, although the methods for these more general systems can be used to effectively stabilize the closed-loop system, they provide little information regarding the behavior of the resulting system relevant to the solvability of the Sylvester equation. We restrict our attention to the single-input single-output case, because in this situation it is possible to stabilize the closed-loop system using a technique that subsequently allows us to derive easily verifiable sufficient conditions for the assumptions of Lemma 3.10 to be satisfied.

Taking into account these simplifications the problem we are considering has the following form.

The Robust Output Regulation Problem on W_α. Let $\alpha \geq 0$. Find $(\mathscr{G}_1, \mathscr{G}_2, K)$ such that the following are satisfied:

1. The closed-loop system operator A_e generates a strongly stable C_0-semigroup on X_e, we have $\sigma(A_e) \cap \sigma(S) = \varnothing$ and
$$\sup_{k\in\mathbb{Z}} \, (1+\omega_k^2)^{-\alpha} \|R(i\omega_k, A_e)\| < \infty. \tag{5.1}$$
2. For all initial states $v_0 \in W_\alpha$ and $x_{e0} \in X_e$ the regulation error goes to zero asymptotically, i.e., $\lim_{t\to\infty} e(t) = 0$.
3. If the parameters (A, B, C, D, E, F) are perturbed to (A', B', C', D', E', F') in such a way that the new closed-loop system (A_e', B_e', C_e', D_e') is strongly stable and it satisfies $\sigma(A_e') \cap \sigma(S) = \varnothing$ and (5.1), then $\lim_{t\to\infty} e(t) = 0$ for all initial states $v_0 \in W_\alpha$ and $x_{e0} \in X_e$.

■

This version of the robust output regulation problem differs from the original statement in Section 3.5 only in that here we search for a controller for which the Sylvester equation is guaranteed to have a solution, and that we restrict our attention to perturbations known to preserve this property. Therefore, it is clear that we can still apply

Theorem 3.12 to show that if the controller is conditionally robust and stabilizes the closed-loop system in such a way that the first part of the problem is satisfied, then it solves our version of the robust output regulation problem.

In the course of this chapter it will become clear that in general the best we can hope for is indeed the strong stability of the closed-loop system. The reason for this is that the internal model containing the copy of the exosystem in the controller must be stabilized with a bounded feedback. If the exosystem has an infinite number of eigenvalues on the imaginary axis, then the exponential stabilization of the closed-loop system is in general impossible even if these eigenvalues are all simple [31, Cor. 3.58].

The chapter is concluded with a detailed example concerning the construction of robust controllers. To illustrate the use of the theoretic results we consider the problem of steering the output of a scalar system to the reference signals generated by an infinite-dimensional exosystem. We use the methods presented in the preceding sections to strongly stabilize the closed-loop system. We will then see that the use of these particular methods allows us to easily determine the values of $\alpha \geq 0$ for which the controller solves the robust output regulation problem on W_α.

The organization of the chapter and an account of the main contributions presented in each of the sections is given in the following.

Section 5.1 In this section we introduce the basic form of the robust observer-based controller. We show that this controller satisfies the $\mathscr{G}$-conditions considered in Section 4.2. The remaining parameters of the controller are fixed in the subsequent sections.

Section 5.2 We fix the parameters of the controller to achieve strong stability of the closed-loop system. The result presented in this section *assumes* that the internal model in the controller can be stabilized using a bounded feedback. This rather involved part of the stabilization of the closed-loop system requires additional assumptions and is treated separately in Section 5.3.

Section 5.3 In this section we consider the stabilization of the internal model in the controller. In particular we show that the internal model can be stabilized using a bounded feedback if the values of the transfer function of the stabilized plant at the frequencies $i\omega_k$ of the exosystem decay at most polynomially as $|k| \to \infty$.

Section 5.4 Subsequent to the stabilization of the closed-loop system, we consider assumptions guaranteeing the solvability of the Sylvester equation. We show that thanks to our method of stabilizing the internal model we can derive easily verifiable sufficient conditions for the controller to satisfy (5.1).

Section 5.5. The chapter is concluded with an example on the construction of robust controllers. We design an error feedback controller to steer the output of a scalar system to the signals generated by an infinite-dimensional exosystem.

The construction of the controller solving the robust output regulation problem generalizes the method presented in [15], where the problem of robust output regulation is studied for infinite-dimensional diagonal exosystems. In particular, we use similar choices for the parameters of the observer-based controller in stabilizing the closed-loop system.

5.1 An Observer Based Controller Satisfying the $\mathcal{G}$-Conditions

In this section we introduce the structure of the error feedback controller that we use to solve the robust output regulation problem. We show that the form of the controller guarantees that it satisfies the $\mathcal{G}$-conditions considered in Section 4.2. Using the results presented in Chapter 4 we can then under suitable assumptions conclude that the controllers having this structure are conditionally robust. The remaining parameters of the controller are fixed in Sections 5.2 and 5.3 to stabilize the closed-loop system.

We will consider the construction of the controller under the following standing assumptions. Unfortunately, these are not yet sufficient to guarantee the solvability of the robust output regulation problem. In particular, in order to stabilize the internal model of the exosystem in the controller, we need conditions not only on the structure of the controller, but also on the choices of its individual parameters. These additional assumptions are stated in Theorems 5.4 and 5.6.

Assumption 5.1. *Assume the following.*

1. *The infinite part of the spectrum* $\sigma(S) = \{i\omega_k\}_{k\in\mathbb{Z}}$ *consists only of simple eigenvalues and has a uniform gap, i.e., there exists* $N \in \mathbb{N}$ *such that*

$$\inf_{k\neq l} |\omega_k - \omega_l| > 0,$$

   where $|k|, |l| \geq N$.

2. *The input and output spaces of the system are one-dimensional, i.e.,* $Y = U = \mathbb{C}$.

3. *The pair* (A, B) *is exponentially stabilizable and the pair* (C, A) *exponentially detectable.*

The assumption that the infinite part of the spectrum of S has a uniform gap is not crucial to our approach to the stabilization of the closed-loop system. It can be replaced with the requirement that the spectrum of S does not have any finite accumulation points, if we have an asymptotic lower bound for the distances of the neighboring eigenvalues. However, the price of this added generality is that the conditions for the stabilizability of the closed-loop system become more complicated. The effect of relaxing this assumption will be discussed later in Section 5.4.

The structure of our observer-based error feedback controller is specified in the following definition.

Definition 5.2. The parameters of the error feedback controller $(\mathscr{G}_1, \mathscr{G}_2, K)$ on the space $Z = X \times W$ are chosen to be of the form

$$\mathscr{G}_1 = \begin{pmatrix} A + BK_1 + L(C + DK_1) & (B + LD)K_2 \\ 0 & S \end{pmatrix}, \qquad \mathscr{G}_2 = \begin{pmatrix} -L \\ G_2 \end{pmatrix}, \qquad K = \begin{pmatrix} K_1 & K_2 \end{pmatrix},$$

where $G_2 = g_2 \in \mathscr{L}(Y, W)$ satisfies $\langle g_2, \phi_k^{n_k} \rangle \neq 0$ for all $k \in \mathbb{Z}$, and where $K_1 \in \mathscr{L}(X, U)$, $K_2 \in \mathscr{L}(W, U)$, and $L \in \mathscr{L}(Y, X)$. ■

The copy of the operator S in $\mathscr{G}_1$ is loosely called the *internal model* of the exosystem in the controller. As we saw in the previous chapter, the dimension of the output space determines the number of copies of the dynamics of the signal generator the controller must contain in order for it to be conditionally robust. Since we are only considering systems with a single output, one copy of the operator S is sufficient. In the case of a p-dimensional output space the operator S in $\mathscr{G}_1$ would have to be replaced with an operator G_1 copying the dynamics of the exosystem p times [15].

We conclude this section by showing that if the operators K_1, K_2 and L are chosen in such a way that the spectra of the closed-loop system and the exosystem are disjoint, then the controller satisfies the $\mathscr{G}$-conditions. Using the results presented in Chapter 4 we can then immediately conclude that for such choices of parameters the controller is conditionally robust. The problem of choosing these operators to strongly stabilize the closed-loop system is considered in the next two sections.

Theorem 5.3. *If $\sigma(A_e) \cap \sigma(S) = \varnothing$, then the controller $(\mathscr{G}_1, \mathscr{G}_2, K)$ in Definition 5.2 satisfies the $\mathscr{G}$-conditions and is therefore conditionally robust.*

Proof. Since $g_2 \neq 0$ we have that $\mathscr{G}_2 y \neq 0$ for all $y \in Y$ and thus $\mathscr{N}(\mathscr{G}_2) = \{0\}$.

Let $k \in \mathbb{Z}$ and assume $(x \;\; v)^T \in \mathscr{R}(i\omega_k I - \mathscr{G}_1) \cap \mathscr{R}(\mathscr{G}_2)$. Definition 5.2 then implies that there exist $x_1 \in \mathscr{D}(A)$, $v_1 \in \mathscr{D}(S)$, and $y \in Y$ such that

$$\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} i\omega_k I - A - BK_1 - L(C + DK_1) & -(B + LD)K_2 \\ 0 & i\omega_k I - S \end{pmatrix} \begin{pmatrix} x_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} -L \\ G_2 \end{pmatrix} y.$$

The second line of this equation shows that $(i\omega_k I - S)v_1 = G_2 y$, and using the structure of the operator S further implies

$$\langle g_2, \phi_k^{n_k} \rangle y = \langle G_2 y, \phi_k^{n_k} \rangle = \langle (i\omega_k I - S)v_1, \phi_k^{n_k} \rangle = (i\omega_k - i\omega_k)\langle v_1, \phi_k^{n_k} \rangle = 0.$$

Since $\langle g_2, \phi_k^{n_k}\rangle \neq 0$ by definition, we must have $y = 0$. This further concludes that $(x \;\; v)^T = \mathscr{G}_2 y = 0$, and thus $\mathscr{R}(i\omega_k I - \mathscr{G}_1) \cap \mathscr{R}(\mathscr{G}_2) = \{0\}$.

Let $k \in \mathbb{Z}$ be such that $n_k = d_k$. Let $(x \;\; v)^T \in \mathscr{N}(i\omega_k I - \mathscr{G}_1)^{d_k - 1}$. Due to the triangular structure of the operator $\mathscr{G}_1$, this clearly also implies $v \in \mathscr{N}(i\omega_k I - S)^{d_k-1}$. Since we assumed $\sigma(A_e) \cap \sigma(S) = \varnothing$, we have from Lemma 4.12 that $\mathscr{R}(i\omega_k I - \mathscr{G}_1) + \mathscr{R}(\mathscr{G}_2) = Z$ and thus there exist $x_1 \in \mathscr{D}(A)$, $v_1 \in \mathscr{D}(S)$, and $y \in Y$ such that

$$\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} i\omega_k I - A - BK_1 - L(C + DK_1) & -(B + LD)K_2 \\ 0 & i\omega_k I - S \end{pmatrix} \begin{pmatrix} x_1 \\ v_1 \end{pmatrix} + \begin{pmatrix} -L \\ G_2 \end{pmatrix} y.$$

The second line of this equation further implies $v = (i\omega_k I - S)v_1 + G_2 y$. Using the structure of the operator S_k we can see that

$$\begin{aligned}
(i\omega_k I - S_k) &= -\sum_{l=2}^{d_k} \langle \cdot, \phi_k^l \rangle \phi_k^{l-1}, \\
(i\omega_k I - S_k)^2 &= \sum_{j=2}^{d_k} \langle \sum_{l=2}^{d_k} \langle \cdot, \phi_k^l \rangle \phi_k^{l-1}, \phi_k^j \rangle \phi_k^{j-1} = \sum_{l=3}^{d_k} \langle \cdot, \phi_k^l \rangle \phi_k^{l-2} \\
&\;\;\vdots \\
(i\omega_k I - S_k)^{d_k-1} &= (-1)^{d_k-1} \langle \cdot, \phi_k^{d_k} \rangle \phi_k^1,
\end{aligned}$$

and finally $(i\omega_k I - S_k)^{d_k} = 0$. Since we have $v \in \mathscr{N}(i\omega_k I - S)^{d_k-1}$, the properties of the projection P_k and $v = (i\omega_k I - S)v_1 + G_2 y$ can be used to further show that

$$\begin{aligned}
0 &= P_k (i\omega_k I - S)^{d_k-1} v = (i\omega_k I - S_k)^{d_k-1} v \\
&= (i\omega_k I - S_k)^{d_k-1} \left((i\omega_k I - S)v_1 + G_2 y \right) \\
&= (i\omega_k I - S_k)^{d_k} v_1 + (i\omega_k I - S_k)^{d_k-1} G_2 y = (-1)^{d_k-1} y \langle g_2, \phi_k^{d_k} \rangle \phi_k^1.
\end{aligned}$$

Since $\langle g_2, \phi_k^{d_k} \rangle \neq 0$, we must have $y = 0$. This immediately implies

$$\begin{pmatrix} x \\ v \end{pmatrix} = (i\omega_k I - \mathscr{G}_1) \begin{pmatrix} x_1 \\ v_1 \end{pmatrix} \in \mathscr{R}(i\omega_k I - \mathscr{G}_1),$$

and thus $\mathscr{N}(i\omega_k I - \mathscr{G}_1)^{d_k-1} \subset \mathscr{R}(i\omega_k I - \mathscr{G}_1)$.

This concludes that the controller satisfies the $\mathscr{G}$-conditions. Since $\sigma(A_e) \cap \sigma(S) = \varnothing$, by Theorems 4.7 and 4.4 it is also conditionally robust. □

5.2 Stabilization of the Closed-Loop System

We now turn to the problem of choosing the parameters K_1, K_2 and L of the observer-based controller in Definition 5.2 in such a way that the closed-loop is strongly stable and $\sigma(A_e)\cap\sigma(S)=\varnothing$. In this section we show that it is possible to reduce this problem to the feedback stabilization of the internal model, which is then considered separately in Section 5.3. This is confirmed by the following theorem, which also lists the appropriate choices for the parameters of the controller.

Theorem 5.4. *Choose $K_{11}\in\mathscr{L}(X,U)$ and $L\in\mathscr{L}(Y,X)$ in such a way that $A+BK_{11}$ and $A+LC$ are exponentially stable. Then the Sylvester equation*

$$SH_{e1}=H_{e1}(A+BK_{11})+G_2(C+DK_{11}) \tag{5.2}$$

on $\mathscr{D}(A)$ has a unique solution $H_{e1}\in\mathscr{L}(X,W)$ satisfying $H_{e1}(\mathscr{D}(A))\subset\mathscr{D}(S)$.

Denote $B_1=H_{e1}B+G_2D$ and assume $K_2\in\mathscr{L}(W,U)$ can be chosen in such a way that the semigroup generated by the operator $S+B_1K_2$ is strongly stable and $\sigma(S+B_1K_2)\cap\sigma(S)=\varnothing$. If the parameter K_1 is chosen as $K_1=K_{11}+K_2H_{e1}$, then the closed-loop system is strongly stable and $\sigma(A_e)\cap\sigma(S)=\varnothing$.

We will first consider the solvability of the Sylvester equation in the theorem. We will also need a similar result later in Section 5.3. For this reason, the following lemma is presented for more general operators $\tilde{A}$ and $\tilde{G}$ in place of $A+BK_{11}$ and $G_2(C+DK_{11})$, respectively.

Lemma 5.5. *Assume the operator $\tilde{A}:\mathscr{D}(\tilde{A})\subset\tilde{X}\to\tilde{X}$ generates an exponentially stable semigroup and $\tilde{G}\in\mathscr{L}(\tilde{X},W)$. Then the Sylvester equation $SH=H\tilde{A}+\tilde{G}$ has a unique solution $H\in\mathscr{L}(\tilde{X},W)$ satisfying $H(\mathscr{D}(\tilde{A}))\subset\mathscr{D}(S)$. The operator H is given by*

$$H=\begin{pmatrix}\cdots & H_{-1}^T & H_0^T & H_1^T & \cdots\end{pmatrix}^T,\qquad H_k=\sum_{l=1}^{n_k}(-1)^{l-1}J_{n_k}^{l-1}P_k\tilde{G}R(i\omega_k,\tilde{A})^l,$$

where $J_{n_k}\in\mathscr{L}\left(\operatorname{span}\left\{\phi_k^l\right\}_{l=1}^{n_k}\right)$ is an operator corresponding to a single $n_k\times n_k$ Jordan block associated to an eigenvalue 0.

Proof. Since $\tilde{A}$ generates an exponentially stable semigroup and since the growth bound of the semigroup generated by $-S$ polynomially bounded, we have from [43] that the Sylvester equation has a unique solution $H\in\mathscr{L}(\tilde{X},W)$ and $H(\mathscr{D}(\tilde{A}))\subset\mathscr{D}(S)$. It remains to show that this solution has the given form.

Applying the projection operators P_k to the both sides of the Sylvester equation we see that for every $k\in\mathbb{Z}$

$$P_kSH=P_kH\tilde{A}+P_k\tilde{G},$$

and for all $s \in \rho(\tilde{A}) \cap \rho(S)$ we have (denoting $H_k = P_k H$ and $\tilde{G}_k = P_k \tilde{G}$)

$$\begin{aligned} & P_k S H = P_k H \tilde{A} + P_k \tilde{G} \\ \Rightarrow \quad & S_k H_k = H_k \tilde{A} + \tilde{G}_k \\ \Rightarrow \quad & (sI - S_k) H_k = H_k (sI - \tilde{A}) - \tilde{G}_k \\ \Rightarrow \quad & H_k R(s,\tilde{A}) = R(s,S_k) H_k - R(s,S_k) \tilde{G}_k R(s,\tilde{A}). \end{aligned}$$

Let Γ_k be positively oriented circles with the same radius centered at points $i\omega_k$ in such a way that for every $k \in \mathbb{Z}$ the set $\sigma(\tilde{A}) \cup (\sigma(S) \setminus \{i\omega_k\})$ lies outside Γ_k. Let $x \in \mathcal{D}(\tilde{A})$ and $k \in \mathbb{Z}$. Integrating over Γ_k we obtain

$$\int_{\Gamma_k} H_k R(s,\tilde{A}) x\, ds = \int_{\Gamma_k} R(s,S_k) H_k x\, ds - \int_{\Gamma_k} R(s,S_k) \tilde{G}_k R(s,\tilde{A}) x\, ds, \tag{5.3}$$

and since $R(s,\tilde{A})x$ is analytic inside Γ_k, the left-hand side of (5.3) vanishes. If we denote by $J_{n_k} \in \mathcal{L}\left(\operatorname{span}\{\phi_k^l\}_{l=1}^{n_k}\right)$ a finite-dimensional operator corresponding to an $n_k \times n_k$ Jordan block associated to an eigenvalue 0, we have

$$\begin{aligned} R(s,S_k) = (sI - S_k)^{-1} &= \left((s - i\omega_k)I - J_{n_k}\right)^{-1} = \frac{1}{s - i\omega_k}\left(I - \frac{J_{n_k}}{s - i\omega_k}\right)^{-1} \\ &= \frac{1}{s - i\omega_k} \sum_{l=0}^{n_k - 1} \frac{J_{n_k}^l}{(s - i\omega_k)^l} = \sum_{l=0}^{n_k-1} \frac{J_{n_k}^l}{(s - i\omega_k)^{l+1}}. \end{aligned} \tag{5.4}$$

The Cauchy integral formula [49], [7, Sec. A.5] states that if f is a function that is analytic inside Γ_k, then

$$\int_{\Gamma_k} \frac{f(s)}{(s - i\omega_k)^{l+1}} = \frac{2\pi i}{l!} f^{(l)}(i\omega_k).$$

Using the expression (5.4) for $R(s,S_k)$ and applying the above integral formula to the function $f(s) \equiv H_k x$ imply that the first term on the right-hand side of (5.3) is equal to $2\pi i H_k x$. Equation (5.3) and another application of the Cauchy integral formula further show that

$$\begin{aligned} H_k x &= \frac{1}{2\pi i} \int_{\Gamma_k} \sum_{l=0}^{n_k-1} \frac{J_{n_k}^l}{(s - i\omega_k)^{l+1}} \tilde{G}_k R(s,\tilde{A}) x\, ds = \frac{1}{2\pi i} \sum_{l=0}^{n_k-1} \int_{\Gamma_k} \frac{J_{n_k}^l}{(s - i\omega_k)^{l+1}} \tilde{G}_k R(s,\tilde{A}) x\, ds \\ &= \sum_{l=0}^{n_k-1} \frac{1}{l!} J_{n_k}^l \tilde{G}_k \left(\frac{d^l}{ds^l} R(s,\tilde{A}) x \right)(i\omega_k) = \sum_{l=0}^{n_k-1} (-1)^l J_{n_k}^l \tilde{G}_k R(i\omega_k,\tilde{A})^{l+1} x. \end{aligned}$$

This concludes the proof. □

We can now complete the proof of Theorem 5.4 by showing that if the parameters of the controller are chosen as suggested, then the closed-loop system is strongly stable and $\sigma(A_e) \cap \sigma(S) = \varnothing$.

Proof of Theorem 5.4. The solvability of the Sylvester equation in the theorem follows directly from Lemma 5.5.

If the error feedback controller has the structure described in Definition 5.2, the system operator of the closed-loop system is given by

$$A_e = \begin{pmatrix} A & BK \\ \mathcal{G}_2 C & \mathcal{G}_1 + \mathcal{G}_2 DK \end{pmatrix} = \begin{pmatrix} A & BK_1 & BK_2 \\ -LC & A+BK_1+LC & BK_2 \\ G_2C & G_2DK_1 & S+G_2DK_2 \end{pmatrix}.$$

If we choose a similarity transform $Q_e \in \mathcal{L}(X \times X \times W, X \times W \times X)$ satisfying

$$Q_e = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & I \\ -I & I & 0 \end{pmatrix} \qquad \text{and} \qquad Q_e^{-1} = \begin{pmatrix} I & 0 & 0 \\ I & 0 & I \\ 0 & I & 0 \end{pmatrix},$$

we can then define an operator $\tilde{A}_e$ on the space $X \times W \times X$ by

$$\tilde{A}_e = Q_e A_e Q_e^{-1} = \begin{pmatrix} A+BK_1 & BK_2 & BK_1 \\ G_2(C+DK_1) & S+G_2DK_2 & G_2DK_1 \\ 0 & 0 & A+LC \end{pmatrix}.$$

It is well-known that $\tilde{A}_e$ generates a C_0-semigroup, and that this semigroup is strongly stable if and only if the semigroup $T_e(t)$ generated by A_e is. The triangular structure of $\tilde{A}_e$ further implies that since $A+LC$ is exponentially stable, this operator generates a strongly stable semigroup if the semigroup generated by the operator

$$\tilde{A}_{e1} = \begin{pmatrix} A+BK_1 & BK_2 \\ G_2(C+DK_1) & S+G_2DK_2 \end{pmatrix} = \begin{pmatrix} A & 0 \\ G_2C & S \end{pmatrix} + \begin{pmatrix} B \\ G_2D \end{pmatrix} \begin{pmatrix} K_1 & K_2 \end{pmatrix}$$

is strongly stable [15, Lem. 20]. Using $K_1 = K_{11} + K_2 H_{e1}$ shows that

$$\tilde{A}_{e1} = \begin{pmatrix} A+BK_{11} & 0 \\ G_2(C+DK_{11}) & S \end{pmatrix} + \begin{pmatrix} B \\ G_2D \end{pmatrix} \begin{pmatrix} K_2H_{e1} & K_2 \end{pmatrix}.$$

We can now use the solution H_{e1} of the Sylvester equation (5.2) to choose a similarity transform $Q_{e1} \in \mathcal{L}(X \times W)$ in such a way that

$$Q_{e1} = \begin{pmatrix} I & 0 \\ H_{e1} & I \end{pmatrix}, \qquad Q_{e1}^{-1} = \begin{pmatrix} I & 0 \\ -H_{e1} & I \end{pmatrix}.$$

A direct computation shows that since H_{e1} is a solution of the Sylvester equation (5.2), we have

$$\begin{aligned} Q_{e1} \begin{pmatrix} A+BK_{11} & 0 \\ G_2(C+DK_{11}) & S \end{pmatrix} Q_{e1}^{-1} &= \begin{pmatrix} A+BK_{11} & 0 \\ H_{e1}(A+BK_{11}) + G_2(C+DK_{11}) - SH_{e1} & S \end{pmatrix} \\ &= \begin{pmatrix} A+BK_{11} & 0 \\ 0 & S \end{pmatrix}. \end{aligned}$$

Therefore, if we denote $B_1 = H_{e1}B + G_2D$ and define $A_{e1} = Q_{e1}\tilde{A}_{e1}Q_{e1}^{-1}$, we then have

$$\begin{aligned} A_{e1} = Q_{e1}\tilde{A}_{e1}Q_{e1}^{-1} &= \begin{pmatrix} A+BK_{11} & 0 \\ 0 & S \end{pmatrix} + \begin{pmatrix} B \\ H_{e1}B + G_2D \end{pmatrix} \begin{pmatrix} K_2H_{e1} - K_2H_{e1} & K_2 \end{pmatrix} \\ &= \begin{pmatrix} A+BK_{11} & BK_2 \\ 0 & S+B_1K_2 \end{pmatrix} \end{aligned}$$

Since the operators $A + BK_{11}$ and $S + B_1K_2$ generate exponentially and strongly stable semigroups, respectively, the operator A_{e1} generates a strongly stable semigroup by [15, Lem. 20]. Using this and the earlier arguments we can conclude that the closed-loop system is strongly stable.

The operator K_2 was chosen in such a way that $\sigma(S + B_1K_2) \cap \sigma(S) = \varnothing$, and since the operators $A + BK_{11}$ and $A + LC$ are generators of exponentially stable semigroups, we also have

$$\sigma(A+BK_{11}) \cap \sigma(S) = \varnothing \qquad \text{and} \qquad \sigma(A+LC) \cap \sigma(S) = \varnothing.$$

We can now use Lemma A.1 and the similarities between the operators to deduce that

$$\begin{aligned} \sigma(A_{e1}) \cap \sigma(S) = \varnothing \quad &\Rightarrow \quad \sigma(\tilde{A}_{e1}) \cap \sigma(S) = \varnothing \quad \Rightarrow \quad \sigma(\tilde{A}_e) \cap \sigma(S) = \varnothing \\ &\Rightarrow \quad \sigma(A_e) \cap \sigma(S) = \varnothing. \end{aligned}$$

This concludes the proof. □

5.3 Stabilization of the Internal Model

In this section we complete the construction of our observer-based controller by stabilizing the internal model with a bounded feedback. Our main goal is therefore to choose a state feedback $K_2 \in \mathcal{L}(W, U)$ in such a way that the operator

$$S + B_1K_2$$

generates a strongly stable semigroup on W. Here the operator $B_1 = H_{e1}B + G_2D$ is defined as in Theorem 5.4. In choosing the feedback K_2 we apply pole placement of an infinite spectrum [58, 17, 61, 47]. Using this particular approach enables us to directly verify that the condition $\sigma(S + B_1K_2) \cap \sigma(S) = \varnothing$ is satisfied, and to obtain asymptotic estimates for the behavior of the resolvent operator of the closed-loop system. We will see in the next section that the latter is essential to determining on which of the scale spaces W_α the controller solves the robust output regulation problem.

The first one of the conditions in Assumption 5.1 tells us that there exists a finite set $I_S \subset \mathbb{Z}$ of indices and a constant $d > 0$ such that $n_k = 1$ for all $k \in \mathbb{Z} \setminus I_S$ and

$$|\omega_k - \omega_l| \geq d > 0$$

for $k,l \in \mathbb{Z} \setminus I_S$ with $k \neq l$. This further implies that the operators S, B_1 and K_2 can be decomposed as

$$S = \begin{pmatrix} S_f & 0 \\ 0 & S_i \end{pmatrix}, \qquad B_1 = \begin{pmatrix} B_f \\ B_i \end{pmatrix}, \qquad K_2 = \begin{pmatrix} K_f & K_i \end{pmatrix} \tag{5.5}$$

according to the decomposition $W = W^f \times W^i$ of the state space of the exosystem, where

$$W^f = \operatorname{span}\left\{ \phi_k^l \;\middle|\; k \in I_S,\ l = 1,\dots,n_k \right\}, \qquad W^i = \overline{\operatorname{span}}\left\{ \phi_k^1 \right\}_{k\in\mathbb{Z}\setminus I_S}.$$

Here the labels 'f' and 'i' stand for 'finite' and 'infinite' parts of the spaces and operators. The parts S_f, B_f and K_f are operators on finite-dimensional spaces and S_i is an infinite-dimensional diagonal operator

$$S_i v = \sum_{k\in\mathbb{Z}\setminus I_S} i\omega_k \langle v, \phi_k^1 \rangle, \qquad \mathscr{D}(S_i) = \left\{ v \in W^i \;\middle|\; \sum_{k\in\mathbb{Z}\setminus I_S} \omega_k^2 |\langle v, \phi_k^1 \rangle|^2 < \infty \right\}.$$

Our standing assumptions concerning the spectrum of the exosystem also imply that we can safely assume that the frequencies are ordered in such a way that $\omega_k \leq \omega_l$ for all $k,l \in \mathbb{Z} \setminus I_S$ with $k \leq l$. For notational convenience, we also assume $0 \in I_S$.

Theorem 5.6 below completes the set of conditions required to strongly stabilize the closed-loop system in such a way that $\sigma(A_e) \cap \sigma(S) = \varnothing$. To state the assumptions we need the complex-valued function $P_K(\cdot)$ defined by

$$P_K(\lambda) = (C + DK_{11})R(\lambda, A + BK_{11})B + D \tag{5.6}$$

for all $\lambda \in \rho(A + BK_{11})$. This is precisely the transfer function of the original plant that has been stabilized by choosing the input u as $u = K_{11}x + \tilde{u}$. It is well-known that the invertibility of a transfer function is preserved under this kind of feedback. Because of this, our assumption on the invertibility of the operators $P(i\omega_k)$ made in Section 3.1 also implies that we have $P_K(i\omega_k) \neq 0$ for all $k \in \mathbb{Z}$.

Since we have some freedom in choosing the parameter $G_2 \in \mathscr{L}(Y, W)$ of the controller, the theorem states that the stabilization of the internal model can be achieved using bounded feedback, provided that the values $P_K(i\omega_k)$ of the transfer function of the stabilized plant decay to zero at a rate that is at most polynomial. Moreover, it also shows that this rate is reflected in the behavior of the resolvent of the stabilized closed-loop system at the eigenvalues $i\omega_k$ of the exosystem. For finite-dimensional systems the assumption on the polynomial decay of the transfer function is always satisfied. However, in the case of infinite-dimensional systems the situation is more complicated, and in particular the values of the transfer function can approach zero at a faster rate.

Theorem 5.6. *Assume that there exist $\beta, c > 0$ such that*

$$|P_K(i\omega_k)| \cdot |\langle g_2, \phi_k^1 \rangle| \geq \frac{c}{|k|^\beta} \tag{5.7}$$

for large enough $|k|$. Then the operator $K_2 \in \mathcal{L}(W, U)$ can be chosen in such a way that the semigroup generated by the operator $S+B_1K_2$ is strongly stable and $\sigma(S+B_1K_2) \cap \sigma(S) = \emptyset$. Furthermore, for any $\gamma > \beta + \frac{1}{2}$ the operator K_2 can be chosen in such a way that the asymptotic behavior of the resolvent operator of the closed-loop system satisfies

$$\|R(i\omega_k, A_e)\| = \mathcal{O}(|k|^\gamma).$$

Proof. We will first show that the pair (S_f, B_f) is controllable. Since the matrix S_f consists of Jordan blocks, it is sufficient to show that $\langle B_f, \phi_k^{n_k} \rangle \neq 0$ for all $k \in I_S$. Using the formula $B_1 = H_{e1}B + G_2D$, Lemma 5.5 and (5.6) we have

$$\begin{aligned}
\langle B_1, \phi_k^{n_k} \rangle &= \langle H_k B + G_2 D, \phi_k^{n_k} \rangle \\
&= \left\langle \sum_{l=1}^{n_k} (-1)^{l-j} J_{n_k}^{l-1} P_k G_2 (C + DK_{11}) R(i\omega_k, A + BK_{11})^l B, \phi_k^{n_k} \right\rangle + \langle g_2, \phi_k^{n_k} \rangle D \\
&= \sum_{l=1}^{n_k} \left[(-1)^{l-j} \langle J_{n_k}^{l-1} g_2, \phi_k^{n_k} \rangle (C + DK_{11}) R(i\omega_k, A + BK_{11})^l B \right] + \langle g_2, \phi_k^{n_k} \rangle D \\
&= \langle g_2, \phi_k^{n_k} \rangle (C + DK_{11}) R(i\omega_k, A + BK_{11}) B + \langle g_2, \phi_k^{n_k} \rangle D \\
&= \langle g_2, \phi_k^{n_k} \rangle P_K(i\omega_k) \neq 0
\end{aligned}$$

for all $k \in \mathbb{Z}$. This concludes that the pair (S_f, B_f) is controllable. Since S_f and B_f are finite-dimensional operators, we can now choose an operator $K_{f1} \in \mathcal{L}(W^f, U)$ in such a way that $S_f + B_f K_{f1}$ is exponentially stable.

Since S_i is a diagonal operator, we can use Lemma 5.5 to show that the Sylvester equation

$$S_i H = H(S_f + B_f K_{f1}) + B_i K_{f1} \tag{5.8}$$

has a unique solution $H \in \mathcal{L}(W^f, W^i)$ given by

$$Hv = \sum_{k \in \mathbb{Z} \setminus I_S} \left\langle B_i K_{f1} R(i\omega_k, S_f + B_f K_{f1}) v, \phi_k^1 \right\rangle \phi_k^1.$$

We now choose $K_f = K_{f1} + K_i H$. For this operator we have

$$S + B_1 K_2 = \begin{pmatrix} S_f + B_f K_{f1} & 0 \\ B_i K_{f1} & S_f \end{pmatrix} + \begin{pmatrix} B_f \\ B_i \end{pmatrix} \begin{pmatrix} K_i H & K_i \end{pmatrix},$$

and as in the proof of Theorem 5.4 we can use the fact that the operator H is the solution of the Sylvester equation (5.8) to show

$$\begin{aligned}
&\begin{pmatrix} I & 0 \\ H & I \end{pmatrix} (S + B_1 K_2) \begin{pmatrix} I & 0 \\ -H & I \end{pmatrix} \\
&= \begin{pmatrix} S_f + B_f K_{f1} & 0 \\ 0 & S_i \end{pmatrix} + \begin{pmatrix} B_f \\ HB_f + B_i \end{pmatrix} \begin{pmatrix} K_i H - K_i H & K_i \end{pmatrix} \\
&= \begin{pmatrix} S_f + B_f K_{f1} & B_f K_i \\ 0 & S_i + B_2 K_i \end{pmatrix}. \qquad (5.9)
\end{aligned}$$

Here we have denoted $B_2 = HB_f + B_i \in \mathscr{L}(U, W^i)$. For any $u \in U = \mathbb{C}$ we have

$$\begin{aligned}
B_2 u = (HB_f + B_i)u &= \sum_{k \in \mathbb{Z} \setminus I_S} \langle B_i \big(K_{f1} R(i\omega_k, S_f + B_f K_{f1}) B_f u + u \big), \phi_k^1 \rangle \phi_k^1 \\
&= \sum_{k \in \mathbb{Z} \setminus I_f} \big(K_{f1} R(i\omega_k, S_f + B_f K_{f1}) B_f u + u \big) \langle B_i, \phi_k^1 \rangle \phi_k^1. \qquad (5.10)
\end{aligned}$$

The triangular form in (5.9) implies that since $S_f + B_f K_{f1}$ is exponentially stable, the operator $S + B_1 K_2$ can be stabilized by choosing $K_i \in \mathscr{L}(W^i, U)$ in such a way that $S_i + B_2 K_i$ generates a strongly stable semigroup on W^i [15, Lem. 20].

We will choose the stabilizing operator K_i using pole placement of an infinite spectrum [58, 54]. Let $\gamma > \beta + \frac{1}{2}$ and choose

$$\mu_k = -\frac{1}{|k|^\gamma} + i\omega_k$$

for $k \in \mathbb{Z} \setminus I_S$ (recall that we assumed $0 \in I_S$). In particular we will show that there exists an operator $K_i \in \mathscr{L}(W^i, U)$ such that $\sigma(S_i + B_2 K_i) = \{\mu_k\}_k$ and the operator $S_i + B_2 K_i$ is a strongly stable Riesz-spectral operator with at most finite number of nonsimple eigenvalues. Denote

$$d = \inf_{k \neq l} |\omega_k - \omega_l| > 0,$$

where $k, l \in \mathbb{Z} \setminus I_S$. For all $\lambda \in \mathbb{C}$ such that $\operatorname{dist}(\lambda, i\omega_k) > \frac{1}{3} d$ we have

$$\sum_{k \in \mathbb{Z} \setminus I_S} \left| \frac{\langle B_2, \phi_k^1 \rangle}{\lambda - i\omega_k} \right|^2 \le \frac{3}{d} \sum_{k \in \mathbb{Z} \setminus I_S} \left| \langle B_2, \phi_k^1 \rangle \right|^2 \le \frac{3}{d} \|B_2\|^2 < \infty, \qquad (5.11a)$$

$$\sum_{\substack{k \in \mathbb{Z} \setminus I_S \\ k \neq l}} \left| \frac{\langle B_2, \phi_k^1 \rangle}{i\omega_l - i\omega_k} \right|^2 \le \frac{1}{d} \sum_{\substack{k \in \mathbb{Z} \setminus I_S \\ k \neq l}} \left| \langle B_2, \phi_k^1 \rangle \right|^2 \le \frac{1}{d} \|B_2\|^2 < \infty. \qquad (5.11b)$$

Our next step is to derive a lower bound for the behavior of the terms $\left| \langle B_2, \phi_k^1 \rangle \right|$ as $|k| \to \infty$. We can first observe that we must have $K_{f1} R(i\omega_k, S_f) B_f \neq 1$ for all $k \in \mathbb{Z} \setminus I_S$, since otherwise we would have

$$(i\omega_k I - S_f - B_f K_{f1})R(i\omega_k, S_f)B_f = (i\omega_k I - S_f)R(i\omega_k, S_f)B_f - B_f K_{f1}R(i\omega_k, S_f)B_f = 0,$$

i.e., $i\omega_k \in \sigma(S_f + B_f K_{f1})$. This, however, is impossible since $S_f + B_f K_{f1}$ is exponentially stable. The application of the well-known Sherman-Morrison formula therefore implies

$$K_f R(i\omega_k, S_f + B_f K_{f1})B_f + 1 = \frac{1}{1 - K_{f1}R(i\omega_k, S_f)B_f} \neq 0 \tag{5.12}$$

for all $k \in \mathbb{Z} \setminus I_S$. On the other hand, using the form of the operator $B_1 = H_{e1}B + G_2 D$ and Lemma 5.5 we can see that for all $k \in \mathbb{Z} \setminus I_S$

$$\langle B_i, \phi_k^1 \rangle = \langle B_1, \phi_k^1 \rangle = \langle G_2(C + DK_{11})R(i\omega_k, A + BK_{11})B + G_2 D, \phi_k^1 \rangle = \langle g_2, \phi_k^1 \rangle P_K(i\omega_k).$$

By Definition 5.2 and the property $P_K(i\omega_k) \neq 0$ we also see that these terms must be nonzero for all $k \in \mathbb{Z} \setminus I_S$. This and (5.12) together with the formula (5.10) for the operator B_2 imply that for all $k \in \mathbb{Z} \setminus I_S$ we have

$$\langle B_2, \phi_k^1 \rangle = (K_{f1}R(i\omega_k, S_f + B_f K_{f1})B_f + 1)\langle B_i, \phi_k^1 \rangle \neq 0. \tag{5.13}$$

Furthermore, since the norms $\|R(i\omega_k, S_f)\|$ decay to zero as $|k| \to \infty$, we can use (5.12) to show an estimate

$$|\langle B_2, \phi_k^1 \rangle| = \left|K_{f1}R(i\omega_k, S_f + B_f K_{f1})B_f + 1\right| \cdot |\langle g_2, \phi_k^1 \rangle P_K(i\omega_k)| \geq \frac{|\langle g_2, \phi_k^1 \rangle| \cdot |P_K(i\omega_k)|}{1 + |K_{f1}R(i\omega_k, S_f)B_f|}$$

$$\geq \frac{|\langle g_2, \phi_k^1 \rangle| \cdot |P_K(i\omega_k)|}{1 + \|K_{f1}\| \cdot \|R(i\omega_k, S_f)\| \cdot \|B_f\|} \geq \frac{1}{2}|\langle g_2, \phi_k^1 \rangle| \cdot |P_K(i\omega_k)|$$

for all $k \in \mathbb{Z} \setminus I_S$ with $|k|$ large enough. Our assumption (5.7) finally implies that there exist a constant $c > 0$ such that for all $k \in \mathbb{Z} \setminus I_S$ with $|k|$ large enough

$$|\langle B_2, \phi_k^1 \rangle| \geq \frac{1}{2}|\langle g_2, \phi_k^1 \rangle| \cdot |P_K(i\omega_k)| \geq c|k|^{-\beta}$$

and thus for a large enough $N \in \mathbb{N}$ we also have

$$\sum_{|k| \geq N} \left| \frac{\mu_k - i\omega_k}{\langle B_2, \phi_k^1 \rangle} \right|^2 \leq \frac{1}{c^2} \sum_{|k| \geq N} \frac{|k|^{2\beta}}{|k|^{2\gamma}} \leq \frac{1}{c^2} \sum_{|k| \geq N} \frac{1}{|k|^{2(\gamma - \beta)}} < \infty, \tag{5.14}$$

since $2(\gamma - \beta) > 1$.

Since the conditions (5.11), (5.13), and (5.14) are satisfied, by [58, Thm. 1] there exists an operator $K_i \in \mathcal{L}(W^i, U)$ such that $S_i + B_2 K_i$ is a strongly stable Riesz-spectral operator with eigenvalues $\{\mu_k\}_{k \in \mathbb{Z} \setminus I_S}$, and at most a finite number of these eigenvalues are nonsimple. Since

$$\sigma(S + B_1 K_2) \subset \sigma(S_f + B_f K_{f1}) \cup \sigma(S_i + B_i K_i) = \sigma(S_f + B_f K_{f1}) \cup \{\mu_k\}_{k \in \mathbb{Z} \setminus I_S},$$

where $S_f + B_f K_{f1}$ is exponentially stable, we have $\sigma(S+B_1K_2)\cap\sigma(S)=\varnothing$. This concludes that the internal model can be stabilized using a bounded feedback K_2. The infinite part K_i of this operator is obtained by choosing $K_i = \langle \cdot, h\rangle$, where $h \in W^i$ is given by

$$h = \sum_{k\in\mathbb{Z}\setminus I_S} h_k \phi_k^1,$$

$$\overline{h_k} = \frac{\mu_k - i\omega_k}{\langle B_2, \phi_k^1\rangle} \prod_{\substack{l\in\mathbb{Z}\setminus I_S\\ l\neq k}} \frac{i\omega_k - \mu_l}{i\omega_k - i\omega_l} = \frac{1}{|k|^\gamma \langle B_2, \phi_k^1\rangle} \prod_{\substack{l\in\mathbb{Z}\setminus I_S\\ l\neq k}} \left(1 + i\frac{1}{|l|^\gamma(\omega_l - \omega_k)}\right).$$

In the remaining part of the proof we derive the estimate for the asymptotic behavior of the resolvent operator of the closed-loop system. To estimate the resolvent operators of the various composite operators we will use the fact that if X_1 and X_2 are Banach spaces, and if $A_{11} \in \mathscr{L}(X_1)$, $A_{12} \in \mathscr{L}(X_2, X_1)$ and $A_{22} \in \mathscr{L}(X_2)$, we then have

$$\left\| \begin{pmatrix} A_{11} & A_{11}A_{12}A_{22} \\ 0 & A_{22} \end{pmatrix} \right\| \leq \left(\|A_{11}\|+1\right)\left(\|A_{12}\|+1\right)\left(\|A_{22}\|+1\right). \tag{5.15}$$

This follows directly from the estimate

$$\begin{aligned} \left\| \begin{pmatrix} A_{11} & A_{11}A_{12}A_{22} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| &\leq \|A_{11}x_1 + A_{11}A_{12}A_{22}x_2\| + \|A_{22}x_2\| \\ &\leq \left(\|x_1\| + \|x_2\|\right)\left(\|A_{11}\| + \|A_{11}\|\cdot\|A_{12}\|\cdot\|A_{22}\| + \|A_{22}\|\right) \\ &\leq \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| \cdot \max\left\{\|A_{12}\|, 1\right\} \cdot \left(\|A_{11}\|(1+\|A_{22}\|) + \|A_{22}\|\right) \\ &\leq \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| \cdot \left(\|A_{11}\|+1\right)\left(\|A_{12}\|+1\right)\left(\|A_{22}\|+1\right). \end{aligned}$$

Here we used the norm $\|(x_1 \;\; x_2)^T\| = \|x_1\| + \|x_2\|$ on the composite space $X_1 \times X_2$. A different choice for the norm would have only resulted in a constant $M > 0$ on the right-hand side of the estimate in (5.15).

We will start with the asymptotic behavior of $R(i\omega_k, S_i + B_2K_i)$. The fact that $S_i + B_2K_i$ is a Riesz-spectral operator and the infinite part of its spectrum consists only of simple eigenvalues implies that there exists an isomorphism $Q_i \in \mathscr{L}(W^i)$ such that

$$S_i + B_2K_i = Q_i \begin{pmatrix} S_i^{fin} & 0 \\ 0 & S_i^{inf} \end{pmatrix} Q_i^{-1},$$

where S_i^{fin} is a finite-dimensional exponentially stable operator and $S_i^{inf} = \mathrm{diag}(\mu_k)_{|k|\geq N}$ for some $N \in \mathbb{N}$. This means that the resolvent operator of $S_i + B_2K_i$ satisfies

$$\begin{aligned} \left\|R(i\omega_k, S_i + B_2K_i)\right\| &= \left\| Q_i \begin{pmatrix} R(i\omega_k, S_i^{fin}) & 0 \\ 0 & R(i\omega_k, S_i^{inf}) \end{pmatrix} Q_i^{-1} \right\| \\ &\leq \|Q_i\|\|Q_i^{-1}\| \cdot \max\left\{ \left\|R(i\omega_k, S_i^{fin})\right\|, \left\|R(i\omega_k, S_i^{inf})\right\| \right\} \end{aligned}$$

for all $k \in \mathbb{Z} \setminus I_S$. For $k \in \mathbb{Z}$ with $|k| \geq N$ the norm of $R(i\omega_k, S_i^{fin})$ is uniformly bounded and

$$\left\|R(i\omega_k, S_i^{inf})\right\| = \frac{1}{|i\omega_k - \mu_k|} = |k|^\gamma.$$

This immediately implies

$$\left\|R(i\omega_k, S_i + B_2K_i)\right\| = \mathcal{O}(|k|^\gamma).$$

We can now turn to considering the asymptotic behavior of the resolvent operator of $S + B_1K_2$. Similarly as in the derivation of the estimate (5.15) we can easily see that

$$\left\|\begin{pmatrix} I & 0 \\ H & I \end{pmatrix}\right\| \leq \|H\| + \|I\| + \|I\| = \|H\| + 2.$$

Using this and (5.9) we see that the resolvent operator of the strongly stabilized internal model $S + B_1K_2$ satisfies

$$\begin{aligned}
&\left\|R(i\omega_k, S + B_1K_2)\right\| \\
\leq\ & \left(\|H\|+2\right)^2 \left\|\begin{pmatrix} R(i\omega_k, S_f + B_fK_{f1}) & R(i\omega_k, S_f + B_f)B_fK_iR(i\omega_k, S_i + B_2K_i) \\ & R(i\omega_k, S_i + B_2K_i) \end{pmatrix}\right\| \\
\leq\ & \left(\|H\|+2\right)^2 \left(\|B_fK_i\|+1\right)\left(\|R(i\omega_k, S_f + B_fK_{f1})\|+1\right)\left(\|R(i\omega_k, S_i + B_2K_i)\|+1\right)
\end{aligned}$$

Since $\|R(i\omega_k, S_f + B_fK_{f1})\|$ is uniformly bounded with respect to $k \in \mathbb{Z} \setminus I_S$, this estimate implies

$$\left\|R(i\omega_k, S + B_1K_2)\right\| = \mathcal{O}(|k|^\gamma).$$

This, in turn, can be used to estimate the behavior of the resolvent $R(i\omega_k, \tilde{A}_{e1})$. Using the definition of the operator A_{e1} in the proof of Theorem 5.4 and the estimate (5.15), we obtain

$$\begin{aligned}
&\left\|R(i\omega_k, \tilde{A}_{e1})\right\| = \left\|Q_{e1}^{-1}R(i\omega_k, A_{e1})Q_{e1}\right\| \\
\leq\ & \|Q_{e1}^{-1}\|\|Q_{e1}\| \left\|\begin{pmatrix} R(i\omega_k, A+BK_{11}) & R(i\omega_k, A+BK_{11})BK_2R(i\omega_k, S+B_1K_2) \\ 0 & R(i\omega_k, S+B_1K_2) \end{pmatrix}\right\| \\
\leq\ & \|Q_{e1}^{-1}\|\|Q_{e1}\|\left(\|R(i\omega_k, A+BK_{11})\|+1\right)\left(\|BK_2\|+1\right)\left(\|R(i\omega_k, S+B_1K_2)\|+1\right).
\end{aligned}$$

Since $A+BK_{11}$ generates an exponentially stable semigroup, the terms $\|R(i\omega_k, A+BK_{11})\|$ are uniformly bounded with respect to k and thus

$$\left\|R(i\omega_k, \tilde{A}_{e1})\right\| = \mathcal{O}(|k|^\gamma).$$

Finally, we can estimate the behavior of the resolvent operators $R(i\omega_k, A_e)$ of the closed-loop system. Similarly as above, we can use the definition of the operator $\tilde{A}_e$ in the proof of Theorem 5.4 and the estimate (5.15) to show

$$\begin{aligned}
&\left\|R(i\omega_k, A_e)\right\| = \left\|Q_e^{-1}R(i\omega_k, \tilde{A}_e)Q_e\right\| \\
\leq\ &\|Q_e^{-1}\|\|Q_e\| \left\| \begin{pmatrix} R(i\omega_k, \tilde{A}_{e1}) & R(i\omega_k, \tilde{A}_{e1}) \begin{pmatrix} B \\ G_2 D \end{pmatrix} K_1 R(i\omega_k, A+LC) \\ 0 & R(i\omega_k, A+LC) \end{pmatrix} \right\| \\
\leq\ &\|Q_e^{-1}\|\|Q_e\| \big(\|R(i\omega_k, \tilde{A}_{e1})\| + 1\big) \left(\left\| \begin{pmatrix} B \\ G_2 D \end{pmatrix} K_1 \right\| + 1 \right) \big(\|R(i\omega_k, A+LC)\| + 1\big).
\end{aligned}$$

Since $A + LC$ generates an exponentially stable semigroup, the norms $\|R(i\omega_k, A + LC)\|$ are uniformly bounded with respect to $k \in \mathbb{Z}$. Because of this, the above estimate implies

$$\left\|R(i\omega_k, A_e)\right\| = \mathcal{O}(|k|^\gamma).$$

This concludes the proof. □

5.4 The Solvability of the Robust Output Regulation Problem

We conclude the study of our controller by determining the scale spaces W_α on which it solves the robust output regulation problem. In particular we will see that, provided the values $P_K(i\omega_k)$ considered in the previous section decay to zero at a rate that is at most polynomial, such a scale space always exists. Our main result also shows a concrete connection between the rate of this decay and the smoothness of the exogeneous signals the controller is capable of tracking and rejecting. At the end of this section we will also discuss relaxing the standing assumptions on the spectrum of our exosystem.

In the light of the results presented in the earlier sections we can see that it is sufficient to determine the values of the parameter $\alpha \geq 0$ for which the controller we have constructed satisfies (5.1). The role of this condition in the robust output regulation problem is to provide a sufficient condition for the solvability of the Sylvester equation

$$\Sigma S = A_e \Sigma + B_e$$

on $W_{\alpha+1}$ through the use of Lemma 3.10. It was already shown in Theorem 5.3 that our controller is conditionally robust, and in Theorems 5.4 and 5.6 that under the given assumptions the closed-loop system can be strongly stabilized in such a way that $\sigma(A_e) \cap \sigma(S) = \varnothing$. Therefore, once we have chosen $\alpha \geq 0$ in such a way that condition (5.1) is satisfied, we can conclude from Theorem 3.12 that the controller solves the robust output regulation problem on W_α.

The following theorem is the main result of this section. It uses the estimate derived in Theorem 5.6 to determine the scale space on which our controller solves the robust output regulation problem.

Theorem 5.7. *Assume that there exist constants $\beta, c > 0$ such that*

$$|P_K(i\omega_k)| \cdot |\langle g_2, \phi_k^1 \rangle| \geq \frac{c}{|k|^\beta}$$

for large enough $|k|$. If the parameters of the controller are chosen as described earlier in this chapter for some $\gamma > \beta + \frac{1}{2}$ in Theorem 5.6, then the controller solves the robust output regulation problem on W_γ.

Proof. We will first verify that condition (5.1) is satisfied for $\alpha = \gamma$. Theorem 5.6 and our assumption that the infinite part of the spectrum of S has a uniform gap imply that there exist constants $N \in \mathbb{N}$ and $c, M > 0$ such that

$$\|R(i\omega_k, A_e)\| \leq M|k|^\alpha \qquad \text{and} \qquad |\omega_k| \geq c|k|$$

for all $k \in \mathbb{Z}$ with $|k| \geq N$. This implies that for all such $k \in \mathbb{Z}$ we have

$$\frac{\|R(i\omega_k, A_e)\|^2}{(1+\omega_k^2)^\alpha} \leq \frac{M^2|k|^{2\alpha}}{(1+c^2k^2)^\alpha} \leq \frac{M^2}{c^{2\alpha}}\frac{|k|^{2\alpha}}{|k|^{2\alpha}} = \frac{M^2}{c^{2\alpha}} < \infty,$$

which concludes that condition (5.1) is satisfied.

Since $\sigma(A_e) \cap \sigma(S) = \varnothing$ and since by Theorem 5.3 the controller is conditionally robust, Theorem 3.12 concludes that the controller solves the robust output regulation problem on W_γ. □

The above theorem also illustrates a close connection between the asymptotic behavior the values $P_K(i\omega_k)$ of the transfer function of the stabilized plant and the minimal level of smoothness of the signals our controller is capable of tracking and rejecting. In particular this is visible in the case of τ-periodic reference and disturbance signals generated by a diagonal exosystem with frequencies

$$(\omega_k)_{k\in\mathbb{Z}} = \left(\frac{2\pi k}{\tau}\right)_{k\in\mathbb{Z}}.$$

Combining Theorems 5.7 and 2.6 shows us that for such exosystems and for the above choices of parameters, our controller is only guaranteed to be capable of tracking and rejecting τ-periodic signals belonging to the space $H^\gamma_{per}(0,\tau)$.

The applicability of Theorem 5.7 has an evident limitation arising from the fact that since the parameter G_2 of the controller is a bounded operator, we necessarily have

$$\sum_{k\in\mathbb{Z}} |\langle g_2, \phi_k^1\rangle|^2 < \infty.$$

This immediately implies that regardless of the behavior of the transfer function $P_K(\cdot)$ of the stabilized plant, the constant β in the theorem is always larger than $\frac{1}{2}$. Therefore it is impossible to use this result to guarantee the existence of a controller solving the robust output regulation problem on any space W_α with $\alpha \leq 1$.

Throughout this chapter we have assumed that the infinite part of the spectrum of the exosystem consists of eigenvalues having a uniform gap. This assumption is not crucial to the construction of the observer-based controller, and can be relaxed at the cost of added complexity in the results. In particular our approach can be applied if the infinite part of the spectrum of S consists of simple eigenvalues having no finite accumulation points, and if we have a polynomial bound for the rate at which the neighboring eigenvalues approach each other as they approach infinity. In the stabilization of the internal model $S + B_1K_2$ we would then only need to modify the conditions in such a way that the series in (5.11) converge. If the eigenvalues of S do not have a uniform gap, it is clear that this results in an additional requirement that the terms $|\langle B_2, \phi_k^1\rangle|$ approach zero fast enough as $|k| \to \infty$. Since we have seen in the proof of Theorem 5.6 that for large $|k|$ these terms behave like $|\langle g_2, \phi_k^1\rangle| \cdot |P_K(i\omega_k)|$, the condition could be expressed as a requirement that the asymptotic decay of these terms is sufficiently fast.

On the other hand, if the eigenvalues of S approach infinity faster than at a constant rate as $|k| \to \infty$, any known bound for this rate can be used to improve Theorem 5.7. More precisely, if there exist constants $\eta > 1$ and $c > 0$ such that for large $|k|$ we have

$$|\omega_k| \geq c|k|^\eta, \tag{5.16}$$

it is then easy to show that our controller in fact solves the robust output regulation problem on W_α for $\alpha = \gamma/\eta$. Since the infinite part of the spectrum of S still has a uniform gap, the construction of the controller can be carried out exactly as described in the earlier sections. Therefore we only need to verify that condition (5.1) is satisfied. This, however, is easily accomplished using (5.16), since for all $k \in \mathbb{Z}$ with large enough $|k|$ we then have

$$\frac{\|R(i\omega_k, A_e)\|^2}{(1+\omega_k^2)^\alpha} \leq \frac{M^2|k|^{2\gamma}}{(1+c^2|k|^{2\eta})^\alpha} \leq \frac{M^2}{c^{2\alpha}}\frac{|k|^{2\gamma}}{|k|^{2\alpha\eta}} = \frac{M^2}{c^{2\alpha}} < \infty.$$

This concludes the theoretical part of this chapter. In the next section we will study a concrete example on choosing the parameters of the controller to achieve robust output regulation and disturbance rejection of signals generated by an infinite-dimensional exosystem.

5.5 Robust Controller for a Scalar System

In this section we consider the robust output regulation of a scalar system of the form

$$\begin{aligned} \dot{x}(t) &= ax(t) + bu(t), \qquad x(0) = x_0 \in \mathbb{C} \\ y(t) &= cx(t) + du(t) \end{aligned}$$

on the space $X = \mathbb{C}$. We consider the single-input single-output case and assume $b \neq 0$ and $c \neq 0$. Since the infinite-dimensional exosystem want to consider has 0 as an eigenvalue, we must also require $a \neq 0$.

As the signal generator we choose an infinite-dimensional exosystem

$$\begin{aligned} \dot{v}(t) &= Sv(t), \qquad v(0) = v_0 \in W \\ y_{ref}(t) &= F_r v(t) \end{aligned}$$

capable of generating the step signal depicted in Figure 2.4 with period $\tau = 2\pi$. We have already considered such an exosystem as an example at the end of Section 2.2. We saw that a suitable choice for the state space W of the signal generator is

$$W = \overline{\operatorname{span}} \left\{ \phi_0^1, \phi_0^2, \{\phi_k\}_{k \in \mathbb{Z} \setminus \{0\}} \right\}.$$

As the system operator S we choose

$$S = \langle \cdot, \phi_0^2 \rangle \phi_0^1 + \sum_{k \in \mathbb{Z} \setminus \{0\}} ik \langle \cdot, \phi_k \rangle \phi_k, \qquad \mathscr{D}(S) = \left\{ v \in W \;\middle|\; \sum_{k \in \mathbb{Z} \setminus \{0\}} k^2 |\langle v, \phi_k \rangle|^2 < \infty \right\}.$$

The operator S has eigenvalues $i\omega_k = ik$ for $k \in \mathbb{Z}$ and the only nontrivial Jordan block in the exosystem is the 2×2-block associated to $i\omega_0 = 0$. The output operator $F_r \in \mathscr{L}(W, \mathbb{C})$ corresponding to the reference signal is chosen in such a way that

$$F_r \phi_0^1 = 1, \qquad F_r \phi_0^2 = 0, \qquad \text{and} \qquad F_r \phi_k = \frac{1}{k}, \quad \forall k \neq 0.$$

With these choices of the parameters of the exosystem, the step function in Figure 2.4 is generated with an initial value $v_0 \in W$ satisfying

$$\langle v_0, \phi_0^1 \rangle = \frac{\pi}{2}, \qquad \langle v_0, \phi_0^2 \rangle = 1, \qquad \text{and} \qquad \langle v_0, \phi_k \rangle = \begin{cases} 0 & k \neq 0 \text{ and } k \text{ even} \\ -\frac{2}{\pi k} & k \text{ odd.} \end{cases}$$

and $v_0 \in W_\alpha$ for all $\alpha < \frac{1}{2}$.

We also remarked in Section 2.2 that if $\gamma \geq 1$, then with the appropriate choice of the initial state $v_0 \in W_{\gamma-1}$ this exosystem can be used to generate any reference signal of the form

$$y_{ref}(t) = y_1 t + y_0(t),$$

where $y_1 \in \mathbb{C}$ and $y_0(\cdot) \in H^{\gamma}_{per}(0, 2\pi)$.

The transfer function of our system is given by

$$P(\lambda) = \frac{cb}{\lambda - a} + d$$

for all $\lambda \neq a$. We assume the parameters of the plant are such that $P(i\omega_k) \neq 0$ for all $k \in \mathbb{Z}$. In particular this requires that $ad \neq bc$.

With these choices of parameters the system and the signal generator satisfy the conditions stated in Assumption 5.1. We can therefore use the method presented earlier in this chapter to construct a controller solving the robust output regulation problem related to these systems.

Choosing the Parameters of the Controller

We can now choose the first parameters in the operators $(\mathcal{G}_1, \mathcal{G}_2, K)$ of the observer-based error feedback controller in Definition 5.2. As the stabilizing feedback and output injection of the pairs (A,B) and (C,A), respectively, we will choose

$$K_{11} = -\frac{a+1}{b}, \qquad \Leftrightarrow \qquad A + BK_{11} = -1$$
$$L = -\frac{a+1}{c}, \qquad \Leftrightarrow \qquad A + LC = -1.$$

The values of the transfer function $P_K(\cdot)$ of the stabilized plant at the frequencies $i\omega_k = ik$ of the exosystem are given by

$$P_K(i\omega_k) = \frac{bc - (a+1)d}{ik+1} + d = \frac{bc + (ik-a)d}{ik+1}.$$

We choose the parameter $g_2 \in W$ of the controller as

$$g_2 = \phi_0^2 + \sum_{k \neq 0} \frac{1}{|k|^{3/2}} \phi_k.$$

This choice clearly satisfies the requirement that $\langle g_2, \phi_k^{n_k} \rangle \neq 0$ for all $k \in \mathbb{Z}$.

By Lemma 5.5 the Sylvester equation $SH = H(A+BK_{11}) + G_2(C+DK_{11})$ has a unique solution $H_{e1} \in W$ given by

$$H_{e1} = H_0^1 \phi_0^1 + H_0^2 \phi_0^2 + \sum_{k \neq 0} H_k \phi_k,$$

where

$$\begin{aligned} H_0 = \begin{pmatrix} H_0^1 \\ H_0^2 \end{pmatrix} &= \sum_{l=1}^{2} (-1)^{l-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^{l-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (C + DK_{11}) R(0, A + BK_{11})^l \\ &= (C + DK_{11}) \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \frac{1}{0 - (-1)} + (-1) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \left(\frac{1}{0 - (-1)} \right)^2 \right] \\ &= \left(c - \frac{(a+1)d}{b} \right) \begin{pmatrix} -1 \\ 1 \end{pmatrix} \end{aligned}$$

and

$$H_k = \langle g_2, \phi_k\rangle (C + DK_{11}) R(i\omega_k, A + BK_{11}) = \left(c - \frac{(a+1)d}{b} \right) \frac{1}{|k|^{3/2}} \cdot \frac{1}{ik+1}$$

for all $k \neq 0$. As was also implied by Lemma 5.5, we clearly have $H_{e1} \in \mathcal{D}(S)$.

Solvability of the Robust Output Regulation Problem

We will first estimate the asymptotic behavior of $\langle g_2, \phi_k\rangle$ and $P_K(ik)$ in order to determine the scale spaces W_α on which it is possible to solve the robust output regulation problem. Using the formulas for these terms we can see that for all $k \neq 0$ with large enough $|k|$ we have

$$|\langle g_2, \phi_k\rangle| \cdot |P_K(ik)| = \frac{1}{|k|^{3/2}} \left| \frac{bc - (a+1)d}{ik+1} + d \right| \geq \frac{1}{|k|^{3/2}} \left(|d| - \frac{|bc - (a+1)d|}{|k|} \right).$$

This shows that if $d \neq 0$, then the conditions of Theorem 5.7 are satisfied for $\beta = 3/2$, and if $d = 0$ we can choose $\beta = 5/2$. The results presented earlier in this chapter show for any $\alpha > \beta + \frac{1}{2}$ the parameters of the controller can be chosen in such a way that the robust output regulation problem is solved on W_α. In particular this suggests that Theorem 5.7 does not guarantee that we can choose the parameters of the controller to asymptotically track the step signal, since this reference signal is corresponds to an initial state $v_0 \notin W_{1/2}$ of the exosystem.

In the following we will assume $d \neq 0$ and stabilize the closed-loop system in such a way that the robust output regulation problem is solved on W_α for $\alpha = 5/2$. Due to the properties of the exosystem the controller will then be able to steer the output of the scalar system to any reference signal

$$y_{ref}(t) = y_1 t + y_0(t),$$

where $y_1 \in \mathbb{C}$ and $y_0(\cdot) \in H^{7/2}_{per}(0, 2\pi)$.

Stabilization of the Internal Model

The decomposition of the exosystem into exponentially stabilizable and diagonal parts can be accomplished by choosing $I_S = \{0\}$. We then have

$$S_f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B_f = \left(c - \frac{(a+1)d}{b} \right) \begin{pmatrix} -1 \\ 1 \end{pmatrix} b + \begin{pmatrix} 0 \\ 1 \end{pmatrix} d = \begin{pmatrix} ad - bc + d \\ -(ad - bc) \end{pmatrix},$$

and pair (S_f, B_f) is stabilizable since we have assumed $ad \neq bc$. On the other hand, the infinite-dimensional diagonal part is given by

$$S_i = \sum_{k\neq 0} ik\langle \cdot, \phi_k\rangle \phi_k, \qquad \mathscr{D}(S_i) = \left\{ v \in W \;\middle|\; \sum_{k\neq 0} k^2 |\langle v, \phi_k\rangle|^2 < \infty \right\}$$

$$B_i = \sum_{k\neq 0} \langle g_2, \phi_k\rangle P_K(i\omega_k)\phi_k = \sum_{k\neq 0} \frac{bc + (ik-a)d}{ik+1} \cdot \frac{1}{|k|^{3/2}} \cdot \phi_k.$$

We will first stabilize the pair (S_f, B_f) with a feedback K_{f1}. For this purpose we will choose

$$K_{f1} = \frac{1}{(ad-bc)^2}\begin{pmatrix} 2(ad-bc) & 5(ad-bc)+2d \end{pmatrix}.$$

It is well-known that since $\sigma(S_f) = \{0\}$, a value $\lambda \neq 0$ is an eigenvalue of $S_f + B_f K_{f1}$ if and only if $K_{f1}R(\lambda, S_f)B_f = 1$. A direct computation shows that

$$1 - K_{f1}R(\lambda, S_f)B_f = 1 - \frac{1}{\lambda^2}K_{f1}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} B_f = 1 + \frac{3\lambda+2}{\lambda^2} = \frac{\lambda^2+3\lambda+2}{\lambda^2} = \frac{(\lambda+2)(\lambda+1)}{\lambda^2},$$

and therefore $\sigma(S_f + B_f K_{f1}) = \{-1, -2\}$. This concludes that $S_f + B_f K_{f1}$ is exponentially stable.

From the proof of Theorem 5.6 we have that for $k \neq 0$

$$\langle B_2, \phi_k\rangle = \left(K_{f1}R(ik, S_f + B_f K_{f1})B_f + 1\right)\langle B_i, \phi_k\rangle = \frac{\langle g_2, \phi_k\rangle P_K(ik)}{1 - K_{f1}R(ik, S_f)B_f}$$

$$= -\frac{1}{|k|^{3/2}} \cdot \frac{bc + (ik-a)d}{ik+1} \cdot \frac{k^2}{(ik+2)(ik+1)} = -\frac{|k|^{1/2}(bc + (ik-a)d)}{(ik+1)^2(ik+2)}.$$

As in the proof of Theorem 5.6, we stabilize the infinite part of the internal model with a feedback operator of the form $K_i = \langle \cdot, h\rangle$ with

$$h = \sum_{k\neq 0} h_k \phi_k,$$

where we choose the parameter $\gamma = 5/2$ and

$$\overline{h_k} = \frac{1}{|k|^{5/2}\langle B_2, \phi_k\rangle} \prod_{l\neq 0,k} \left(1 + i\frac{1}{|l|^{5/2}(\omega_l - \omega_k)}\right)$$

$$= -\frac{(ik+1)^2(ik+2)}{|k|^3(bc + (ik-a)d)} \prod_{l\neq 0,k} \left(1 + i\frac{1}{|l|^{5/2}(l-k)}\right).$$

With these choices of the parameters Theorem 5.6 states that the resolvent operator of the closed-loop system has asymptotic behavior

$$\|R(ik, A_e)\| = \mathscr{O}(|k|^{5/2}), \tag{5.17}$$

and by Theorem 5.7 the dynamic error feedback controller solves the robust output regulation problem on W_α with $\alpha = 5/2$.

Robustness Properties of the Controller

By construction, the controlled system is capable of tracking the reference signals and rejecting the disturbances despite perturbations to the parameters A, B, C, D, E, and F of the system, provided that the strong stability of the closed-loop system and the solvability of the associated Sylvester equation are preserved. In particular this allows changing the constants a, b, c, and d in such a way that the stability of the closed-loop system is preserved, the condition $\sigma(A_e) \cap \sigma(S) = \varnothing$ remains valid, and the resolvent operator of the closed-loop system still has the asymptotic behavior (5.17).

As was mentioned earlier, the chosen controller is capable of steering the output of the scalar system to any reference signal of the form

$$y_{ref}(t) = y_1 t + y_0(t),$$

where $y_1 \in \mathbb{C}$ and $y_0(\cdot) \in H^{7/2}_{per}(0, 2\pi)$. The robustness of the controller with respect to perturbations to the operator F_r further enlarges this class of signals. Since this operator (and the operator E) do not appear in the system operator A_e of the closed-loop system, they do not affect the stability of the closed-loop system or the sufficient conditions for the solvability of the Sylvester equation. This means that the operator F_r can be replaced with an arbitrary operator F_r' as long as it satisfies the condition $(F_r'\phi_k) \in \ell^2(\mathbb{C})$ imposed on the output operator of the infinite-dimensional exosystem. Therefore a similar consideration as in Example 2.8 allows us to observe that our controlled system is actually capable of tracking any reference signal of the above form where $y_0(\cdot) \in H^{\gamma}_{per}(0, 2\pi)$ for some $\gamma > 5/2$.

The same conclusion also applies to the perturbations of the operator $E = 0$. This means that even though we were originally not interested in rejecting disturbance signals affecting the state of the system, our controller is still able to handle any such signals generated by the infinite-dimensional exosystem with an output operator $E \in \mathscr{L}(W, \mathbb{C})$ satisfying $(E\phi_k)_k \in \ell^2(\mathbb{C})$.

Chapter 6

The Infinite-Dimensional Sylvester Differential Equation

Before moving on to the final part of this thesis where we study the theory of output regulation for linear systems with periodic exosystems, we will consider one of the cornestones of the theory separately. This chapter is dedicated to the study of the infinite-dimensional *Sylvester differential equation*

$$\dot{\Sigma}(t) + \Sigma(t)S(t) = A_e(t)\Sigma(t) + B_e(t). \tag{6.1}$$

In the course of the subsequent chapters we will see that the role of this time-dependent operator differential equation in the study of periodic output regulation is to take the place of the Sylvester operator equation in the time-dependent regulator equations. We consider the solvability of (6.1) separately to avoid this question from interrupting the development of the output regulation theory in Chapter 7. This arrangement also allows us to, for the reader's convenience, take the time to recall the theory of strongly continuous evolution families and nonautonomous abstract Cauchy problems. These concepts will be essential in the study of both the solvability of the Sylvester differential equation and the problem of output regulation of infinite-dimensional systems with periodic exosystems.

In particular in the context of output regulation, one of the key differences between the theories of Sylvester differential equations and Sylvester operator equations is that in the time-dependent case the mild solvability of the equation no longer implies its solvability in a stronger sense. As we saw in Lemma 3.6, this is always true for Sylvester operator equations. We will also see that in order to guarantee the solvability of (6.1) in the classical sense we need considerably stronger assumptions than for its mild solvability. However — as is also customary in the theory of time-invariant distributed parameter systems — we will mainly be interested in the mild states of the infinite-dimensional systems encountered in the following chapters. For this reason, the mild solvability of the Sylvester differential equation (6.1) is sufficient for the purposes of periodic output regulation. Even so, to better understand these types of equations we will use the main part of this chapter to study their classical solutions.

We take the advantage of considering the Sylvester differential equation in a separate chapter and study its solvability under more general assumptions than those related to the output regulation problem in the next chapter. In particular, we will not use the fact that $S(\cdot)$, being related to the periodic exosystem, is a matrix-valued function. Instead, we consider equation (6.1) in a situation where $(S(t), \mathscr{D}(S(t)))$ is a family of unbounded operators on a Banach space.

The organization and the main contributions of this chapter are outlined in the following.

Section 6.1. In this section we recall the definition of a strongly continuous evolution family related to a nonautonomous abstract Cauchy problem and state the standing assumptions on the families of operators $(A_e(t), \mathscr{D}(A_e(t)))$ and $(S(t), \mathscr{D}(S(t)))$.

Section 6.2. We first consider the solvability of the infinite-dimensional Sylvester differential equation on an interval $[0, \tau]$. We present sufficient conditions for classical solvability of the equation, and use the form of the classical solution to define its mild solution.

Section 6.3. We conclude the chapter by considering the periodic version of the Sylvester differential equation. We present conditions under which it has a unique periodic mild solution.

The treatment of the Sylvester differential equation presented in this chapter generalizes the results on the solvability of finite-dimensional equations of this type [60, 28] and the solvability of the infinite-dimensional equation in the special case where the operators $A(t) \equiv A$ and $S(t) \equiv S$ are generators of strongly continuous semigroups [9, 8]. For our purposes it is also relevant that in the time-invariant case the equation becomes an infinite-dimensional Sylvester operator equation [2, 43, 44]. Our approach to solving the Sylvester differential equation (6.1) generalizes the method used in [43].

6.1 Strongly Continuous Evolution Families

In this section we recall the definition and certain fundamental properties of strongly continuous evolution families related to nonautonomous abstract Cauchy problems. We will see that these operator-valued functions are instrumental in studying nonautonomous infinite-dimensional differential equations. They can, in fact, be used in much the same way as strongly continuous semigroups in the time-invariant case. However, the theory of evolution families is at times considerably weaker than the theory of semigroups. In particular, the differentiation rules for evolution families can be used only under fairly strong assumption, whereas the corresponding properties are guaranteed for all semigroups. For a more detailed introduction to the theory of evolution families the reader is referred to [42, Ch. 5], [10, Sec. VI.9], and [50].

Definition 6.1 (A Strongly Continuous Evolution Family). A family of bounded linear operators $(U(t,s))_{t\geq s} \subset \mathscr{L}(X)$ is called a *strongly continuous evolution family* if it satisfies the conditions

(a) $U(s,s) = I$ for $s \in \mathbb{R}$.

(b) $U(t,s) = U(t,r)U(r,s)$ for $t \geq r \geq s$.

(c) $\left\{ (t,s) \in \mathbb{R}^2 \;\middle|\; t \geq s \right\} \ni (t,s) \mapsto U(t,s)$ is a strongly continuous mapping.

A strongly continuous evolution family is called *exponentially bounded* if there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|U(t,s)\| \leq Me^{\omega(t-s)}$$

for all $t \geq s$. The evolution family is called *periodic* (with period τ) if for all $t \geq s$

$$U(t+\tau, s+\tau) = U(t,s)$$

■

Whereas strongly continuous semigroups are related to autonomous abstract Cauchy problems, the strongly continuous evolution families are related to their nonautonomous counterparts. Consider such an equation

$$\dot{x}(t) = A(t)x(t) + f(t) \tag{6.2a}$$

$$x(s) = x_s \in X, \tag{6.2b}$$

and assume $U(t,s)$ is the associated strongly continuous evolution family. Then, if for some $s \in \mathbb{R}$ this equation has a classical solution $x(\cdot) \in C^1([s,\infty), X)$ satisfying $x(t) \in \mathscr{D}(A(t))$ for all $t \geq s$, this solution is given by

$$x(t) = U(t,s)x_s + \int_s^t U(t,r)f(r)ds \tag{6.3}$$

for all $t \geq s$. This relationship reveals that we have, in fact, already come across a strongly continuous evolution family in this thesis. Indeed, if the underlying space is finite-dimensional, then the evolution family associated to the family of operators is precisely the fundamental matrix of the corresponding nonautonomous differential equation. This was exactly the case in Section 2.3, where the state of the periodic exosystem was determined by the evolution family $U_S(t,s)$.

The relationship between the strongly continuous evolution families and the corresponding nonautonomous Cauchy problems, however, is not as clear as in the case of autonomous equations [32]. For time-invariant equations the connection between the semigroup and the associated differential equation is an immediate consequence of the differentiation rule

$$\frac{d}{dt}T_A(t)x = AT_A(t)x = T_A(t)Ax \qquad \forall x \in \mathcal{D}(A).$$

This property of the semigroup implies that the orbits $t \mapsto T_A(t)x$ are classical solutions of the associated abstract Cauchy problem for $x \in \mathcal{D}(A)$, and it is also well-known that they are mild solutions of the equation for all $x \in X$. However, such differentiation rules are satisfied only by certain special classes of evolution families under fairly restrictive assumptions [51], [42, Ch. 5].

In this thesis we are mainly interested in the mild solutions of the differential equation (6.2). For this the main difficulty is in finding an appropriate sense in which the orbits $t \mapsto U(t,s)x_s$ are solutions of the homogeneous equation

$$\dot{x}(t) = A(t)x(t) \tag{6.4a}$$

$$x(s) = x_s \in X. \tag{6.4b}$$

If this is made clear, then the formula (6.3) can be used to define a mild solution of the inhomogeneous equation (6.2). We choose to associate an evolution family $U(t,s)$ to the nonautonomous abstract Cauchy problem (6.4) by requiring that the evolution family and the family $(A(t),\mathcal{D}(A(t)))$ of operators are related by a certain weak differentiation rule. The definition guarantees that the orbits $t \mapsto U(t,s)x_s$ solve a particular weak version of the differential equation (6.4). This definition is convenient for our purposes, but it should be noted that the theory of output regulation developed in the subsequent chapters does not depend on the precise way of associating the evolution family and the corresponding nonautonomous abstract Cauchy problem.

Definition 6.2. We call $U(t,s)$ *an evolution family associated to the family* $(A(t),\mathcal{D}(A(t)))$ *of operators* if the following are satisfied.

1. There exists a dense subspace $\mathcal{D}^*$ of X^* such that
$$\mathcal{D}^* \subset \bigcap_{t\in\mathbb{R}} \mathcal{D}(A(t)^*).$$
2. For all $s \in \mathbb{R}$ and for any $x \in X$ and $x^* \in \mathcal{D}^*$ the mapping
$$t \mapsto \langle U(t,s)x, x^*\rangle : [s,\infty) \to \mathbb{C}$$
is absolutely continuous and for all almost all $t \in (s,\infty)$ we have
$$\frac{d}{dt}\langle U(t,s)x, x^*\rangle = \langle U(t,s)x, A(t)^*x^*\rangle.$$

■

We will see in Chapter 8 that for the purposes of output regulation with periodic exosystems it is important that Definition 6.2 covers the situation where the operators $A(t)$ are of the form

$$A(t) = A_0 + A_1(t), \qquad \mathscr{D}(A(t)) \equiv \mathscr{D}(A_0), \tag{6.5}$$

where $A_0 : \mathscr{D}(A_0) \subset X \to X$ generates a strongly continuous semigroup $T_0(t)$ on X and the function $A_1(\cdot) \in C(\mathbb{R}, \mathscr{L}(X))$ is periodic. We will see in Theorem 8.3 that in the case of (6.5) there indeed exists a strongly continuous evolution family $U(t,s)$ associated to the family $(A(t), \mathscr{D}(A(t)))$ of operators in the sense of Definition 6.2, and that we can choose $\mathscr{D}^* = \mathscr{D}(A_0^*)$. For such operators the evolution family $U(t,s)$ can actually be considered as a perturbation of the semigroup $T_0(t)$, and it in particular satisfies the integral equation [10, Thm. VI.9.19]

$$U(t,s)x = T_0(t-s)x + \int_s^t T_0(t-r)A_1(r)U(r,s)x\,dr, \qquad x \in X.$$

As we mentioned above, we define the mild solution of the nonhomogeneous differential equation using the formula (6.3).

Definition 6.3 (Mild solution)**.** If $U(t,s)$ is a strongly continuous evolution family associated to the family $(A(t), \mathscr{D}(A(t)))$ of operators and if $f \in C(\mathbb{R}, X)$, then for any $s \in \mathbb{R}$ the function $x(\cdot) \in C([s,\infty), X)$ defined by (6.3) is called the *mild solution* of the nonautonomous abstract Cauchy problem (6.2). ■

In Chapters 7 and 8 we will mainly be interested in the case where the families $(A_e(t), \mathscr{D}(A_e(t)))$ and $(S(t), \mathscr{D}(S(t)))$ of unbounded operators are periodic. For this reason it is also worthwhile to note that if a family of operators is periodic, then also the corresponding evolution family is periodic in the sense of Definition 6.1, and the lengths of their periods are equal.

Throughout this chapter — as well as the next two chapters — we assume that there exist strongly continuous evolution families $U_e(t,s)$ and $U_S(t,s)$ associated to our families of operators. We also make an assumption corresponding to the property that the nonautonomous Cauchy problem associated to the family $(S(t), \mathscr{D}(S(t)))$ of operators can be solved forward and backwards in time. The exact conditions are listed in the following.

Assumption 6.4. *There exist exponentially bounded strongly continuous evolution families $U_e(t,s)$ and $U_S(t,s)$ associated to the families $(A_e(t), \mathscr{D}(A_e(t)))$ and $(S(t), \mathscr{D}(S(t)))$ of operators, respectively. The evolution family $U_S(t,s)$ satisfies the conditions of Definition* 6.1 *for all $t,s,r \in \mathbb{R}$. The operator-valued function $B_e(\cdot)$ is strongly continuous.*

6.2 The Infinite-Dimensional Sylvester Differential Equation

In this section we consider the infinite-dimensional Sylvester differential equation

$$\dot{\Sigma}(t) + \Sigma(t)S(t) = A_e(t)\Sigma(t) + B_e(t), \qquad \Sigma(0) = \Sigma_0 \in \mathscr{L}(W, X_e) \tag{6.6}$$

on the interval $[0, \tau]$. We mainly concentrate on finding conditions under which the equation is solvable in the classical sense. We can subsequently use the form of this solution to define the mild solution of (6.6). The results presented in this section will be used in the next section to study the periodic Sylvester differential equation.

If the spaces X_e and W are finite-dimensional and if the operator-valued functions appearing in (6.6) are continuous — as is the situation in [60] — the solvability of the Sylvester differential equation is a fairly straightforward matter. In fact, in this situation equation (6.6) is nothing more than a system of ordinary differential equations. In the infinite-dimensional case, however, the solvability of the equation becomes a much more interesting problem. For example, in the strict sense equation (6.6) is only meaningful if for all $t \in [0, \tau]$ we have $\Sigma(t)v \in \mathscr{D}(A_e(t))$ whenever $v \in \mathscr{D}(S(t))$. We also need more sophisticated conditions to be able to apply the Leibniz integral rule, which is the core of the proof for the classical solvability of (6.6). The use of this result in the finite-dimensional case requires continuity of the functions $A_e(\cdot)$, $S(\cdot)$ and $B_e(\cdot)$. For families of unbounded operators the concept of continuity is not as straightforward a matter as it is for matrix-valued functions. We will see that the solvability of the Sylvester differential equation (6.6) in this classical sense requires strong assumptions on the evolution families and functions involved. Fortunately, as already remarked, it turns out that for the purposes of periodic output regulation problem it is sufficient to consider a weaker type of solution of this equation, more precisely the mild solution defined at the end of this section.

We begin by defining the classical solution of the Sylvester differential equation (6.6) on the interval $[0, \tau]$. For simplicity we consider this type of solvability only in the case where the domains of the operators $S(t)$ are independent of time, i.e., $\mathscr{D}(S(t)) =: \mathscr{D}(S)$ for all $t \in \mathbb{R}$.

Definition 6.5. A strongly continuous function $\Sigma(\cdot) \in C([0, \tau], \mathscr{L}_s(W, X_e))$ that satisfies $\Sigma(\cdot)v \in C^1([0, \tau], X_e)$ and $\Sigma(t)v \in \mathscr{D}(A_e(t))$ for all $v \in \mathscr{D}(S)$ and $t \in [0, \tau]$ is called *a classical solution* of the Sylvester differential equation (6.6) if $\Sigma(0) = \Sigma_0$ and if for all $v \in \mathscr{D}(S)$

$$\frac{d}{dt}\Sigma(t)v + \Sigma(t)S(t)v = A_e(t)\Sigma(t)v + B_e(t)v$$

on the interval $[0, \tau]$. ■

The following theorem is the main result of this section. It presents sufficient conditions for the classical solvability of the Sylvester differential equation and describes the form of the solution. The *parabolic conditions* which the family $(A_e(t), \mathscr{D}(A_e(t)))$ is assumed to satisfy essentially require that for all $t \in [0,\tau]$ the operators $A_e(t)$ are generators of analytic semigroups on X_e.

Theorem 6.6. *Assume the following are satisfied.*

1. *There exists* $\mu \in \mathbb{R}$ *such that* $U_e(t,s)$ *satisfies the* parabolic conditions:

   (P$_1$) *The domain* $\mathscr{D}(A_e(t)) =: \mathscr{D}(A_e)$ *is independent of* $t \in [0,\tau]$ *and dense in* X_e.

   (P$_2$) *We have* $\{ \lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geq \mu \} \subset \rho(A_e(t))$ *for every* $t \in [0,\tau]$. *Furthermore, there exists a constant* $M \geq 1$ *such that*
   $$\left\|R(\lambda, A_e(t))\right\| \leq \frac{M}{|\lambda - \mu| + 1}$$
   for all $t \in [0,\tau]$ *and for all* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda \geq \mu$.

   (P$_3$) *There exists a constant* $L \geq 0$ *such that for all* $t,s,r \in [0,\tau]$
   $$\left\|(A_e(t) - A_e(s))R(\mu, A_e(r))\right\| \leq L|t-s|.$$

2. *The domain* $\mathscr{D}(A_e(t)^*) =: \mathscr{D}(A_e^*)$ *is independent of* $t \in [0,\tau]$ *and dense in* X_e^*. *For all* $x \in X_e$ *and* $x^* \in \mathscr{D}(A_e^*)$ *the mapping*
   $$t \mapsto \langle x, A_e(t)^* x^* \rangle$$
   is continuous on $[0,\tau]$.

3. *The domain* $\mathscr{D}(S(t)) =: \mathscr{D}(S)$ *is independent of* $t \in [0,\tau]$ *and dense in* W. *For every* $v \in \mathscr{D}(S)$ *the function* $S(\cdot)v$ *is continuous, we have* $U_S(t,s)v \in \mathscr{D}(S)$, *and the evolution family* $U_S(t,s)$ *satisfies the differentiation rules*
   $$\frac{\partial}{\partial s} U_S(t,s)v = -U_S(t,s)S(s)v, \qquad \frac{\partial}{\partial t} U_S(t,s)v = S(t)U_S(t,s)v.$$
   for all $t,s \in [0,\tau]$.

4. *For every* $v \in W$ *the function* $B_e(\cdot)v$ *is Hölder continuous on* $[0,\tau]$.

5. *The operator* $\Sigma_0 \in \mathscr{L}(W, X_e)$ *satisfies* $\Sigma_0(\mathscr{D}(S)) \subset \mathscr{D}(A_e)$.

Then the infinite-dimensional Sylvester differential equation (6.6) *on* $[0,\tau]$ *has a unique classical solution* $\Sigma(\cdot)$ *given by the formula*

$$\Sigma(t)v = U_e(t,0)\Sigma_0 U_S(0,t)v + \int_0^t U_e(t,s)B_e(s)U_S(s,t)v\,ds, \qquad v \in W \tag{6.7}$$

for $t \in [0,\tau]$.

Proof. Since $U_e(t,s)$ satisfies the parabolic conditions, we have from [42, Sec. 5.6] that for all $x \in X_e$, $y \in \mathscr{D}(A_e)$, and $t > s$

$$\frac{\partial}{\partial t}U_e(t,s)x = A_e(t)U_e(t,s)x, \qquad \frac{\partial}{\partial s}U_e(t,s)y = -U_e(t,s)A_e(s)y.$$

Let $v \in \mathscr{D}(S)$, $x^* \in \mathscr{D}(A_e^*)$, and $s \in [0,\tau]$. Using the above differentiation rules we see that for any $t \in (s,\tau]$ we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\langle U_e(t,s)B_e(s)U_S(s,t)v, x^*\rangle \\
=\ &\langle A_e(t)U_e(t,s)B_e(s)U_S(s,t)v, x^*\rangle - \langle U_e(t,s)B_e(s)U_S(s,t)S(t)v, x^*\rangle \\
=\ &\langle U_e(t,s)B_e(s)U_S(s,t)v, A_e(t)^*x^*\rangle - \langle U_e(t,s)B_e(s)U_S(s,t)S(t)v, x^*\rangle \\
&\frac{\partial}{\partial t}\langle U_e(t,0)\Sigma_0 U_S(0,t)v, x^*\rangle \\
=\ &\langle A_e(t)U_e(t,0)\Sigma_0 U_S(0,t)v, x^*\rangle - \langle U_e(t,0)\Sigma_0 U_S(0,t)S(t)v, x^*\rangle \\
=\ &\langle U_e(t,0)\Sigma_0 U_S(0,t)v, A_e(t)^*x^*\rangle - \langle U_e(t,0)\Sigma_0 U_S(0,t)S(t)v, x^*\rangle.
\end{aligned}$$

We will use the Leibniz integral rule to show the function $\Sigma(\cdot)$ defined by (6.7) is a solution of the Sylvester differential equation. This result states that if a function

$$f : \left\{ (t,s) \;\middle|\; 0 \le s \le t \le \tau \right\} \to \mathbb{C}$$

is continuous in t and s, and if $\frac{\partial}{\partial t}f(t,s)$ exists and is continuous and uniformly bounded on the set $\left\{ (t,s) \;\middle|\; 0 \le s < t \le \tau \right\}$, then the mapping $t \mapsto \int_0^t f(t,s)ds$ is differentiable on $(0,\tau)$ and

$$\frac{d}{dt}\int_0^t f(t,s)ds = f(t,t) + \int_0^t \frac{\partial}{\partial t}f(t,s)ds.$$

Our assumptions imply that the function

$$(t,s) \to f(t,s) = \langle U_e(t,s)B_e(s)U_S(s,t)v, x^*\rangle$$

is continuous for $0 \le s \le t \le \tau$, and the computation above shows that it is continuously differentiable with respect to t. Since the mappings $(t,s) \to U_e(t,s)$ and $(t,s) \to U_S(s,t)$ are strongly continuous, there exist constants $M_e, M_S > 0$ such that

$$\max_{0\le s\le t\le\tau}\|U_e(t,s)\| \le M_e, \qquad \max_{0\le s\le t\le\tau}\|U_S(s,t)\| \le M_S.$$

These properties conclude that we can indeed use the Leibniz integral rule, since

$$\begin{aligned}
\left|\frac{\partial}{\partial t}f(t)\right| &\le \|U_e(t,s)B_e(s)U_S(s,t)v\| \cdot \|A_e(t)^*x^*\| + \|U_e(t,s)B_e(s)U_S(s,t)S(t)v\| \cdot \|x^*\| \\
&\le \|U_e(t,s)\| \cdot \|B_e(s)\| \cdot \|U_S(s,t)\| \left(\|v\| \cdot \|A_e(t)^*x^*\| + \|S(t)v\| \cdot \|x^*\|\right) \\
&\le M_e M_S \cdot \max_{r\in[0,\tau]}\|B_e(r)\| \Big(\|v\| \cdot \max_{r\in[0,\tau]}\|A_e(r)^*x^*\| + \|x^*\| \cdot \max_{r\in[0,\tau]}\|S(r)v\|\Big) < \infty.
\end{aligned}$$

For the function $\Sigma(\cdot)$ defined by (6.7) we now have

$$\frac{d}{dt}\langle\Sigma(t)v,x^*\rangle = \frac{\partial}{\partial t}\langle U_e(t,0)\Sigma_0 U_S(0,t)v,x^*\rangle + \frac{\partial}{\partial t}\int_0^t \langle U_e(t,s)B_e(s)U_S(s,t)v,x^*\rangle ds$$

$$= \langle U_e(t,0)\Sigma_0 U_S(0,t)v, A_e(t)^*x^*\rangle - \langle U_e(t,0)\Sigma_0 U_S(0,t)S(t)v,x^*\rangle$$

$$+ \int_0^t \Big(\langle U_e(t,s)B_e(s)U_S(s,t)v, A_e(t)^*x^*\rangle - \langle U_e(t,s)B_e(s)U_S(s,t)S(t)v,x^*\rangle\Big)ds$$

$$+ \langle U_e(t,t)B_e(t)U_S(t,t)v,x^*\rangle$$

$$= \langle\Sigma(t)v, A_e(t)^*x^*\rangle - \langle\Sigma(t)S(t)v,x^*\rangle + \langle B_e(t)v,x^*\rangle. \tag{6.8}$$

This shows that the function $\Sigma(\cdot)$ satisfies the Sylvester differential equation (6.6) in a certain weaker sense. In order to show that it is a classical solution of this equation, we will need to study the differentiability properties of $\Sigma(\cdot)$ in greater detail.

We will next show that the mapping $t \mapsto \Sigma(t)v$ is continuously differentiable on the interval $(0,\tau)$, and that $\Sigma(t)v \in \mathcal{D}(A_e)$ for all $t \in [0,\tau]$. We will do this by first considering the nonautonomous Cauchy problem

$$\dot{x}(t) = A_e(t)x(t) + B_e(t)U_S(t,0)w, \qquad x(0) = \Sigma_0 w \in \mathcal{D}(A_e)$$

for $w \in \mathcal{D}(S)$. Since $x(0) \in \mathcal{D}(A_e)$ and since $t \mapsto B_e(t)U_S(t,0)w$ is Hölder continuous on the interval $[0,\tau]$, we have from [42, Thm. 5.7.1] that this equation has a unique classical solution on $[0,\tau]$ given by

$$x(t) = U_e(t,0)\Sigma_0 w + \int_0^t U_e(t,s)B_e(s)U_S(s,0)w\,ds.$$

The fact that it is a classical solution implies that $x(\cdot)$ is continuously differentiable on $(0,\tau)$ and $x(t) \in \mathcal{D}(A_e)$ for all $t \in [0,\tau]$. If we denote by $H(\cdot) : [0,\tau] \to \mathcal{L}(W,X_e)$ the strongly continuous mapping $x(t) = H(t)w$ defined by the above formula, then for all $w \in \mathcal{D}(S)$ the function $t \mapsto H(t)w$ is continuously differentiable on $(0,\tau)$ and $H(t)w \in \mathcal{D}(A_e)$ for all $t \in [0,\tau]$. Since the mapping $t \mapsto U_S(0,t)v$ is continuously differentiable, the choice $w = U_S(0,t)v \in \mathcal{D}(S)$ and a straightforward computation finally show that the function

$$\begin{aligned} t \mapsto H(t)U_S(0,t)v &= U_e(t,0)\Sigma_0 U_S(0,t)v + \int_0^t U_e(t,s)B_e(s)U_S(s,0)U_S(0,t)v\,ds \\ &= U_e(t,0)\Sigma_0 U_S(0,t)v + \int_0^t U_e(t,s)B_e(s)U_S(s,t)v\,ds \\ &= \Sigma(t)v \end{aligned}$$

is continuously differentiable on $(0,\tau)$ and $\Sigma(t)v \in \mathcal{D}(A_e)$ for all $[0,\tau]$. Using these properties we can write equation (6.8) as

$$\langle \frac{d}{dt}\Sigma(t)v,x^*\rangle + \langle\Sigma(t)S(t)v,x^*\rangle = \langle A_e(t)\Sigma(t)v,x^*\rangle + \langle B_e(t)v,x^*\rangle.$$

Since $x^* \in \mathscr{D}(A_e^*)$ was arbitrary and since $\mathscr{D}(A_e^*)$ is dense in X_e^*, this further implies

$$\frac{d}{dt}\Sigma(t)v + \Sigma(t)S(t)v = A_e(t)\Sigma(t)v + B_e(t)v.$$

Since $v \in W$ was arbitrary, this concludes that $\Sigma(\cdot)$ is a classical solution of the Sylvester differential equation (6.6).

To prove the uniqueness of the solution, let $\Sigma_1(\cdot) \in C^1([0,\tau], \mathscr{L}_s(W,X))$ be a classical solution of the Sylvester differential equation (6.6). Letting $v \in \mathscr{D}(S)$ and applying both sides of equation (6.6) to $U_S(s,t)v \in \mathscr{D}(S)$ for $t > s$ we obtain

$$\left[\frac{d}{dr}\Sigma_1(r)U_S(s,t)v\right]_{r=s} + \Sigma_1(s)S(s)U_S(s,t)v = A_e(s)\Sigma_1(s)U_S(s,t)v + B_e(s)U_S(s,t)v$$

$$\Rightarrow \quad \left[\frac{d}{dr}U_e(t,s)\Sigma_1(r)U_S(s,t)v\right]_{r=s} + U_e(t,s)\Sigma_1(s)S(s)U_S(s,t)v$$
$$= U_e(t,s)A_e(s)\Sigma_1(s)U_S(s,t)v + U_e(t,s)B_e(s)U_S(s,t)v$$

$$\Rightarrow \quad \frac{d}{ds}\big(U_e(t,s)\Sigma_1(s)U_S(s,t)v\big) = U_e(t,s)B_e(s)U_S(s,t)v.$$

Integrating both sides of the last equation from 0 to t and using $\Sigma_1(0) = \Sigma_0$ shows that

$$\begin{aligned}\int_0^t U_e(t,s)B_e(s)U_S(s,t)v\,ds &= U_e(t,t)\Sigma_1(t)U_S(t,t)v - U_e(t,0)\Sigma_1(0)U_S(0,t)v \\ &= \Sigma_1(t)v - U_e(t,0)\Sigma_0 U_S(0,t)v,\end{aligned}$$

and thus $\Sigma_1(\cdot) = \Sigma(\cdot)$ on $\mathscr{D}(S)$. Since $\mathscr{D}(S)$ is dense in W, this concludes the proof. □

The following example illustrates the nature of the parabolic conditions which the family $(A_e(t), \mathscr{D}(A_e(t)))$ of operators was assumed to satisfy in Theorem 6.6.

Example 6.7. Let $\alpha(\cdot), \gamma(\cdot) \in C([0,\tau],\mathbb{R})$ be Lipschitz continuous functions and assume that $\alpha(t) > 0$ for all $t \in [0,\tau]$. Consider a one-dimensional heat equation with time-varying coefficients

$$\begin{aligned}\frac{\partial x}{\partial t}(z,t) &= \alpha(t)\frac{\partial^2 x}{\partial z^2}(z,t) + \gamma(t)x(z,t), \\ x(z,0) &= x_0(z) \\ x(0,t) &= x(1,t) = 0\end{aligned}$$

on the interval $[0,1]$. This equation can be written as a nonautonomous abstract Cauchy problem

$$\dot{x}(t) = A(t)x(t), \qquad x(t) = x_0 \in X$$

on the space $X = L^2(0,1)$, where the family $(A(t), \mathcal{D}(A))$ of unbounded operators is such that

$$A(t)x = \alpha(t)x'' + \gamma(t)x,$$

$$\mathcal{D}(A) = \left\{ x \in X \;\middle|\; x,\, x' \text{ absolutely cont.},\; x'' \in L^2(0,1),\; x(0) = x(1) = 0 \right\}.$$

The operators $A(t)$ have spectral decompositions [7, Ex. A.4.26]

$$A(t)x = \sum_{n=1}^{\infty} \lambda_n(t)\langle x, \phi_n\rangle \phi_n, \qquad x \in \mathcal{D}(A) = \left\{ x \in X \;\middle|\; \sum_{n=1}^{\infty} n^4 |\langle x, \phi_n\rangle|^2 < \infty \right\},$$

where the eigenvalues are given by $\lambda_n(t) = -\alpha(t)n^2\pi^2 + \gamma(t)$ and the corresponding eigenvectors $\phi_n = \sqrt{2}\sin(n\pi\cdot)$ form an orthonormal basis of X. These decompositions and the fact that $\alpha(\cdot)$ and $\gamma(\cdot)$ are Lipschitz continuous functions can be used to verify that the parabolic conditions are satisfied.

We can also show that this family of operators satisfies the second condition in Theorem 6.6. Since the operators $A(t)$ are self-adjoint, we can achieve this by showing that the mapping $t \mapsto A(t)x$ is continuous for all elements $x \in \mathcal{D}(A)$. If we define an operator $A_0 : \mathcal{D}(A) \subset X \to X$ by $A_0 x = x''$, we can write

$$A(t)x = \alpha(t)A_0 x + \gamma(t)x, \qquad x \in \mathcal{D}(A).$$

Since $\alpha(\cdot)$ and $\gamma(\cdot)$ are continuous, we can conclude that $t \mapsto A(t)x$ is continuous for all $x \in \mathcal{D}(A)$, and thus the second condition in Theorem 6.6 is satisfied. ■

Families of operators satisfying the conditions on $(S(t), \mathcal{D}(S))$ include, for example, all functions $S(\cdot) \in C([0,\tau], \mathcal{L}_s(W))$ and the case where $S(t) \equiv S$ is a generator of a strongly continuous group on W.

As was already mentioned, the parabolic conditions in Theorem 6.6 require that for all $t \in [0,\tau]$ the operators $A_e(t)$ generate analytic semigroups on X_e. In the next chapter it will become clear that in output regulation this is only possible if the system operator A of the plant is a generator of an analytic semigroup. Fortunately the following weaker form of solvability of Sylvester differential equations will be sufficient for the purposes of output regulation of infinite-dimensional systems with periodic exosystems.

Definition 6.8. Under the conditions of Assumption 6.4 the operator-valued function $\Sigma(\cdot) \in C([0,\tau], \mathcal{L}_s(W, X_e))$ defined by (6.7) is called the *mild solution* of the Sylvester differential equation (6.6) on $[0,\tau]$. ■

We will now turn to our main interest, the periodic version of the Sylvester differential equation. In the next chapter we will see that the periodic solutions of this equation can be used to characterize the solvability of an output regulation problem related to a distributed parameter system with a periodic exosystem. The existence of such solutions is the main topic of the next section.

6.3 The Periodic Sylvester Differential Equation

In this section we consider the *periodic* Sylvester differential equation, meaning the equation

$$\dot{\Sigma}(t) + \Sigma(t)S(t) = A_e(t)\Sigma(t) + B_e(t) \tag{6.9}$$

on $\mathbb{R}$ in a situation where the families $(A_e(t), \mathscr{D}(A_e(t)))$ and $(S(t), \mathscr{D}(S(t)))$ of operators and the operator-valued function $B_e(\cdot)$ are periodic with the same period $\tau > 0$. A periodic solution of this equation is a mild or a classical solution $\Sigma(\cdot)$ of the Sylvester differential equation (6.6) corresponding to an initial condition for which $\Sigma(\cdot)$ is a periodic function. This is made more precise in the following.

Definition 6.9. A *periodic solution* of the Sylvester differential equation (6.9) is a periodic function $\Sigma(\cdot)$ that is a solution of (6.6) with $\Sigma_0 = \Sigma(0)$ on all intervals $[0, \tau']$ with $\tau' > 0$.

The following theorem is the main result of this section. It states that under the standing assumptions and a suitable condition on the growths of the evolution families, the periodic Sylvester differential equation (6.9) has a unique periodic mild solution, and that the length of the period of this solution is τ.

Theorem 6.10. *Let the evolution families $U_e(t,s)$ and $U_S(t,s)$ and the function $B_e(\cdot)$ be periodic with period $\tau > 0$. If there exist constants $M_e, M_S \geq 1$ and $\omega_e, \omega_S \in \mathbb{R}$ such that $\omega_e + \omega_S < 0$ and*

$$\|U_e(t,s)\| \leq M_e e^{\omega_e(t-s)} \qquad \text{and} \qquad \|U_S(s,t)\| \leq M_S e^{\omega_S(t-s)}$$

for all $t \geq s$, then the periodic Sylvester differential equation (6.9) *has a unique periodic mild solution $\Sigma_\infty(\cdot) \in C_\tau(\mathbb{R}, \mathscr{L}_s(W, X_e))$ given by the formula*

$$\Sigma_\infty(t)v = \int_{-\infty}^{t} U_e(t,s)B_e(s)U_S(s,t)v\,ds, \qquad v \in W. \tag{6.10}$$

Proof. We will begin by showing that $\Sigma_\infty(\cdot)$ is a mild solution of the Sylvester differential equation (6.6). Since for every $v \in W$ we have

$$\Sigma_\infty(t)v = U_e(t,0)\int_{-\infty}^{0} U_e(0,s)B_e(s)U_S(s,0)U_S(0,t)v\,ds + \int_0^t U_e(t,s)B_e(s)U_S(s,t)v\,ds,$$

it suffices to show that the linear operator $\Sigma_\infty(0): W \to X_e$ defined by

$$\Sigma_\infty(0)v = \int_{-\infty}^{0} U_e(0,s)B_e(s)U_S(s,0)v\,ds, \qquad v \in W$$

is bounded. For all $v \in W$ we have

$$\int_{-\infty}^{0} \|U_e(0,s)B_e(s)U_S(s,0)v\|ds \leq M_e M_S \|B_e\|_\infty \int_{-\infty}^{0} e^{(\omega_e+\omega_S)s} ds \cdot \|v\| =: M\|v\|, \tag{6.11}$$

where $M < \infty$. This concludes that $\Sigma_\infty(0)$ is a well-defined operator and since

$$\left\| \int_{-\infty}^{0} U_e(0,s)B_e(s)U_S(s,0)v ds \right\| \leq \int_{-\infty}^{0} \|U_e(0,s)B_e(s)U_S(s,0)v\|ds \leq M\|v\|,$$

we have $\Sigma_\infty(0) \in \mathcal{L}(W, X_e)$. This concludes that $\Sigma_\infty(\cdot)$ is a mild solution of the Sylvester differential equation (6.6) corresponding to the initial condition $\Sigma_0 = \Sigma_\infty(0)$.

To prove the periodicity of $\Sigma_\infty(\cdot)$, let $t \in \mathbb{R}$. For any $v \in W$ we then have

$$\begin{aligned}
\Sigma_\infty(t+\tau)v &= \int_{-\infty}^{t+\tau} U_e(t+\tau,s)B_e(s)U_S(s,t+\tau)v ds \\
&= \int_{-\infty}^{t} U_e(t+\tau,s+\tau)B_e(s+\tau)U_S(s+\tau,t+\tau)v ds \\
&= \int_{-\infty}^{t} U_e(t,s)B_e(s)U_S(s,t)v ds = \Sigma_\infty(t)v.
\end{aligned}$$

This concludes that the function $\Sigma_\infty(\cdot)$ is τ-periodic.

It remains to show that the periodic Sylvester differential equation (6.9) has no other periodic solutions. To this end, let $\Sigma(\cdot)$ be a periodic mild solution of the equation corresponding to an arbitrary initial condition $\Sigma(0) = \Sigma_0 \in \mathcal{L}(W, X_e)$, i.e.,

$$\Sigma(t)v = U_e(t,0)\Sigma_0 U_S(0,t)v + \int_0^t U_e(t,s)B_e(s)U_S(s,t)v ds, \qquad v \in W.$$

For any $v \in W$ and $t \geq 0$ the difference $\Delta(t) = \Sigma_\infty(t) - \Sigma(t)$ satisfies

$$\begin{aligned}
\Delta(t)v &= \int_{-\infty}^{t} U_e(t,s)B_e(s)U_S(s,t)v ds - U_e(t,0)\Sigma_0 U_S(0,t)v - \int_0^t U_e(t,s)B_e(s)U_S(s,t)v ds \\
&= \int_{-\infty}^{0} U_e(t,s)B_e(s)U_S(s,t)v ds - U_e(t,0)\Sigma_0 U_S(0,t)v \\
&= U_e(t,0)\Sigma_\infty(0)U_S(0,t)v - U_e(t,0)\Sigma_0 U_S(0,t)v = U_e(t,0)\Delta(0)U_S(0,t)v.
\end{aligned}$$

This and our assumption $\omega_e + \omega_S < 0$ further imply

$$\|\Delta(t)\| \leq \|U_e(t,0)\| \cdot \|\Delta(0)\| \cdot \|U_S(0,t)\| \leq M_e M_S e^{(\omega_e+\omega_S)t} \|\Delta(0)\| \longrightarrow 0$$

as $t \to \infty$. Since $\Sigma(\cdot)$ and $\Sigma_\infty(\cdot)$ are periodic functions and since $\|\Sigma(t) - \Sigma_\infty(t)\| \to 0$ as $t \to \infty$, we must have $\Sigma(t) \equiv \Sigma_\infty(t)$. This concludes that $\Sigma_\infty(\cdot)$ is the only periodic solution of the Sylvester differential equation (6.9). □

We conclude this chapter and the treatment of the solvability of the Sylvester differential equation by showing that under the conditions of Theorem 6.6 and the growth assumptions in Theorem 6.10 the periodic Sylvester differential equation has a unique periodic classical solution.

Theorem 6.11. *Let the evolution families $U_e(t,s)$ and $U_S(t,s)$ and the function $B_e(\cdot)$ be periodic with period $\tau > 0$ and assume that they satisfy the conditions of Theorem 6.6. If there exist constants $M_e, M_S \geq 1$ and $\omega_e, \omega_S \in \mathbb{R}$ such that $\omega_e + \omega_S < 0$ and such that the estimates in Theorem 6.10 are satisfied for all $t \geq s$, then the periodic Sylvester differential equation (6.9) has a unique periodic classical solution $\Sigma_\infty(\cdot) \in C_\tau(\mathbb{R}, \mathscr{L}_s(W, X_e))$ such that*

$$\Sigma_\infty(\cdot)v \in C^1(\mathbb{R}, X_e) \quad \textit{and} \quad \Sigma(t)v \in \mathscr{D}(A_e) \tag{6.12}$$

for all $v \in \mathscr{D}(S)$ and $t \in \mathbb{R}$. In this case the solution $\Sigma_\infty(\cdot)$ is given by (6.10).

Proof. It is sufficient to show that in addition to the properties shown in the proof of Theorem 6.10, the operator-valued function $\Sigma_\infty(\cdot)$ satisfies (6.12) for all $v \in \mathscr{D}(S)$ and for all $t \in \mathbb{R}$. This can in fact be done by showing that $\Sigma_\infty(0)v \in \mathscr{D}(A_e)$ for all $v \in \mathscr{D}(S)$. Then, in the same way as in the proof of Theorem 6.10, we can show that $\Sigma_\infty(\cdot)$ is a classical solution of the Sylvester differential equation on all intervals $[0, \tau']$ with $\tau' > 0$. This and the periodicity of $\Sigma_\infty(\cdot)$ then conclude that (6.12) are satisfied for all $v \in \mathscr{D}(S)$ and $t \in \mathbb{R}$.

Let $v \in \mathscr{D}(S)$. To show $\Sigma_\infty(0)v \in \mathscr{D}(A_e)$ we write

$$\Sigma_\infty(0)v = \int_{-\infty}^{-1} U_e(0,s)B_e(s)U_S(s,0)v\,ds + \int_{-1}^{0} U_e(0,s)B_e(s)U_S(s,0)v\,ds =: x_0 + x_1.$$

We will first show that $x_0 \in \mathscr{D}(A_e)$. If we denote $f(s) = U_e(0,s)B_e(s)U_S(s,0)v$, then the fact that $U_e(t,s)$ satisfies the parabolic conditions implies that $f(s) \in \mathscr{D}(A_e)$ for all $s < 0$ and the estimate (6.11) implies $f \in L^1((-\infty,-1), X_e)$. We have from [42, Thm. 5.6.1] that $A_e(0)U_e(0,-1) \in \mathscr{L}(X_e)$ and thus

$$\int_{-\infty}^{-1} \|A_e(0)U_e(0,s)B_e(s)U_S(s,0)v\|ds \leq \|A_e(0)U_e(0,-1)\| \int_{-\infty}^{-1} \|U_e(-1,s)B_e(s)U_S(s,0)v\|ds$$

$$\leq M_e M_S \max_{r\in[0,\tau]} \|B_e(r)\| \cdot \|A_e(0)U_e(0,-1)\| \cdot \|v\| \cdot e^{-\omega_e} \int_{-\infty}^{-1} e^{-(\omega_e+\omega_S)s} ds < \infty.$$

This shows that $A_e(0)f \in L^1((-\infty,-1), X_e)$ and since $A_e(0)$ — as a generator of an analytic semigroup — is a closed linear operator, we have that $x_0 \in \mathscr{D}(A_e(0)) = \mathscr{D}(A_e)$.

It remains to show $x_1 \in \mathscr{D}(A_e)$. As in the proof of Theorem 6.6, we have that since the mapping $t \mapsto B_e(t)U_S(t,0)v$ is Hölder continuous on $[-1,0]$, the nonautonomous abstract Cauchy problem

$$\dot{x}(t) = A_e(t)x(t) + B_e(t)U_S(t,0)v, \qquad x(-1) = 0$$

has a unique classical solution

$$x(t) = \int_{-1}^{t} U_e(t,s)B_e(s)U_S(s,0)v\,ds$$

on $[-1,0]$. Because of this, we also have $x_1 = x(0) \in \mathcal{D}(A_e)$. Combining the above results shows that we have $\Sigma_\infty(0)v = x_0 + x_1 \in \mathcal{D}(A_e)$. Since $v \in \mathcal{D}(S)$ was arbitrary, this concludes the proof. □

This concludes our treatment of the solvability of Sylvester differential equations. In the next chapter we will apply these results to obtain a characterization for the solvability of the problem of output regulation and disturbance rejection related to a distributed parameter system with a periodic exosystem.

Chapter 7

Periodic Output Regulation

We now turn to studying the output regulation of distributed parameter systems in a situation where the reference and disturbance signals are generated using the periodic exosystem introduced in Section 2.3. In the course of the following two chapters we will see that the periodic output regulation problem has both significant differences and surprising similarities compared to the problem of tracking signals generated with time-invariant exosystems. In particular, we will see that using a periodic signal generator leads us very naturally to the use of periodically time-dependent control laws to accomodate for the dynamics of the nonautonomous exosystem. Even so, we will be able to approach the periodic output regulation problem using methods that are very similar to those employed in studying the output regulation problem for infinite-dimensional exosystems in Chapter 3.

Despite the differences arising from the use of the time-dependent exosystem, the systems involved still have characteristics similar to the ones familiar from the case of an autonomous exosystem. Most notably, the plant and the controller can be written as a periodically time-dependent closed-loop system which still greatly resembles the one we have studied in the earlier chapters. We will see that the parameters of this composite system can again be used to characterize the solvability of the output regulation problem. This way we are once more in a position where we can develop the theory of output regulation simultaneously for different types of controllers. These general results can subsequently be used to derive conditions for the existence of particular types of controllers achieving output regulation.

Remarkably, only a few adjustments are necessary in order to accommodate for the added dependence on time. The most significant one of them is that we need to replace the Sylvester operator equation appearing in the regulator equations with the corresponding Sylvester differential equation

$$\dot{\Sigma}(t) + \Sigma(t)S(t) = A_e(t)\Sigma(t) + B_e(t).$$

However, with this modification, all of the main results in Chapter 3 have their counterparts in the theory of periodic output regulation. In particular, the state of the nonautonomous closed-loop system can be expressed using the solution of the Sylvester differential equation, and this formula can again be used to study the asymptotic behavior of the regulation error.

The simplicity and ease of these modifications, however, is partly superficial. The fact that we consider time-dependent controllers leads to an infinite-dimensional nonautonomous closed-loop system. These types of abstract differential equations can be studied using the theory of the strongly continuous evolution families but, as we have seen, the question of their well-posedness in the classical sense is a far more complicated subject than for their time-invariant counterparts. Because of this difficulty, we concentrate on the mild states of our systems. Later in Chapter 8 we will see that for the actual periodic control laws we construct in this thesis the state of the closed-loop system has some additional regularity properties.

Despite the additional complexity arising from the time-dependence of the exosystem, we will during the course of this chapter see that using the periodic exosystem actually leads to conditions that are simpler than the ones in the corresponding results for infinite-dimensional autonomous exosystems. Much of this is due to the fact that in the theory of periodic output regulation we can again consider exponentially stabilizable closed-loop systems. Many of the auxiliary assumptions used in connection with the infinite-dimensional exosystem were required because exponential stability of the closed-loop system was unachievable.

In the following we outline the contents of this chapter and the main results in each of the sections.

Section 7.1. In this section we state the basic assumptions on the system, on the static state feedback law and the dynamic error feedback controller, and on the closed-loop system.

Section 7.2. We formulate the periodic output regulation problem and state the main result of the chapter. This theorem shows that the solvability of the output regulation problem related to a periodic exosystem can be characterized using the behavior of periodic mild solutions of a certain periodic Sylvester differential equation. The theorem is proved in Section 7.3.

Section 7.3. In this section we show that the behavior of the state of the closed-loop system is related to the periodic Sylvester differential equation in a way that is analogous to the case of time-invariant exosystems studied in Chapter 3. In particular we show that the Sylvester differential equation has a periodic mild solution if and only if the closed-loop system can be written in a special form. This relationship is used to prove the Theorem 7.1 presented in the preceding section.

The theory of periodic output regulation problem developed in this chapter generalizes the theory presented in [60, 20] concerning output regulation of finite-dimensional systems with time-dependent signal generators. The main generalization is to allow the system to be controlled to be infinite-dimensional. Also the exosystem considered in this chapter is more general than the one used in [60] in the sense that $S(\cdot)$ is not required to be a smooth function.

7.1 The System and Two Types of Controllers

We begin by stating the basic assumptions on our system, the reference and disturbance signals generated by the periodic exosystem, and the control laws. We consider two particular types of periodic controllers, the *static state feedback law* and the *periodic error feedback controller*. We introduce the forms of these controllers and state the standing assumptions on the resulting periodic closed-loop systems.

We consider a linear distributed parameter system

$$\dot{x}(t) = Ax(t) + Bu(t) + w_s(t), \qquad x(0) = x_0 \in X \tag{7.1a}$$

$$y(t) = Cx(t) + Du(t) + w_m(t) \tag{7.1b}$$

on a Banach space X. Here $x(t) \in X$ is the state of the system, $u(t) \in U$ is the input, and $y(t) \in Y$ the output. The input space U and the output space Y are Hilbert spaces. We assume that the operator $A : \mathcal{D}(A) \subset X \to X$ generates a strongly continuous semigroup $T_A(t)$ on X, and that the rest of the operators are bounded, $B \in \mathcal{L}(U,X)$, $C \in \mathcal{L}(X,Y)$, and $D \in \mathcal{L}(U,Y)$. The terms $w_s(t)$ and $w_m(t)$ denote the disturbances to the state and the measurement, respectively. These signals and the reference signal $y_{ref}(t)$ are generated by a periodic exosystem

$$\dot{v}(t) = S(t)v(t), \qquad v(0) = v_0 \in W \tag{7.2}$$

on $W = \mathbb{C}^q$. The reference and disturbance signals are obtained as outputs of this system,

$$w_s(t) = E(t)v(t), \qquad w_m(t) = F_m(t)v(t), \qquad y_{ref}(t) = F_r(t)v(t).$$

The functions related to the periodic exosystem are assumed to satisfy the conditions stated in Definition 2.9, i.e., all of the functions are periodic with period $\tau > 0$, and we have $S(\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathcal{L}(W))$,

$$E(\cdot) \in C_\tau(\mathbb{R}, \mathcal{L}(W,X)), \qquad \text{and} \qquad F_m(\cdot),\ F_r(\cdot) \in C_\tau(\mathbb{R}, \mathcal{L}(W,Y)).$$

As in Chapter 2, the fundamental matrix related to the periodic exosystem is denoted by $U_S(t,s)$. To exclude asymptotically decaying and exponentially growing signals, we assume that Assumption 2.10 is satisfied.

Defining $F(\cdot) \in C_\tau(\mathbb{R}, \mathscr{L}(W,Y))$ by $F(t) = F_m(t) - F_r(t)$ for all $t \in [0,\tau]$, the system (7.1) can be written in a standard form as

$$\dot{x}(t) = Ax(t) + Bu(t) + E(t)v(t), \qquad x(0) = x_0 \in X \tag{7.3a}$$

$$e(t) = Cx(t) + Du(t) + F(t)v(t) \tag{7.3b}$$

with the exosystem (7.2). Here $e(t) = y(t) - y_{ref}(t) \in Y$ denotes the regulation error.

The Two Controller Types

In this thesis we consider two particular types of τ-periodic controllers, a static state feedback law and a dynamic error feedback controller. When using static state feedback we assume that we can use the state $x(t)$ of the system (7.3) and the state $v(t)$ of the periodic exosystem in the control law. In this case the control signal is constructed in such a way that

$$u(t) = Kx(t) + L(t)v(t),$$

where $K \in \mathscr{L}(X,U)$ and $L(\cdot) \in C_\tau(\mathbb{R}, \mathscr{L}(W,X))$ are the fixed parameters of the control law.

The dynamic error feedback controller is a periodically time-dependent system of the form

$$\dot{z}(t) = \mathscr{G}_1(t)z(t) + \mathscr{G}_2(t)e(t), \qquad z(0) = z_0 \in Z \tag{7.4a}$$

$$u(t) = K(t)z(t) \tag{7.4b}$$

on a Banach space Z. We assume $(\mathscr{G}_1(t), \mathscr{D}(\mathscr{G}_1(t)))$ is a τ-periodic family of unbounded operators, and the two operator-valued functions satisfy $\mathscr{G}_2(\cdot) \in C_\tau(\mathbb{R}, \mathscr{L}_s(Y,Z))$ and $K(\cdot) \in C_\tau(\mathbb{R}, \mathscr{L}_s(Z,U))$.

The Nonautonomous Closed-Loop System

We assume the closed-loop system consisting of the plant and the controller can be written in the form

$$\dot{x}_e(t) = A_e(t)x_e(t) + B_e(t)v(t), \qquad x_e(0) = x_{e0} \in X_e \tag{7.5a}$$

$$e(t) = C_e(t)x_e(t) + D_e(t)v(t) \tag{7.5b}$$

on a Banach space X_e, where $(A_e(t), \mathscr{D}(A_e(t)))$ is a τ-periodic family of unbounded operators and the operator-valued functions are such that $B_e(\cdot) \in C_\tau(\mathbb{R}, \mathscr{L}(W,X_e))$, $C_e(\cdot) \in C_\tau(\mathbb{R}, \mathscr{L}_s(X_e,Y))$, and $D_e(\cdot) \in C_\tau(\mathbb{R}, \mathscr{L}(W,Y))$.

In the case of the static state feedback law we choose the state space and the state of the closed-loop system as $X_e = X$ and $x_e(t) = x(t)$, respectively, and the parameters in (7.5) are given by

$$A_e(t) \equiv A + BK, \qquad B_e(t) = BL(t) + E(t),$$
$$C_e(t) \equiv C + DK, \qquad D_e(t) = DL(t) + F(t).$$

On the other hand, in the case of the periodic error feedback controller we choose the state space and the state of (7.5) as $X_e = X \times Z$ and $x_e(t) = (x(t) \;\; z(t))^T \in X_e$, respectively, and the appropriate parameters of the closed-loop systems are such that for all $t \in \mathbb{R}$ we have

$$A_e(t) = \begin{pmatrix} A & BK(t) \\ \mathcal{G}_2(t)C & \mathcal{G}_1(t) + \mathcal{G}_2(t)DK(t) \end{pmatrix}, \qquad B_e(t) = \begin{pmatrix} E(t) \\ \mathcal{G}_2(t)F(t) \end{pmatrix},$$

$C_e(t) = \begin{pmatrix} C & DK(t) \end{pmatrix}$ and $D_e(t) = F(t)$. It is clear that the assumptions on the parameters of the closed-loop system are satisfied for both of these controller types.

We restrict our attention to the controllers for which there exists an exponentially bounded strongly continuous evolution family $U_e(t,s)$ associated to the family $(A_e(t), \mathcal{D}(A_e(t)))$ of operators in the sense of Definition 6.2. Under this assumption the closed-loop system has a well-defined mild state

$$x_e(t) = U_e(t,0)x_{e0} + \int_0^t U_e(t,s)B_e(s)v(s)ds$$

for initial states $x_{e0} \in X_e$ and $v_0 \in W$ of the closed-loop system and the controller, respectively. In the next chapter we will see that we can always choose the parameters of the controllers in such a way that this condition is satisfied. In fact, any static state feedback law has this property and the evolution family is given by

$$U_e(t,s) = T_{A+BK}(t-s),$$

where $T_{A+BK}(t)$ is the C_0-semigroup generated by the operator $A + BK$.

7.2 The Periodic Output Regulation Problem

We can now formulate the periodic output regulation problem consisting of the output tracking and disturbance rejection for the plant (7.3) together with the periodic exosystem (7.2). It is worthwhile to re-emphasize that the objectives are again stated using only the parameters of the closed-loop system. This allows a uniform treatment of the problem for all controller types for which the closed-loop system can be written in the form (7.5). The periodic controllers introduced in the previous section are considered separately in the next chapter, where we use our general results to derive conditions for the solvability of the periodic output regulation problem for these particular types of controllers.

The Periodic Output Regulation Problem. Choose the parameters of a τ-periodic controller in such a way that the following are satisfied.

1. The evolution family $U_e(t,s)$ is exponentially stable, i.e., there exist constants $M_e \geq 1$ and $\omega_e > 0$ such that
$$\|U_e(t,s)\| \leq M_e e^{-\omega_e(t-s)}$$
for all $t \geq s$.

2. For all initial states $x_{e0} \in X_e$ and $v_0 \in W$ of the closed-loop system (7.5) and the exosystem (7.2), respectively, the regulation error satisfies
$$\lim_{t\to\infty} e(t) = 0.$$

■

The main result of this chapter is stated in the following theorem. It characterizes the controllers solving the periodic output regulation problem in terms of the behavior of the periodic solution of an infinite-dimensional Sylvester differential equation. This result has the same form as Theorem 3.2 relating the solvability of the output regulation problem to the solvability of the well-known regulator equations. We can therefore consider the Sylvester differential equation (7.6) together with the time-dependent regulation constraint (7.7) as the *periodic regulator equations* related to output regulation of infinite-dimensional linear systems with nonautonomous exosystems.

Theorem 7.1. *Assume the closed-loop system is exponentially stable. Then the periodic Sylvester differential equation*

$$\dot{\Sigma}(t) + \Sigma(t)S(t) = A_e(t)\Sigma(t) + B_e(t) \tag{7.6}$$

has a unique periodic mild solution $\Sigma_\infty(\cdot) \in C_\tau(\mathbb{R}, \mathscr{L}(W, X_e))$ given by the formula

$$\Sigma_\infty(t)v = \int_{-\infty}^{t} U_e(t,s)B_e(s)U_S(s,t)v\,ds, \qquad v \in W.$$

The controller solves the periodic output regulation problem if and only if this solution satisfies

$$C_e(t)\Sigma_\infty(t) + D_e(t) = 0, \qquad \forall t \in [0,\tau]. \tag{7.7}$$

We can first of all observe that the results on the mild solvability of the Sylvester differential equation (7.6) established in the previous chapter can be readily applied under our standing assumptions. Furthermore, since the state space $W = \mathbb{C}^q$ of the exosystem is finite-dimensional, the strong continuity coincides with the continuity in the uniform operator topology for the operator-valued functions

$$S(\cdot):\mathbb{R}\to\mathscr{L}(W),\quad \Sigma(\cdot),\, B_e(\cdot):\mathbb{R}\to\mathscr{L}(W,X_e),\quad \text{and}\quad D_e(\cdot):\mathbb{R}\to\mathscr{L}(W,X_e).$$

The possibility of characterizing the solvability of the periodic output regulation problem using properties of a Sylvester differential equation is based on the connection between (7.6) and the behavior of the state of the closed-loop system. This relationship is studied in the next section.

7.3 The Dynamic Steady State of the Closed-Loop System

In this section we show that the state of the closed-loop system can be described using the periodic mild solution of the associated Sylvester equation. This connection also allows us to study the asymptotic behaviors of the state of the closed-loop system and the regulation error, and ultimately to prove Theorem 7.1. The following theorem shows us that any periodic mild solution of the Sylvester differential equation (7.6) can be used to express the state of the closed-loop system and, conversely, that any $\Sigma(\cdot)$ for which the state of the closed-loop system is of this form must necessarily be a mild solution of the Sylvester differential equation.

Theorem 7.2. *Let $\Sigma(\cdot)\in C_\tau(\mathbb{R},\mathscr{L}(W,X_e))$. Then the following are equivalent.*

(a) *The function $\Sigma(\cdot)$ is a periodic mild solution of the Sylvester differential equation* (7.6).

(b) *For all initial states $x_{e0}\in X_e$ and $v_0\in W$ of the closed-loop system and the exosystem the state of the closed-loop system can be written as*

$$x_e(t)=U_e(t,0)(x_{e0}-\Sigma(0)v_0)+\Sigma(t)v(t). \tag{7.8}$$

If (b) *is satisfied, then the regulation error is given by*

$$e(t)=C_e(t)U_e(t,0)(x_{e0}-\Sigma(0)v_0)+(C_e(t)\Sigma(t)+D_e(t))v(t),\qquad t\geq 0 \tag{7.9}$$

for all initial states $x_{e0}\in X_e$ and $v_0\in W$ of the closed-loop system and the exosystem.

Proof. For any initial conditions $x_{e0}\in X_e$ and $v_0\in W$ and for any $t\geq 0$ the mild state of the closed-loop system (7.5) is given by

$$x_e(t)=U_e(t,0)x_{e0}+\int_0^t U_e(t,s)B_e(s)U_S(s,0)v_0ds.$$

Using this and $v(t) = U_S(t,0)v_0$ we can see that if $\Sigma(\cdot) \in C_\tau(\mathbb{R}, \mathcal{L}(W, X_e))$, then the state of the closed-loop having representation (7.8) for all initial states is equivalent to

$$U_e(t,0)x_{e0} + \int_0^t U_e(t,s)B_e(s)U_S(s,0)v_0 ds = U_e(t,0)(x_{e0} - \Sigma(0)v_0) + \Sigma(t)v(t)$$

for all $x_{e0} \in X_e$ and $v_0 \in W$ or, in turn,

$$\int_0^t U_e(t,s)B_e(s)U_S(s,0)v_0 ds = -U_e(t,0)\Sigma(0)v_0 + \Sigma(t)U_S(t,0)v_0, \qquad \forall v_0 \in W.$$

Since the matrix $U_S(0,t)$ is invertible, every $v_0 \in W$ can be written as $v_0 = U_S(0,t)w_0$ for some $w_0 \in W$. Thus the above condition is equivalent the fact that

$$\int_0^t U_e(t,s)B_e(s)U_S(s,0)U_S(0,t)w_0 ds = -U_e(t,0)\Sigma(0)U_S(0,t)w_0 + \Sigma(t)U_S(t,0)U_S(0,t)w_0$$

$$\Leftrightarrow \qquad \int_0^t U_e(t,s)B_e(s)U_S(s,t)w_0 ds = -U_e(t,0)\Sigma(0)U_S(0,t)w_0 + \Sigma(t)w_0,$$

for all $w_0 \in W$. This is precisely the condition that $\Sigma(\cdot)$ is a mild solution of the Sylvester differential equation (7.6) corresponding to an initial condition $\Sigma(0)$. Since $\Sigma(\cdot)$ is a periodic function, this is finally equivalent to the fact that it is a periodic mild solution of the Sylvester differential equation (7.6).

If the state of the closed-loop system has the representation (7.8), then for all initial states $x_{e0} \in X_e$ and $v_0 \in W$ the regulation error is given by

$$e(t) = C_e(t)x_e(t) + D_e v(t) = C_e(t)U_e(t,0)(x_{e0} - \Sigma(0)v_0) + (C_e(t)\Sigma(t) + D_e(t))v(t).$$

This concludes the proof. □

Analogously to the situation in Section 3.3, we can see that the formula (7.8) can be used to study the asymptotic behavior of the state of the closed-loop. In particular, if the closed-loop system is exponentially stable, then the first term in (7.8) decays to zero with time for all initial states. This shows that the state of an exponentially stable closed-loop system behaves asymptotically as

$$x_e(t) \sim \Sigma(t)v(t).$$

Therefore, in the case of the periodic exosystem the *periodic dynamic steady state* of the closed-loop system is described by the asymptotic behavior of the mapping $t \mapsto \Sigma(t)v(t)$. The behavior of the regulation error corresponding to this steady state is in turn given by

$$e(t) \sim (C_e(t)\Sigma(t) + D_e(t))v(t),$$

since the first term in the formula (7.9) decays to zero asymptotically due to the stability of the closed-loop system.

It is now indeed clear that the main difference in describing the asymptotic behavior of the state of the closed-loop system compared to the situation in Chapter 3 is that the solution $\Sigma \in \mathscr{L}(W_\alpha, X_e)$ of the Sylvester equation has been replaced with the solution $\Sigma(\cdot) \in C_\tau(\mathbb{R}, \mathscr{L}(W, X_e))$ of the corresponding Sylvester differential equation. Now the steady state behavior of the closed-loop system has a stronger dependence on time, but the periodicity of the function $\Sigma(\cdot)$ implies that it is still in a certain sense regular. In particular, Lemma 2.14 implies that the dynamic steady state must be polynomially bounded. Furthermore, if the state $v(t)$ of the periodic exosystem is periodic $v(\cdot) \in C_\tau(\mathbb{R}, W)$, then also the dynamic steady state is periodic $\Sigma(\cdot)v(\cdot) \in C_\tau(\mathbb{R}, X_e)$ and thus uniformly bounded.

The following corollary summarizes the results on the asymptotic behaviors of the state of the closed-loop system and the regulation error.

Corollary 7.3. *Assume the closed-loop system is exponentially stable and the Sylvester differential equation* (7.6) *has a unique periodic mild solution* $\Sigma(\cdot) \in C_\tau(\mathbb{R}, \mathscr{L}(W, X_e))$*. Then for all initial states* $x_{e0} \in X_e$ *and* $v_0 \in W$ *of the closed-loop system and the exosystem, respectively, the state of the closed-loop system and the regulation error satisfy*

$$\lim_{t\to\infty} \|x_e(t) - \Sigma(t)v(t)\| = 0 \quad \text{and} \quad \lim_{t\to\infty} \|e(t) - (C_e(t)\Sigma(t) + D_e(t))v(t)\| = 0. \tag{7.10}$$

Before we can collect the results presented in this section to prove Theorem 7.1, we still need to show that the asymptotic decay of the regulation error is equivalent to the regulation constraint (7.7) being satisfied. This relationship relies on our assumption that the periodic exosystem does not generate decaying orbits.

Lemma 7.4. *Let* $\Sigma(\cdot) \in C_\tau(\mathbb{R}, \mathscr{L}(W, X_e))$*. We have that*

$$\lim_{t\to\infty} (C_e(t)\Sigma(t) + D_e(t))U_S(t,0)v_0 = 0, \qquad \forall v_0 \in W$$

if and only if $C_e(t)\Sigma(t) + D_e(t) = 0$ *for all* $t \in [0, \tau]$.

Proof. It is clearly sufficient to show that the limit implies $C_e(t_0)\Sigma(t_0) + D_e(t_0) = 0$ for all $t_0 \in [0, \tau)$. To this end, let $t_0 \in [0, \tau)$ be arbitrary and denote $t_n = t_0 + n\tau$ for $n \in \mathbb{N}$. Now for all $v_0 \in W$

$$\left\|(C_e(t_0)\Sigma(t_0) + D_e(t_0))U_S(t_n, 0)v_0\right\| = \left\|(C_e(t_n)\Sigma(t_n) + D_e(t_n))U_S(t_n, 0)v_0\right\| \longrightarrow 0$$

as $n \to \infty$. Let $\lambda \in \sigma(U_S(\tau, 0))$ and let $\{\phi_k\}_{k=1}^m$ be a Jordan chain associated to this eigenvalue. By Assumption 2.10 we have $|\lambda| = 1$. Now $U_S(\tau, 0)\phi_1 = \lambda\phi_1$ and

$$U_S(\tau, 0)\phi_k = \lambda\phi_k + \phi_{k-1}, \quad k \in \{2, \ldots, m\}. \tag{7.11}$$

Using the periodicity of the evolution family $U_S(t,s)$ we obtain

$$\begin{aligned} U_S(t_n,0) &= U_S(t_0+n\tau,0) = U_S(t_0+n\tau,n\tau)U_S(n\tau,(n-1)\tau)\cdots U_S(\tau,0) \\ &= U_S(t_0,0)U_S(\tau,0)^n. \end{aligned}$$

Using this we can see that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \left\|(C_e(t_0)\Sigma(t_0)+D_e(t_0))U_S(t_n,0)\phi_1\right\| \\ &= \left\|(C_e(t_0)\Sigma(t_0)+D_e(t_0))U_S(t_0,0)\phi_1\right\| \cdot \left(\lim_{n\to\infty}|\lambda|^n\right) \\ &= \left\|(C_e(t_0)\Sigma(t_0)+D_e(t_0))U_S(t_0,0)\phi_1\right\|, \end{aligned}$$

which in turn implies $(C_e(t_0)\Sigma(t_0)+D_e(t_0))U_S(t_0,0)\phi_1 = 0$. Equation (7.11) and a straightforward computation can now be used to verify that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \left\|(C_e(t_0)\Sigma(t_0)+D_e(t_0))U_S(t_n,0)\phi_2\right\| \\ &= \left\|(C_e(t_0)\Sigma(t_0)+D_e(t_0))U_S(t_0,0)\phi_2\right\| \cdot \left(\lim_{n\to\infty}|\lambda|^n\right) \\ &= \left\|(C_e(t_0)\Sigma(t_0)+D_e(t_0))U_S(t_0,0)\phi_2\right\|. \end{aligned}$$

This further implies $(C_e(t_0)\Sigma(t_0)+D_e(t_0))U_S(t_0,0)\phi_2 = 0$. Repeating these steps we can eventually use (7.11) together with the earlier identities to show that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \left\|(C_e(t_0)\Sigma(t_0)+D_e(t_0))U_S(t_n,0)\phi_m\right\| \\ &= \left\|(C_e(t_0)\Sigma(t_0)+D_e(t_0))U_S(t_0,0)\phi_m\right\| \cdot \left(\lim_{n\to\infty}|\lambda|^n\right) \\ &= \left\|(C_e(t_0)\Sigma(t_0)+D_e(t_0))U_S(t_0,0)\phi_m\right\|, \end{aligned}$$

which finally implies $(C_e(t_0)\Sigma(t_0)+D_e(t_0))U_S(t_0,0)\phi_m = 0$. Since the chosen eigenvalue $\lambda \in \sigma(U_S(\tau,0))$ and the associated Jordan chain were arbitrary, we must necessarily have $(C_e(t_0)\Sigma(t_0)+D_e(t_0))U_S(t_0,0) = 0$. The invertibility of $U_S(t_0,0)$ further concludes that $C_e(t_0)\Sigma(t_0)+D_e(t_0) = 0$. Since $t_0 \in [0,\tau)$ was arbitrary, this completes the proof. □

We can now conclude the chapter by presenting the proof of Theorem 7.1.

Proof of Theorem 7.1. Since $U_e(t,s)$ is exponentially stable, there exist constants $\omega_e < 0$ and $M_e \geq 1$ such that

$$\|U_e(t,s)\| \leq M_e e^{\omega_e(t-s)}$$

for all $t \geq s$. On the other hand, since the periodic exosystem satisfies Assumption 2.10, the polynomial bound in Lemma 2.14 implies that for any $0 < \omega_S < |\omega_e|$ we can choose $\tilde{M}_S \geq 1$ in such a way that

$$\|U_S(s,t)\| \leq M_S(|s-t|^{n_S}+1) \leq \tilde{M}_S e^{\omega_S(t-s)}$$

for all $t \geq s$. Since $\omega_e + \omega_S < 0$, Theorem 6.10 concludes that the periodic Sylvester differential equation (7.6) has a unique periodic mild solution $\Sigma_\infty(\cdot) \in C_\tau(\mathbb{R}, \mathcal{L}(W, X_e))$ given by the appropriate formula.

Assume first that the periodic solution $\Sigma_\infty(\cdot)$ of the Sylvester differential equation satisfies the regulation constraint

$$C_e(t)\Sigma_\infty(t) + D_e(t) = 0, \qquad \forall t \in [0,\tau].$$

Since the operator-valued functions $C_e(\cdot)$, $D_e(\cdot)$, and $\Sigma(\cdot)$ are τ-periodic, this is satisfied for all $t \in \mathbb{R}$. Using Corollary 7.3 we can therefore see that for all initial states $x_{e0} \in X_e$ and $v_0 \in W$ the regulation error satisfies

$$\left\|e(t)\right\| = \left\|e(t) - (C_e(t)\Sigma_\infty(t) + D_e(t))v(t)\right\| \longrightarrow 0$$

as $t \to \infty$. This concludes that the controller solves the periodic output regulation problem.

On the other hand, assume that the controller solves the periodic output regulation problem. Corollary 7.3 implies that for all initial states $x_{e0} \in X_e$ and $v_0 \in W$ we have

$$\begin{aligned} &\left\|(C_e(t)\Sigma_\infty(t) + D_e(t))U_S(t,0)v_0\right\| \\ \leq\ &\left\|(C_e(t)\Sigma_\infty(t) + D_e(t))U_S(t,0)v_0 - e(t)\right\| + \left\|e(t)\right\| \longrightarrow 0 \end{aligned}$$

as $t \to \infty$. Since this is satisfied for all $v_0 \in W$, Lemma 7.4 shows that we must have $C_e(t)\Sigma_\infty(t) + D_e(t) = 0$ for all $t \in [0,\tau]$, and thus the regulation constraint (7.7) is satisfied. □

As we have seen, the results presented in this chapter are all independent of the particular form of the controller. We will now turn to applying these results to derive more concrete conditions for the solvability of the periodic output regulation problem using the two controller types introduced in Section 7.1, the static state feedback law and the periodic error feedback controller.

Chapter 8

Controller Design for Periodic Output Regulation

The final chapter dealing with the theory of output regulation in this thesis is dedicated to deriving more concrete conditions for the solvability of the periodic output regulation problem with particular types of controllers. For this we will consider the two controller types — the static state feedback law and the periodic error feedback controller — introduced in the previous chapter. We also present the appropriate choices for the parameters of the controllers achieving output tracking and disturbance rejection. Finally, the study of the periodic output regulation problem is concluded by applying the theoretical results to an example where we design controllers for a scalar system with a delay.

The main difference between the two types of controllers we are considering is that the static state feedback law can only be used if we have complete knowledge of the states of the plant and the exosystem. If these are not available, we need to use a dynamical *observer* to first asymptotically estimate the states based on the information obtained from the output of the plant. Once we have done this, we can use the state feedback law employing the observed states to control the original system. This is precisely the idea behind the observer-based error feedback controller constructed in Section 8.2. Since the state feedback parts of the control laws are identical, some of the conditions for the existence of the controllers are the same for both controller types.

As we discussed at the beginning of the previous chapter, the well-posedness of nonautonomous abstract differential equations is a much more complicated subject than it is in the case of their time-invariant counterparts. In periodic output regulation, however, the sole underlying source of the time-dependence in the systems is the well-behaving finite-dimensional exosystem. For this reason the controller design for periodic signal generators leads to controllers and corresponding closed-loop systems in which the time-dependence has a very specific structure. In particular, we will see that the unbounded parts of the operators $A_e(t)$ remain invariant of time. For these kinds of operator-valued functions the solutions of the nonautonomous abstract differential equations can be studied fairly easily compared to the case where $(A_e(t), \mathcal{D}(A_e(t)))$ is a general family of unbounded operators. To provide a better foundation for considering infinite-dimensional nonautonomous closed-loop systems, we demonstrate some of the regularity properties their states have when the controllers are constructed as suggested in this chapter.

We conclude the treatment of the periodic output regulation problem by presenting a detailed example on construction of periodic controllers for a distributed parameter system. To this end we consider a scalar system with a delay and formulate it as a linear infinite-dimensional system on a Hilbert space. We use the theoretical results presented earlier in the chapter to design a static state feedback law and a dynamic error feedback controller steering the output of this system to the triangle signal considered in Section 2.1. We derive analytic expressions for the parameters of the controllers and show that they can be easily approximated with any given finite accuracy.

The organization of the chapter is outlined in the following.

Section 8.1. In this section we derive sufficient conditions for the solvability of the periodic output regulation problem using a static state feedback law, and present the appropriate choices for the parameters of the controller.

Section 8.2. The topic of this section is the solvability of the periodic output regulation problem using a dynamic error feedback controller. We present sufficient conditions for the existence of such a controller along with the appropriate choices for its parameters.

Section 8.3. In this section we study the regularity properties of the mild state of the closed-loop system in the case where the controller is constructed using one of the methods presented in the preceding sections.

Section 8.4. The chapter is concluded by considering a detailed example where we steer the output of a scalar system with delay to the triangle signal considered in Chapter 2. We consider construction of both a static state feedback law and an error feedback controller solving the problem.

The construction and the conditions for the existence of the controllers generalize the corresponding results for time-invariant finite-dimensional [19] and infinite-dimensional [6, 24] linear systems. In the case of the error feedback controller the stability of the closed-loop system can be determined by modifying a procedure familiar from the theory of autonomous finite-dimensional systems.

8.1 The Static State Feedback Law

In this section we consider choosing the operator $K \in \mathscr{L}(X,U)$ and the operator-valued function $L(\cdot) \in C_\tau(\mathbb{R},\mathscr{L}(W,U))$ in the static state feedback law

$$u(t) = Kx(t) + L(t)v(t).$$

The next theorem presents conditions for the solvability of the periodic output regulation problem with this type of controller and the appropriate choices for its parameters. The conditions involve the solvability of certain constrained Sylvester differential equations. These equations are considerably simpler than the Sylvester differential equation used in Chapter 7 to characterize the solvability of the output regulation problem in terms of the parameters of the closed-loop system. Furthermore, if the function $S(\cdot)$ in the periodic exosystem is a constant matrix $S(t) \equiv S$, then the equations can be reduced to an autonomous abstract Cauchy problem with a constraint. Solving such equations is demonstrated in the example considered in Section 8.4

Theorem 8.1. *Assume the pair (A,B) is exponentially stabilizable and that the constrained Sylvester differential equation*

$$\dot{\Pi}(t) + \Pi(t)S(t) = A\Pi(t) + B\Gamma(t) + E(t) \tag{8.1a}$$

$$0 = C\Pi(t) + D\Gamma(t) + F(t) \tag{8.1b}$$

has a periodic mild solution satisfying $\Pi(\cdot) \in C_\tau(\mathbb{R},\mathscr{L}(W,X))$ and $\Gamma(\cdot) \in C_\tau(\mathbb{R},\mathscr{L}(W,U))$. The periodic output regulation problem is solved by a static state feedback law with parameters $K \in \mathscr{L}(X,U)$ and $L(\cdot) \in C_\tau(\mathbb{R},\mathscr{L}(W,U))$, where K is chosen in such a way that the operator $A+BK$ generates an exponentially stable C_0-semigroup and $L(t) = \Gamma(t) - K\Pi(t)$ for all $t \in [0,\tau]$. The static state feedback law is thus given by

$$u(t) = Kx(t) + (\Gamma(t) - K\Pi(t))v(t).$$

Proof. Since $A_e(t) \equiv A+BK$, we have from the choice of the operator $K \in \mathscr{L}(X,U)$ that the closed-loop system is exponentially stable. Theorem 7.1 shows us that the Sylvester differential equation (7.6) has a unique periodic mild solution $\Sigma_\infty(\cdot) \in C_\tau(\mathbb{R},\mathscr{L}(W,X_e))$ and that the static state feedback law solves the periodic output regulation problem if this solution satisfies the regulation constraint

$$C_e(t)\Sigma_\infty(t) + D_e(t) = 0, \qquad \forall t \in [0,\tau].$$

If we write $L(t) = \Gamma(t) - K\Pi(t)$ and use the fact that we are considering a static state feedback law, the Sylvester differential equation (7.6) becomes

$$\begin{aligned}\dot{\Sigma}(t) + \Sigma(t)S(t) = A_e(t)\Sigma(t) + B_e(t) &= (A+BK)\Sigma(t) + BL(t) + E(t)\\ &= A\Sigma(t) + B\Gamma(t) + E(t) + BK(\Sigma(t) - \Pi(t)).\end{aligned}$$

An operator-valued function $\Sigma(\cdot) \in C_\tau(\mathbb{R}, \mathscr{L}(W,X))$ is a mild solution of this equation on an interval $[0,\tau']$ with $\tau' > 0$ corresponding to an initial condition $\Sigma_0 = \Sigma(0)$ if it satisfies the integral equation

$$\Sigma(t)v = T_A(t)\Sigma(0)U_S(0,t)v + \int_0^t T_A(t-s)(B\Gamma(s) + E(s))U_S(s,t)v\,ds$$
$$+ \int_0^t T_A(t-s)BK(\Sigma(s) - \Pi(s))U_S(s,t)v\,ds.$$

for all $v \in W$ and $t \in [0,\tau']$. For $\Sigma(\cdot) = \Pi(\cdot)$ the second integral vanishes, and $\Sigma(\cdot)$ satisfies the remaining equation due to the fact that $\Pi(\cdot)$ is a mild solution of (8.1a). This implies that $\Sigma(\cdot) = \Pi(\cdot)$ is a periodic mild solution of the Sylvester differential equation (7.6), and since this solution is unique we have $\Sigma_\infty(\cdot) = \Pi(\cdot)$.

Using (8.1b) we can now see that for all $t \in [0,\tau]$ we have

$$C_e(t)\Sigma_\infty(t) + D_e(t) = (C + DK)\Pi(t) + DL(t) + F(t) = C\Pi(t) + D(K\Pi(t) + L(t)) + F(t)$$
$$= C\Pi(t) + D\Gamma(t) + F(t) = 0.$$

As stated above, Theorem 7.1 now implies that the static state feedback law solves the periodic output regulation problem. □

If the states of the plant and the exosystem are not available for us to use in the static state feedback law, we need to use an observer-based error feedback controller in solving the periodic output regulation problem. The controllers of this type are the topic of the next section.

8.2 The Periodic Error Feedback Controller

In this section we consider the construction of observer-based error feedback controllers. The control laws of this type have the advantage that, unlike static state feedback law, they can be used even if we do not have direct access to the states of the plant and the exosystem. The next theorem provides conditions for the solvability of the periodic output regulation problem using a periodic error feedback controller and shows the appropriate choices for its parameters. The condition involving the solvability of the constrained Sylvester differential equation is the same as the one given in Theorem 8.1 for the solvability of the problem using a static state feedback law. The additional conditions present in the theorem guarantee that the states of the plant and the exosystem can be estimated using an asymptotic observer.

Theorem 8.2. *Assume that the pair (A,B) is exponentially stabilizable, that there exists an operator-valued function $L(\cdot) \in C_\tau(\mathbb{R}, \mathcal{L}_s(Y, X \times W))$ such that the evolution family associated to the family*

$$\left(\begin{pmatrix} A & E(t) \\ 0 & S(t) \end{pmatrix} + L(t) \begin{pmatrix} C & F(t) \end{pmatrix}, \mathcal{D}(A) \times W \right) \tag{8.2}$$

of operators is exponentially stable, and assume that the constrained Sylvester differential equation

$$\dot{\Pi}(t) + \Pi(t)S(t) = A\Pi(t) + B\Gamma(t) + E(t) \tag{8.3a}$$

$$0 = C\Pi(t) + D\Gamma(t) + F(t) \tag{8.3b}$$

has a periodic mild solution satisfying $\Pi(\cdot) \in C_\tau(\mathbb{R}, \mathcal{L}(W,X))$ and $\Gamma(\cdot) \in C_\tau(\mathbb{R}, \mathcal{L}(W,U))$. Under these assumptions the periodic output regulation problem is solved by a periodic error feedback controller on the state space $Z = X \times W$, if the family $(\mathcal{G}_1(t), \mathcal{D}(\mathcal{G}_1))$ of operators is chosen in such a way that

$$\mathcal{G}_1(t) = \begin{pmatrix} A & E(t) \\ 0 & S(t) \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} K(t) + L(t) \Big[\begin{pmatrix} C & F(t) \end{pmatrix} + DK(t) \Big], \qquad \mathcal{D}(\mathcal{G}_1) = \mathcal{D}(A) \times W,$$

and if we define $\mathcal{G}_2(t) \equiv -L(t)$ and $K(t) \equiv \begin{pmatrix} K_1 & K_2(t) \end{pmatrix}$. Here the operator $K_1 \in \mathcal{L}(X,U)$ is chosen in such a way that the operator $A + BK_1$ generates an exponentially stable semigroup, and $K_2(t) \equiv \Gamma(t) - K_1\Pi(t)$.

Proof. The first question to arise is whether the closed-loop system has a well-defined mild state for these choices of the parameters of the error feedback controller. This will be confirmed later in Section 8.3, where we investigate the regularity properties of the state of the closed-loop system. There we will see that for the above choices of $\mathcal{G}_1(\cdot)$, $\mathcal{G}_2(\cdot)$, and $K(\cdot)$ there exists an exponentially bounded evolution family associated to the family $(A_e(t), \mathcal{D}(A_e))$ of operators in the sense of Definition 6.2. Here we begin by showing that this evolution family is exponentially stable.

For our choices of the parameters of the error feedback controller the state space of the closed-loop is $X_e = X \times X \times W$. The domains $\mathcal{D}(A_e) = \mathcal{D}(A) \times \mathcal{D}(A) \times W$ of the operators $A_e(t)$ do not depend on time and, if we denote $L(t) = \begin{pmatrix} L_1(t) & L_2(t) \end{pmatrix}^T$, we have

$$A_e(t) = \begin{pmatrix} A & BK_1 & BK_2(t) \\ -L_1(t)C & A + BK_1 + L_1(t)C & E(t) + BK_2(t) + L_1(t)F(t) \\ -L_2(t)C & L_2(t)C & S(t) + L_2(t)F(t) \end{pmatrix}$$

for all $t \in \mathbb{R}$. By applying a time-invariant similarity transform $Q_e \in \mathcal{L}(X_e)$ satisfying

$$Q_e = \begin{pmatrix} I & 0 & 0 \\ -I & I & 0 \\ 0 & 0 & I \end{pmatrix}, \qquad Q_e^{-1} = \begin{pmatrix} I & 0 & 0 \\ I & I & 0 \\ 0 & 0 & I \end{pmatrix},$$

we can define operators $\tilde{A}_e(t)$ with domains $\mathscr{D}(\tilde{A}_e) = \mathscr{D}(A_e)$ by

$$\tilde{A}_e(t) = Q_e A_e(t) Q_e^{-1} = \begin{pmatrix} A + BK_1 & BK_1 & BK_2(t) \\ 0 & A + L_1(t)C & E(t) + L_1(t)F(t) \\ 0 & L_2(t)C & S(t) + L_2(t)F(t) \end{pmatrix}$$

for all $t \in \mathbb{R}$. Since there exists an exponentially bounded strongly continuous evolution family associated to the closed-loop system, the same is true for the family $(\tilde{A}_e(t), \mathscr{D}(\tilde{A}_e))$ of operators. Furthermore, this evolution family is exponentially stable by Lemma A.2 since $BK_2(\cdot) \in C_\tau(\mathbb{R}, \mathscr{L}(W,X))$, since $A + BK_1$ generates an exponentially stable semigroup, and since by assumption the evolution family associated to the family (8.2) of operators is exponentially stable. Because the similarity transformation Q_e is invariant of time, this also implies that the evolution family $U_e(t,s)$ associated to the closed-loop system is exponentially stable.

Theorem 7.1 shows us that the Sylvester differential equation (7.6) has a unique periodic mild solution $\Sigma_\infty(\cdot)$. The same result further states that in order to prove that our controller solves the periodic output regulation problem, it is sufficient to show that the regulation constraint

$$C_e(t)\Sigma_\infty(t) + D_e(t) = 0, \qquad \forall t \in [0,\tau]$$

is satisfied. Writing $\Sigma(t) = \begin{pmatrix} \Sigma_1(t) & \Sigma_2(t) \end{pmatrix}^T$ we can see that for a periodic error feedback controller the Sylvester differential equation (7.6) can be written as a pair of equations

$$\dot{\Sigma}_1(t) + \Sigma_1(t)S(t) = A\Sigma_1(t) + BK(t)\Sigma_2(t) + E(t) \tag{8.4a}$$

$$\dot{\Sigma}_2(t) + \Sigma_2(t)S(t) = \mathscr{G}_1(t)\Sigma_2(t) + \mathscr{G}_2(t)(C\Sigma_1(t) + DK\Sigma_2(t) + F(t)). \tag{8.4b}$$

Let $\Pi(\cdot)$ and $\Gamma(\cdot)$ be components of the solution of equations (8.3). We will show that the unique periodic mild solution $\Sigma_\infty(\cdot)$ of the Sylvester differential equation (7.6) is given by

$$\Sigma_\infty(\cdot) = \begin{pmatrix} \Sigma_1(\cdot) \\ \Sigma_2(\cdot) \end{pmatrix}, \qquad \text{where} \qquad \Sigma_1(\cdot) = \Pi(\cdot), \quad \Sigma_2(\cdot) = \begin{pmatrix} \Pi(\cdot) \\ I \end{pmatrix}.$$

If this is the case, the constraint (8.3b) and the fact that

$$\Gamma(t) = K_1\Pi(t) + K_2(t) = K(t)\begin{pmatrix} \Pi(t) \\ I \end{pmatrix} \tag{8.5}$$

immediately imply

$$\begin{aligned} C_e(t)\Sigma_\infty(t) + D_e(t) &= \begin{pmatrix} C & DK(t) \end{pmatrix}\begin{pmatrix} \Sigma_1(t) \\ \Sigma_2(t) \end{pmatrix} + F(t) = C\Pi(t) + DK(t)\begin{pmatrix} \Pi(t) \\ I \end{pmatrix} + F(t) \\ &= C\Pi(t) + D\Gamma(t) + F(t) = 0. \end{aligned}$$

Theorem 7.1 then concludes that our periodic error feedback controller solves the periodic output regulation problem.

If $\Sigma_2(\cdot) = \begin{pmatrix}\Pi(\cdot) & I\end{pmatrix}^T$, then the identity in (8.5) implies that equation (8.4a) is in fact equivalent to equation (8.3a). Therefore in this situation the function $\Sigma_1(\cdot) = \Pi(\cdot)$ is a mild solution of (8.4a) corresponding to the initial value $\Pi(0) \in \mathscr{L}(W,X)$.

It only remains to show that if $\Sigma_1(\cdot) = \Pi(\cdot)$, then $\Sigma_2(\cdot) = \begin{pmatrix}\Pi(\cdot) & I\end{pmatrix}^T$ is a mild solution of (8.4b). To this end, assume $\Sigma_1(\cdot) = \Pi(\cdot)$. The solution of (8.4b) is of the form $\Sigma_2(\cdot) = \begin{pmatrix}\Sigma_{21}(\cdot) & \Sigma_{22}(\cdot)\end{pmatrix}^T$. If $\Sigma_{22}(\cdot) = I$, then the left-hand side of the equation can be written formally as

$$\dot{\Sigma}(t) + \Sigma(t)S(t) = \begin{pmatrix}\dot{\Sigma}_{21}(t) + \Sigma_{21}(t)S(t)\\ S(t)\end{pmatrix}.$$

On the other hand, using $K_2(t) = \Gamma(t) - K_1\Pi(t)$, the right-hand side of the equation becomes

$$\begin{aligned}
&\mathscr{G}_1(t)\Sigma_2(t) + \mathscr{G}_2(t)\big(C\Pi(t) + DK(t)\Sigma_2(t) + F(t)\big)\\
&= \begin{pmatrix}(A+BK_1)\Sigma_{21}(t) + E(t) + BK_2(t)\\ S(t)\end{pmatrix} + L(t)\Big[\begin{pmatrix}C & F(t)\end{pmatrix} + D\begin{pmatrix}K_1 & K_2(t)\end{pmatrix}\Big]\begin{pmatrix}\Sigma_{21}(t)\\ I\end{pmatrix}\\
&\quad - L(t)\left(C\Pi(t) + DK(t)\begin{pmatrix}\Sigma_{21}(t)\\ I\end{pmatrix} + F(t)\right)\\
&= \begin{pmatrix}(A+BK_1)\Sigma_{21}(t) + BK_2(t) + E(t)\\ S(t)\end{pmatrix} + L(t)C\big(\Sigma_{21}(t) - \Pi(t)\big)\\
&= \begin{pmatrix}A\Sigma_{21}(t) + B\Gamma(t) + E(t)\\ S(t)\end{pmatrix} + \left(\begin{pmatrix}BK_1\\ 0\end{pmatrix} + L(t)C\right)\big(\Sigma_{21}(t) - \Pi(t)\big).
\end{aligned}$$

This shows that if $\Sigma_{22}(\cdot) = I$, then equation (8.4b) is equivalent to the pair

$$\begin{aligned}
\dot{\Sigma}_{21}(t) + \Sigma_{21}(t)S(t) &= A\Sigma_{21}(t) + B\Gamma(t) + E(t) + (BK_1 + L_1(t)C)(\Sigma_{21}(t) - \Pi(t))\\
S(t) &= S(t) + L_2(t)(\Sigma_{21}(t) - \Pi(t))
\end{aligned}$$

of equations. The second equation is clearly satisfied if $\Sigma_{21}(\cdot) = \Pi(\cdot)$. Furthermore, an operator-valued function $\Sigma_{21}(\cdot) \in C_\tau(\mathbb{R}, \mathscr{L}(W,X))$ is a mild solution of the first equation on an interval $[0,\tau']$ with $\tau' > 0$ corresponding to an initial condition $\Sigma_{21}(0)$ if it satisfies the integral equation

$$\begin{aligned}
\Sigma_{21}(t)v &= T_A(t)\Sigma_{21}(0)U_S(0,t) + \int_0^t T_A(t-s)(B\Gamma(s) + E(s))U_S(s,t)v\,ds\\
&\quad + \int_0^t T_A(t-s)(BK_1 + L_1(s)C)(\Sigma_{21}(s) - \Pi(s))U_S(s,t)v\,ds
\end{aligned}$$

for all $v \in W$ and $t \in [0,\tau']$. If $\Sigma_{21}(\cdot) = \Pi(\cdot)$, then the second integral vanishes and the remaining equation is satisfied due to the fact that $\Pi(\cdot)$ is a mild solution of (8.3a).

This finally concludes that $\Sigma(\cdot) = \left(\Pi(\cdot) \;\; \left(\Pi(\cdot) \;\; I\right)^T\right)^T \in C_\tau(\mathbb{R}, \mathscr{L}(W, X_e))$ is a periodic mild solution of the Sylvester differential equation (7.6). Since the closed-loop system is exponentially stable, Theorem 7.1 states that the solution of this equation is unique and thus $\Sigma_\infty(\cdot) = \left(\Pi(\cdot) \;\; \left(\Pi(\cdot) \;\; I\right)^T\right)^T$. As we demonstrated earlier, this solution also satisfies the regulation constraint (7.7). Because of this, Theorem 7.1 concludes that the periodic error feedback controller solves the periodic output regulation problem.
□

The conditions for the solvability of the periodic output regulation problem using an observer-based error feedback controller require that there exists a periodic function $L(\cdot)$ for which the strongly continuous evolution family related to the family (8.2) of operators is exponentially stable. Stabilization of such systems is one of the places where the theory available for nonautonomous systems is significantly weaker than for their time-invariant counterparts. In the example studied in Section 8.4 we will demonstrate achieving the stability of this evolution family in a special case where the plant is exponentially stable and there are no disturbance signals to its state.

8.3 Regularity Properties of the State of the Closed-Loop System

In this section we briefly consider the regularity properties of the state of the closed-loop system in the case where the controller is constructed using one of the results presented in the previous sections. In particular we will verify that for our controllers the closed-loop system has a well-defined mild state given by

$$x_e(t) = U_e(t,s)x_{e0} + \int_0^t U_e(t,s)B_e(s)v(s)dr \tag{8.6}$$

for any initial states of the closed-loop system and the exosystem. We are also interested in any additional differentiability properties of the function $x_e(\cdot)$. We simplify this consideration by assuming that the function $S(\cdot)$ related to the periodic exosystem is continuous and that the output space Y is finite-dimensional. The former assumption does not result in any loss of generality since, as we saw in Section 2.3, any periodic signal generator can be written as a periodic exosystem for which $S(\cdot) \equiv \tilde{S}$ is a constant matrix.

As we already saw in Section 7.1, in the case of the static state feedback law the evolution family related to the closed-loop system is given by

$$U_e(t,s) = T_{A+BK}(t-s),$$

where $T_{A+BK}(t)$ denotes the semigroup generated by the stabilized operator $A + BK$. In this case we can therefore apply the theory of abstract Cauchy problems to determine conditions under which $x_e(\cdot)$ is continuously differentiable. In fact, since the function $t \mapsto v(t) = U_S(t,0)v_0$ has a continuous derivative, the mild state $x_e(t)$ is the classical state of the closed-loop system if [7, Thm. 3.1.3]

$$x_{e0} \in \mathcal{D}(A_e) = \mathcal{D}(A) \qquad \text{and} \qquad B_e(\cdot) \in C_\tau^1(\mathbb{R}, \mathcal{L}(W, X_e)).$$

This means that under these conditions we have $x_e(\cdot) \in C^1([0,\infty), X_e)$ and $x_e(t) \in \mathcal{D}(A_e)$ for all $t \geq 0$. In the case of the static state feedback law the operator-valued function $B_e(\cdot)$ is given by

$$B_e(t) = BL(t) + E(t) = B(\Gamma(t) - K\Pi(t)) + E(t)$$

for all $t \in \mathbb{R}$. We can therefore see that the second one of the above conditions is satisfied if the function $E(\cdot)$ and the components $\Pi(\cdot)$ and $\Gamma(\cdot)$ of the solution of the constrained Sylvester differential equation (8.1) are continuously differentiable.

On the other hand, if we are using a periodic error feedback controller and choose its parameters as described in Section 8.2, the operators $A_e(t)$ do in general depend on time and the situation becomes more complicated. However, the statement of Theorem 8.2 together with our assumptions on $S(\cdot)$ and Y shows that the operators $A_e(t)$ of the resulting closed-loop system are of the form

$$A_e(t) = A_{e0} + A_{e1}(t), \qquad \mathcal{D}(A_e(t)) \equiv \mathcal{D}(A_{e0}), \tag{8.7}$$

where A_{e0} generates a strongly continuous semigroup $T_e(t)$ on X_e and where the time-dependent part satisfies $A_{e1}(\cdot) \in C_\tau(\mathbb{R}, \mathcal{L}(X_e))$. The following theorem shows that in this case the state of the closed-loop system still behaves relatively well compared to the situation where $(A_e(t), \mathcal{D}(A_e(t)))$ is a general family of unbounded operators. In particular for the initial state $v_0 = 0$ of the exosystem we have $v(\cdot) \equiv 0$, and the theorem confirms the existence of a strongly continuous evolution family associated to the family $(A_e(t), \mathcal{D}(A_e(t)))$ of operators in the sense of Definition 6.2. This also further concludes that the closed-loop system has a well-defined mild state.

Theorem 8.3. *For all initial conditions $x_{e0} \in X_e$ and $v_0 \in W$ of the closed-loop system and the exosystem, respectively, and for all $x_e^* \in \mathcal{D}(A_{e0}^*)$ the state of the closed-loop system satisfies*

$$\frac{d}{dt}\langle x_e(t), x_e^*\rangle = \langle x_e(t), A_e(t)^* x_e^*\rangle + \langle B_e(t)v(t), x_e^*\rangle$$

for all $t > 0$. In particular, the mapping $t \mapsto \langle x_e(t), x_e^*\rangle$ is continuously differentiable on $(0,\infty)$.

Proof. Since $A_{e1}(t)$ are bounded linear operators, the structure (8.7) of the operators $A_e(t)$ also implies that for all $t \in \mathbb{R}$ we have

$$A_e(t)^* = A_{e0}^* + A_{e1}(t)^*, \qquad \mathscr{D}(A_e(t)^*) = \mathscr{D}(A_{e0}^*).$$

Moreover, since the operator-valued function $A_{e1}(\cdot)$ is continuous with respect to the uniform operator topology of $\mathscr{L}(X_e)$, also the mapping

$$t \mapsto A_e(t)^* x_e^* = A_{e0}^* x_e^* + A_{e1}(t)^* x_e^*$$

is continuous for all $x_e^* \in \mathscr{D}(A_{e0}^*)$.

We will first express the evolution family $U_e(t,s)$ and show that for any $x_e \in X_e$ and $x_e^* \in \mathscr{D}(A_{e0}^*)$ we have

$$\frac{\partial}{\partial t}\langle U_e(t,s)x_e, x_e^*\rangle = \langle U_e(t,s)x_e, A_e(t)^* x_e^*\rangle \tag{8.8}$$

for all $t > s$. Since $t \mapsto A_e(t)^* x_e^*$ are continuous functions, this will in particular confirm that there exists an evolution family associated to the family $(A_e(t), \mathscr{D}(A_e))$ of operators in the sense of Definition 6.2.

Since the family $(A_e(t), \mathscr{D}(A_e))$ of operators can be seen as a perturbation of the operator A_{e0}, we can also view the evolution family associated to the closed-loop system as a perturbation of the semigroup $T_e(t)$. By [10, Thm. VI.9.19] there exists a unique exponentially bounded evolution family $U_e(t,s)$ satisfying the integral equation

$$U_e(t,s)x_e = T_e(t-s)x_e + \int_s^t T_e(t-r)A_{e1}(r)U_e(r,s)x_e dr, \qquad x_e \in X_e.$$

We will show that $U_e(t,s)$ satisfies (8.8). To this end, let $x_e \in X_e$ and $x_e^* \in \mathscr{D}(A_{e0}^*)$. Since $T_e(t)$ is a strongly continuous semigroup, the mapping $t \mapsto T_e(t)^* x_e^*$ is weakly* differentiable [10, Par. II.2.6]. Using this property we can see that

$$\begin{aligned}\frac{\partial}{\partial t}\left\langle T_e(t-r)A_{e1}(r)U_e(r,s)x_e, x_e^*\right\rangle &= \frac{\partial}{\partial t}\left\langle A_{e1}(r)U_e(r,s)x_e, T_e(t-r)^* x_e^*\right\rangle \\ &= \left\langle T_e(t-r)A_{e1}(r)U_e(r,s)x_e, A_{e0}^* x_e^*\right\rangle\end{aligned}$$

for all $t > r$. Since the functions involved are continuous and uniformly bounded on all finite intervals, we can use the Leibniz integral rule similarly as in the proof of Theorem 6.6 to show that for all $t > s$ we have

$$\begin{aligned}\frac{\partial}{\partial t}\left\langle U_e(t,s)x_e, x_e^*\right\rangle &= \frac{\partial}{\partial t}\left\langle T_e(t-r)x_e, x_e^*\right\rangle + \frac{\partial}{\partial t}\int_s^t \left\langle T_e(t-r)A_{e1}(r)U_e(r,s)x_e, x_e^*\right\rangle dr \\ &= \left\langle T_e(t-r)x_e, A_{e0}^* x_e^*\right\rangle + \left\langle A_{e1}(t)U_e(t,s)x_e, x_e^*\right\rangle + \int_s^t \left\langle T_e(t-r)A_{e1}(r)U_e(r,s)x_e, A_{e0}^* x_e^*\right\rangle dr \\ &= \left\langle T_e(t-r)x_e + \int_s^t T_e(t-r)A_{e1}(r)U_e(r,s)x_e dr, A_{e0}^* x_e^*\right\rangle + \left\langle U_e(t,s)x_e, A_{e1}(t)^* x_e^*\right\rangle \\ &= \left\langle U_e(t,s)x_e, A_e(t)^* x_e^*\right\rangle.\end{aligned}$$

Since $x_e \in X_e$ and $x_e^* \in \mathcal{D}(A_{e0}^*)$ were arbitrary, this concludes that (8.8) is satisfied, and further that $U_e(t,s)$ is an evolution family associated to the family $(A_e(t), \mathcal{D}(A_e))$ of operators in the sense of Definition 6.2. Therefore the closed-loop system has a well-defined mild state $x_e(t)$ given by (8.6). It remains to prove the weak differentiability properties of the mapping $t \mapsto x_e(t)$.

Let $x_{e0} \in X_e$ and $v_0 \in W$ be arbitrary initial states of the closed-loop system and the exosystem, respectively, and let $x_e^* \in \mathcal{D}(A_{e0}^*)$. Using (8.8) we can show that

$$\frac{\partial}{\partial t} \left\langle U_e(t,s)B_e(s)U_S(s,0)v_0, x_e^* \right\rangle = \left\langle U_e(t,s)B_e(s)U_S(s,0)v_0, A_e(t)^* x_e^* \right\rangle$$

for any $t > s$. The continuity of this derivative and the other functions involved implies that we can again use the Leibniz integral rule to show that for any $t > 0$ we have

$$\frac{d}{dt} \left\langle x_e(t), x_e^* \right\rangle = \frac{d}{dt} \left\langle U_e(t,0)x_{e0}, x_e^* \right\rangle + \frac{d}{dt} \int_0^t \left\langle U_e(t,s)B_e(s)U_S(s,0)v_0, x_e^* \right\rangle dr$$

$$= \left\langle U_e(t,0)x_{e0}, A_e(t)^* x_e^* \right\rangle + \left\langle B_e(t)U_S(t,0)v_0, x_e^* \right\rangle + \int_0^t \left\langle U_e(t,s)B_e(s)U_S(s,0)v_0, A_e(t)^* x_e^* \right\rangle dr$$

$$= \left\langle x_e(t), A_e(t)^* x_e^* \right\rangle + \left\langle B_e(t)v(t), x_e^* \right\rangle .$$

The continuity of the mild state of the closed-loop system and the continuities of the mappings

$$t \mapsto A_e(t)^* x_e^* \qquad \text{and} \qquad t \mapsto B_e(t)v(t)$$

imply that the functions on the right-hand side of the previous equation are continuous with respect to t. Since $x_e^* \in \mathcal{D}(A_{e0}^*)$ was arbitrary, this completes the proof. □

This concludes the more theoretical part of the study of the solvability of the periodic output regulation problem using the static state feedback law and and the periodic error feedback controller. In the next section we will consider a more concrete example on the construction of such control laws.

8.4 Periodic Output Regulation of a Delay System

In this section we consider the control of a scalar delay equation. Our goal is to apply the theoretical results presented earlier in this chapter to construct a controller steering the output of the system to the triangle signal considered in Section 2.1. In order to accomplish this, we formulate the delay equation as a distributed parameter system on a Hilbert space, and construct a periodic exosystem generating the desired reference signal. We will then use Theorems 8.1 and 8.2 to construct a static state feedback law and a periodic error feedback controller solving the periodic output regulation problem related to our delay system and the periodic exosystem.

This example in particular illustrates two important steps in construction of the periodic controllers. First of all, we will show how to solve the constrained Sylvester differential equations appearing in Theorems 8.1 and 8.2. Furthermore, we will also demonstrate choosing the stabilizing function $L(\cdot)$ in the family (8.2) of operators in a special case where the original system is exponentially stable and there are no disturbance signals to the state of the plant.

The Plant

We consider a scalar system with delay

$$\dot{x}(t) = -2x(t) + x(t-1) + u(t) \tag{8.9a}$$

$$y(t) = x(t) + u(t), \tag{8.9b}$$

$$x(0) = \alpha \tag{8.9c}$$

$$x(\theta) = f(\theta), \qquad \theta \in [-1,0), \tag{8.9d}$$

where $\alpha \in \mathbb{C}$ and $f \in L^2(-1,0)$. Denote

$$\mathbf{M}_2(-1,0) = \mathbb{C} \times L^2(-1,0).$$

The space $\mathbf{M}_2(-1,0)$ is a Hilbert space with the inner product and the induced norm given by [7, Sec. 2.4]

$$\left\langle \begin{pmatrix} \alpha_1 \\ f_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ f_2 \end{pmatrix} \right\rangle = \alpha_1 \overline{\alpha}_2 + \langle f_1, f_2 \rangle_{L^2}, \qquad \left\| \begin{pmatrix} \alpha \\ f \end{pmatrix} \right\|^2 = |\alpha|^2 + \|f\|_{L^2}^2.$$

The plant (8.9) can be written as a linear system of the form (7.3) on $X = \mathbf{M}_2(-1,0)$ by choosing $Y = U = \mathbb{C}$ and

$$A \begin{pmatrix} \alpha \\ f \end{pmatrix} = \begin{pmatrix} -2\alpha + f(-1) \\ \frac{df}{d\theta} \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad D = 1,$$

$$\mathcal{D}(A) = \left\{ \begin{pmatrix} \alpha \\ f \end{pmatrix} \in \mathbf{M}_2(-1,0) \;\middle|\; f \text{ absolutely cont.}, \ \frac{df}{d\theta} \in L^2(-1,0), \ f(0) = \alpha \right\}.$$

If $\begin{pmatrix} \alpha & f \end{pmatrix}^T \in X$, then the semigroup $T_A(t)$ generated by A satisfies [7, Thm. 2.4.6]

$$T_A(t) \begin{pmatrix} \alpha \\ f \end{pmatrix} = \begin{pmatrix} x(t) \\ x(t+\cdot) \end{pmatrix} \in \mathbf{M}_2(-1,0), \qquad t \geq 0,$$

where $x(t)$ is the solution of the delay equation (8.9) with input $u(t) \equiv 0$. We know from the theory of ordinary differential equations that the function $x(\cdot)$ is determined by the integral equation

$$x(t) = e^{-2t}\alpha + \int_0^t e^{-2(t-r)}x(r-1)dr. \tag{8.10}$$

The function $x(t)$ can be solved from this expression by computing the right-hand side sequentially on intervals $[n, n+1]$, and using the history $x(\theta) = f(\theta)$ for $\theta \in [-1, 0)$ on the interval $[0, 1]$.

The stability of the delay system (8.9) is determined by the location of the roots of the function $\Delta : \mathbb{C} \to \mathbb{C}$ defined by

$$\Delta(\lambda) = \lambda - (-2) - e^{-\lambda}, \qquad \lambda \in \mathbb{C}.$$

More precisely, the plant is exponentially stable if and only if $\Delta(\lambda) \neq 0$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$ [7, Thm. 5.1.7]. We can use this property to show that the semigroup $T_A(t)$ is stable. Indeed, if $\lambda = a + ib$ with $a \geq 0$, then the roots of the above function are determined by

$$\Delta(a + ib) = (a + 2 - e^{-a}\cos(b)) + i(b + e^{-a}\sin(b)) = 0.$$

The imaginary part of this equation implies $b = 0$, since for $b \neq 0$ we would have

$$\operatorname{sinc}(b) = \frac{\sin(b)}{b} = -e^{a} \leq -1.$$

This can never be satisfied since $\operatorname{sinc}(b) > -1$ for all $b \in \mathbb{R}$. On the other hand, if $b = 0$ the real part of the equation becomes $2 = e^{-a} - a$. This, however, is impossible because our assumption $a \geq 0$ implies that the right-hand side is less than or equal to 1. This concludes that $\Delta(\lambda) \neq 0$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$, and further that the semigroup $T_A(t)$ generated by the system operator A of the plant is exponentially stable.

The Exosystem

We will now construct a one-dimensional periodic exosystem capable of generating the triangle signal introduced in Section 2.1. In this example we consider a version with period $\tau = 2$. The signal is depicted in Figure 8.1.

Let $W = \mathbb{C}$, $\tau = 2$, and $S(t) \equiv 0$. We choose $F_r(\cdot) \in C_\tau(\mathbb{R}, \mathscr{L}(W, Y)) = C_\tau(\mathbb{R}, \mathbb{C})$ to be a τ-periodic function satisfying

$$F_r(t) = \begin{cases} t + 1 & 0 \leq t < 1 \\ -t + 3 & 1 \leq t < 2. \end{cases}$$

The signals generated by this exosystem are of the form $y_{ref}(t) = F_r(t)v(t) = F_r(t)v_0$, and the triangle signal in Figure 8.1 corresponds to the initial state $v_0 = 1$ of the signal generator.

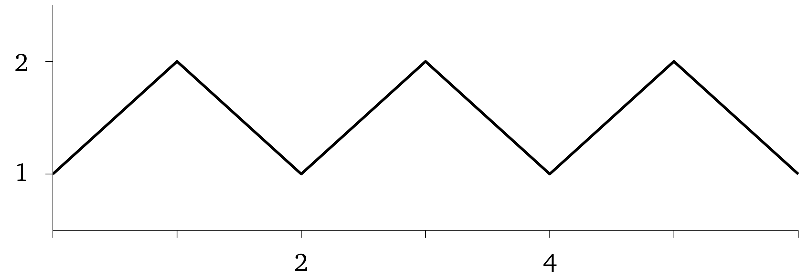

Figure 8.1: The reference signal $y_{ref}(\cdot)$.

We assume that there are no disturbance signals to the state or to the output of the plant, and because of this we choose the corresponding output functions of the exosystem to be identically zero, i.e., $E_d(t) \equiv 0$ and $F_d(t) \equiv 0$. In the standard form of the system the operator-valued functions $E(\cdot) \in C_\tau(\mathbb{R}, \mathscr{L}(W,X))$ and $F(\cdot) \in C_\tau(\mathbb{R}, \mathscr{L}(W,Y))$ are then such that $E(t) = 0$ and $F(t) = -F_r(t)$ for all $t \in \mathbb{R}$.

The Solution of the Constrained Sylvester Differential Equation

We will use Theorems 8.1 and 8.2 to construct a static state feedback law and a periodic error feedback controller solving the periodic output regulation problem related to the plant and the periodic exosystem defined above. To be able to use these results we need to solve the constrained Sylvester differential equations appearing in the statements of the theorems. In the following we demonstrate a method for finding the analytic formulas for $\Pi(\cdot)$ and $\Gamma(\cdot)$. Later in this section we will also derive numerical estimates for these functions in order to simulate the behaviors of the closed-loop systems.

Since $E(t) \equiv 0$ and $S(t) \equiv 0$, the constrained Sylvester differential equation (8.1) becomes

$$\begin{aligned} \dot{\Pi}(t) &= A\Pi(t) + B\Gamma(t) \\ 0 &= C\Pi(t) + D\Gamma(t) + F(t), \end{aligned}$$

where $\Pi(\cdot) \in C_\tau(\mathbb{R}, X)$ and $\Gamma(\cdot) \in C_\tau(\mathbb{R}, \mathbb{C})$. Since $D = 1 \neq 0$, the second equation implies $\Gamma(t) \equiv -C\Pi(t) - F(t)$. Substituting $\Gamma(\cdot)$ into the first equation we obtain

$$\dot{\Pi}(t) = (A - BC)\Pi(t) - BF(t).$$

Since $t \mapsto -BF(t)$ is a continuous periodic function, Theorem 6.10 can be used to show that if the semigroup $T_{A-BC}(t)$ generated by $A - BC$ is exponentially stable, then this equation has a unique periodic mild solution $\Pi(\cdot) \in C_\tau(\mathbb{R}, X)$ given by

$$\Pi(t) = -\int_{-\infty}^{t} T_{A-BC}(t-s)BF(s)ds.$$

Since for all $\begin{pmatrix}\alpha & f\end{pmatrix}^T \in \mathcal{D}(A)$ we have

$$(A - BC)\begin{pmatrix}\alpha \\ f\end{pmatrix} = \begin{pmatrix}-2\alpha + f(-1) \\ \frac{df}{d\theta}\end{pmatrix} - \begin{pmatrix}1 & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix}\alpha \\ f\end{pmatrix} = \begin{pmatrix}-3\alpha + f(-1) \\ \frac{df}{d\theta}\end{pmatrix},$$

it is easy to see that the semigroup $T_{A-BC}(t)$ is of the same form as $T_A(t)$, but the constant -2 is replaced by -3 in the formula (8.10). Using exactly the same procedure as earlier in this section it is easy to verify that this semigroup is exponentially stable. Substituting $\Pi(\cdot)$ into the formula for $\Gamma(\cdot)$ shows that for all $t \in \mathbb{R}$ we have

$$\Gamma(t) = -C\Pi(t) - F(t) = -F(t) + \int_{-\infty}^{t} CT_{A-BC}(t-s)BF(s)ds. \tag{8.11}$$

We can now turn to choosing the parameters of the periodic controllers. For the static state feedback law we already have all the necessary information at our disposal. In the case of the periodic error feedback controller we still need to stabilize the family (8.2) of operators.

The Static State Feedback Law

Since the operator A already generates an exponentially stable semigroup, we can choose $K = 0 \in \mathcal{L}(W,X)$ in Theorem 8.1. The result then states that the static state feedback law solving the periodic output regulation problem is obtained by choosing

$$L(\cdot) = \Gamma(\cdot) - K\Pi(\cdot) = \Gamma(\cdot) \in C_\tau(\mathbb{R},\mathbb{C}).$$

Since for any initial state $v_0 \in \mathbb{C}$ the state of the exosystem is given by $v(t) \equiv v_0$, the resulting static state feedback law is given by

$$u(t) = \Gamma(t)v(t) = \Gamma(t)v_0 = -F(t)v_0 + \int_{-\infty}^{t} CT_{A-BC}(t-s)BF(s)v_0 ds.$$

The Periodic Error Feedback Controller

Since the system (8.9) is exponentially stable, we can choose $K_1 = 0$ in Theorem 8.2. Using this we also see that

$$K_2(\cdot) = \Gamma(\cdot) - K_1\Pi(\cdot) = \Gamma(\cdot) \in C_\tau(\mathbb{R},\mathbb{C}).$$

We still need to choose the exponentially stabilizing function $L(\cdot)$ in the family (8.2) of operators. Since $F(t) = -F_r(t) \le -1$ for all $t \in \mathbb{R}$, the function $t \mapsto F(t)^{-1}$ is a continuous τ-periodic function. If we choose

$$L(t) = \begin{pmatrix}0 \\ -F(t)^{-1}\end{pmatrix}$$

for all $t \in \mathbb{R}$, we then have $L(\cdot) \in C_\tau(\mathbb{R}, \mathscr{L}(Y, X \times W))$ and

$$\begin{pmatrix} A & E(t) \\ 0 & S(t) \end{pmatrix} + L(t)\begin{pmatrix} C & F(t) \end{pmatrix} = \begin{pmatrix} A & 0 \\ -F(t)^{-1}C & -F(t)^{-1}F(t) \end{pmatrix} = \begin{pmatrix} A & 0 \\ -F(t)^{-1}C & -1 \end{pmatrix}.$$

Since the operator A generates an exponentially stable semigroup and since we have $-F(\cdot)^{-1}C \in C_\tau(\mathbb{R}, \mathscr{L}(X, W))$, Lemma A.2 implies that the evolution family associated to the system (8.2) of operators is exponentially stable.

Theorem 8.2 now concludes that our output regulation problem is solved by a periodic error feedback controller if we choose the family $(\mathscr{G}_1(t), \mathscr{D}(\mathscr{G}_1(t)))$ of operators in such a way that $\mathscr{D}(\mathscr{G}_1(t)) \equiv \mathscr{D}(A) \times \mathbb{C}$ and for all $t \in \mathbb{R}$

$$\begin{aligned} \mathscr{G}_1(t) &= \begin{pmatrix} A & E(t) \\ 0 & S(t) \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} \cdot \begin{pmatrix} K_1 & K_2(t) \end{pmatrix} + L(t)\Big[\begin{pmatrix} C & F(t) \end{pmatrix} + DK(t)\Big] \\ &= \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \Gamma(t) \end{pmatrix} - \begin{pmatrix} 0 \\ F(t)^{-1} \end{pmatrix} \Big[\begin{pmatrix} C & F(t) \end{pmatrix} + \begin{pmatrix} 0 & \Gamma(t) \end{pmatrix}\Big] \\ &= \begin{pmatrix} A & B\Gamma(t) \\ -F(t)^{-1}C & -1 - F(t)^{-1}\Gamma(t) \end{pmatrix}, \end{aligned}$$

and if we choose the operator-valued functions as $\mathscr{G}_2(\cdot) = -L(\cdot) = \begin{pmatrix} 0 & F(\cdot)^{-1} \end{pmatrix}^T$ and $K(\cdot) = \begin{pmatrix} 0 & \Gamma(\cdot) \end{pmatrix}$. The error feedback controller consists of a delay system and a one-dimensional ordinary differential equation. Because of this, its initial state is of the form

$$z_0 = \begin{pmatrix} z_0^1 \\ z_0^2 \end{pmatrix}, \qquad z_0^1 = \begin{pmatrix} \alpha \\ f \end{pmatrix},$$

where α and f are the initial value and the history of the delay part of the system, respectively, and z_0^2 is the initial value of the ordinary differential equation part of the system.

Approximation of the Parameters

To simulate the behavior of the controlled systems, we need to find an approximation for the function $\Gamma(\cdot)$ appearing in the parameters of the control laws. Since this is a periodic function with period $\tau = 2$, it is sufficient to compute $\Gamma(t)$ for $t \in [0, 2]$. Moreover, since the function $F(\cdot)$ is known, we only need to consider the integral in (8.11). This term can be divided into two parts as

$$\Gamma_\infty(t) + \Gamma_0(t) = \int_{-\infty}^0 CT_{A-BC}(t-s)BF(s)ds + \int_0^t CT_{A-BC}(t-s)BF(s)ds, \qquad (8.12)$$

where $\Gamma_\infty(\cdot), \Gamma_0(\cdot) \in C([0, \tau], \mathbb{C})$. In the following we will show that it is possible to find an explicit expression for the function $\Gamma_0(\cdot)$, and that we can write $\Gamma_\infty(\cdot)$ in such a way that it is easily approximated numerically with any given finite accuracy.

We will begin by considering $\Gamma_\infty(\cdot)$ in (8.12). Since $F(\cdot)$ is a periodic and even function (i.e., $F(-t)=F(t)$), for $\tau=2$ and for any $t\in[0,2]$ we have

$$\begin{aligned}\Gamma_\infty(t) &= \int_{-\infty}^{0} CT_{A-BC}(t-s)BF(s)ds = \sum_{n=0}^{\infty}\int_{-(n+1)\tau}^{-n\tau} CT_{A-BC}(t-s)BF(s)ds\\ &= \sum_{n=0}^{\infty} -CT_{A-BC}(t)\int_{\tau}^{0} T_{A-BC}(r+n\tau)BF(-r-n\tau)dr\\ &= \sum_{n=0}^{\infty} CT_{A-BC}(t+n\tau)\int_{0}^{\tau} T_{A-BC}(r)BF(r)dr. \end{aligned} \tag{8.13}$$

We can now use the properties of the semigroup $T_{A-BC}(t)$ to show that for the integrand in the last expression we have

$$T_{A-BC}(r)BF(r) = T_{A-BC}(r)\begin{pmatrix}1\\0\end{pmatrix}F(r) = \begin{pmatrix}g(r)\\ g(r+\cdot)\end{pmatrix}F(r),$$

where $g(\cdot)$ is a function satifying $g(\theta)=0$ for $\theta\in[-1,0)$, $g(r)=e^{-3r}$ for $r\in[0,1)$, and

$$g(r) = e^{-3r} + \int_1^r e^{-3(r-s)}e^{-3(s-1)}ds = e^{-3r} + e^{-3(r-1)}(r-1)$$

for $r\in[1,2)$. The value of the integral in (8.13) can be computed using

$$F(t) = -F_r(t) = \begin{cases} -t-1 & 0\le t<1\\ t-3 & 1\le t<2.\end{cases}$$

Indeed, for all $\theta\in[-1,0)$ we have

$$\begin{aligned}&\int_0^\tau T_{A-BC}(r)BF(r)dr = \int_0^\tau \begin{pmatrix}g(r)\\ g(r+\theta)\end{pmatrix}F(r)dr\\ &= \int_0^{-\theta}\begin{pmatrix}e^{-3r}\\0\end{pmatrix}(-r-1)dr + \int_{-\theta}^{1}\begin{pmatrix}e^{-3r}\\ e^{-3(r+\theta)}\end{pmatrix}(-r-1)dr\\ &+\int_1^{1-\theta}\begin{pmatrix}e^{-3r}\left(1+(r-1)e^3\right)\\ e^{-3(r+\theta)}\end{pmatrix}(r-3)dr + \int_{1-\theta}^{2}\begin{pmatrix}e^{-3r}\left(1+(r-1)e^3\right)\\ e^{-3(r+\theta)}\left(1+(r+\theta-1)e^3\right)\end{pmatrix}(r-3)dr\\ &= \frac{1}{27}\begin{pmatrix}-16+13e^{-3}+6e^{-6}\\ 16-6\theta-13e^{-3-3\theta}-6e^{-6-3\theta}-6e^{-3-3\theta}\theta\end{pmatrix} =: \begin{pmatrix}\alpha_1\\ f_1(\theta)\end{pmatrix}.\end{aligned}$$

This and the formula for the semigroup generated by the operator $A-BC$ can be used to find an expression for the function $\Gamma_\infty(\cdot)$. More precisely, we see that for all $t\in[0,2]$

$$\begin{aligned}\Gamma_\infty(t) &= \int_{-\infty}^{0} CT_{A-BC}(t-s)BF(s)ds = \sum_{n=0}^{\infty} CT_{A-BC}(t+n\tau)\int_0^\tau T_{A-BC}(r)BF(r)dr\\ &= \sum_{n=0}^{\infty} h_\infty(t+n\tau),\end{aligned}$$

where the function $h_\infty(\cdot)$ is such that $h_\infty(\theta) = f_1(\theta)$ for $\theta \in [-1,0)$ and for $t \geq 0$ its values can be evaluated sequentially using the integral equation

$$h_\infty(t) = e^{-3t}\alpha_1 + \int_0^t e^{-3(t-s)} h_\infty(s-1)ds.$$

This implies that we can use the series representation above to approximate the function $\Gamma_\infty(\cdot)$ in (8.12) with any given finite accuracy.

It only remains to find an expression for the term $\Gamma_0(\cdot)$ in (8.12). Similarly as in the case of $\Gamma_\infty(\cdot)$ above we can see that for all $t \in [0,2]$ we have

$$\Gamma_0(t) = \int_0^t CT_{A-BC}(t-s)BF(s)ds = \int_0^t h_0(t-s)F(s)ds,$$

where $h_0(\cdot)$ is a function such that $h_0(t) = e^{-3t}$ for $t \in [0,1)$ and

$$h_0(t) = e^{-3t} + \int_1^t e^{-3(t-s)}e^{-3(s-1)}ds = e^{-3t} + e^{-3(t-1)}(t-1)$$

for $t \in [1,2)$. This implies that for any $t \in [0,1)$ (because $t-s<1$) we have

$$\int_0^t CT_{A-BC}(t-s)BF(s)ds = \int_0^t e^{-3(t-s)}(-s-1)ds = -\frac{2}{9}(1-e^{-3t}) - \frac{1}{3}t,$$

and for $t \in [1,2)$

$$\begin{aligned}
&\int_0^t CT_{A-BC}(t-s)BF(s)ds \\
&= \int_0^{t-1} h_0(\overbrace{t-s}^{>1})F(s)ds + \int_{t-1}^1 h_0(\overbrace{t-s}^{<1})F(s)ds + \int_1^t h_0(\overbrace{t-s}^{<1})F(s)ds \\
&= \int_0^{t-1} \big(e^{-3(t-s)} + e^{-3(t-s-1)}(t-s-1)\big)(-s-1)ds + \int_{t-1}^1 e^{-3(t-s)}(-s-1)ds \\
&\quad + \int_1^t e^{-3(t-s)}(s-3)ds \\
&= -\frac{1}{27}\Big[6e^{-3t} + (6t+1)^{-3(t-1)} + (6t-28)\Big].
\end{aligned}$$

Combining these we can see that

$$\Gamma_0(t) = \begin{cases} e^{-3t} + e^{-3(t-1)}(t-1) & t \in [0,1) \\ -\frac{1}{27}\left[6e^{-3t} + (6t+1)^{-3(t-1)} + (6t-28)\right] & t \in [1,2). \end{cases}$$

This explicit expression for the term $\Gamma_0(\cdot)$ and the series representation for $\Gamma_\infty(\cdot)$ conclude that we can easily obtain good approximations for the function $\Gamma(\cdot)$ and, consequently, for the parameters of the controllers.

Simulation

With the aid of the expressions above we can simulate the output of the system (8.9) together with the two controllers we have constructed. Our main intention was to steer the output of the system to the triangle signal depicted in Figure 2.3. We can therefore fix the initial state $v_0 = 1$ of the exosystem.

Figure 8.2 shows the simulated output of the delay system with the static state feedback law for the initial state $x_0 = \begin{pmatrix} 0 & \sin(2\pi\cdot) \end{pmatrix}^T$ of the system. As remarked earlier, this type of controller uses the states of the plant and the exosystem in producing the control signal.

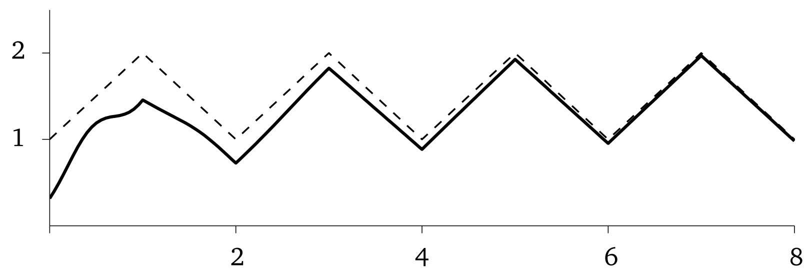

Figure 8.2: Output using the static state feedback law (solid).

If the states of the system and the exosystem are unavailable, the error feedback controller can be used to produce a control signal based on asymptotic estimates of these states. Figure 8.3 shows the simulated output of the system with our periodic error feedback controller for the initial states

$$x_0 = \begin{pmatrix} 0 \\ \sin(2\pi\cdot) \end{pmatrix}, \qquad z_0 = \begin{pmatrix} z_0^1 \\ 1 \end{pmatrix}, \qquad z_0^1 = \begin{pmatrix} 1 \\ \cos(4\pi\cdot) \end{pmatrix}$$

of the system and the controller.

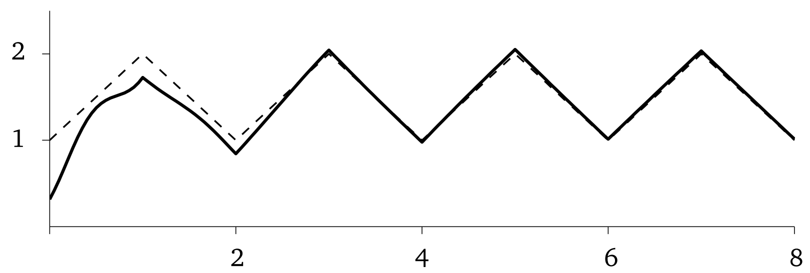

Figure 8.3: Output using the error feedback controller (solid).

This concludes our treatment of the periodic output regulation problem, and at the same time the more theoretical part of this thesis. In the next chapter we will make some concluding remarks concerning the results presented in the earlier chapters and discuss a few of the most interesting topics for further research.

Chapter 9

Conclusions

In this chapter we present concluding remarks regarding the topics considered in this thesis and look into some further directions of research arising from the theory. In particular we are in a position to compare the strengths and weaknesses of the theories of output regulation corresponding to the two different types of exosystems used in this thesis.

The most significant difference between the solutions presented for the output regulation problems for the infinite-dimensional and the periodic exosystem is the fact that the control structure for the infinite-dimensional signal generator guaranteed robustness with respect to perturbations in the parameters of the system. The lack of results concerning this desirable property for periodic controllers creates a serious imbalance between the strengths of the two theories. In hopes of remedying the situation we will discuss the possibilities and difficulties in bridging this gap between our two approaches to tracking continuous periodic signals.

We start by recapitulating the main contributions in each of the chapters. Subsequently we will in Section 9.1 compare the theories of output regulation related to the infinite-dimensional and periodic exosystems. In Section 9.2 we will consider the possibilities and difficulties related to the theory of robust output regulation for periodic exosystems. Finally, in Section 9.3 we will discuss other topics for further research related to the theory presented in this thesis.

Chapter 2: Generation of Nonsmooth Periodic Signals

The first main contribution of this thesis are the two ways of generating reference and disturbance signals of the form

$$y_{ref}(t) = y_n(t)t^n + \cdots + y_1(t)t + y_0(t),$$

where the coefficient functions $y_j(\cdot)$ are nonsmooth and periodic. We showed that it is possible to generate such signals by using either an infinite-dimensional exosystem with a block-diagonal system operator, or a finite-dimensional periodically time-dependent exosystem. The classes of signals generated by each of these exosystems were studied in detail. In particular, in the case of the infinite-dimensional signal generator it was shown that the smoothness properties of the generated signals can be conveniently related to the corresponding initial states using the scale spaces W_α related to the system operator S of the exosystem.

Chapter 3: Output Regulation Theory for Infinite-Dimensional Exosystems

In Chapter 3 we presented theory of output regulation for distributed parameter systems with infinite-dimensional exosystems. Most notably the results in the chapter generalize the characterization of the solvability of the output regulation problem known for diagonal exosystems to the case where the system operator of the exosystem consists of an infinite number of Jordan blocks. Moreover, the theory was generalized to allow the solution of the regulator equations to be an unbounded operator. In the light of the results presented in Chapter 2, the level of unboundedness of this operator could be related in a concrete way to the smoothness properties of the considered reference and disturbance signals. This way it was possible to use the smoothness properties of the exogeneous signals to weaken the conditions for the solvability of the output regulation problem.

Chapter 4: The Internal Model Principle for Distributed Parameter Systems

The main result of this thesis, the generalization of the internal model principle to distributed parameter systems with infinite-dimensional exosystems, was presented in Chapter 4. We also established precise conditions for the equivalences between three separate definitions for the internal model used in the literature. Due to the fact that we considered strong stability of the closed-loop system, we were required to add new assumptions to the statement of the result. These additional conditions are automatically satisfied for exponentially stabilizable systems with finite-dimensional exosystems.

Chapter 5: Robust Controller Design for Infinite-Dimensional Exosystems

Using the internal model principle we constructed an observer-based error feedback controller achieving robust output regulation for infinite-dimensional single-input single-output systems. Our approach of using pole placement to strongly stabilize the internal model allowed us to find sufficient conditions for the solvability of the robust output regulation problem. In particular these conditions relate in a very concrete way the behavior of the transfer function of the stabilized plant on the spectrum of the exosystem, $P_K(i\omega_k)$, and the smoothness of the admissible reference and disturbance signals. More precisely, a faster decay rate of $P_K(i\omega_k)$ as $|k| \to \infty$ requires a higher level of smoothness from the exogeneous signals in order for the results to guarantee the existence of a controller solving the robust output regulation problem.

Chapter 6: Solvability of the Infinite-Dimensional Sylvester Differential Equation

In Chapter 6 we studied the infinite-dimensional Sylvester differential equation, which was subsequently used in the treatment of the periodic output regulation problem in the later chapters. We defined two different types of solvability for this equation and presented conditions for the existence of a unique classical solution. In addition, we also derived sufficient conditions for the existence of a unique periodic solution to a periodic version of this equation.

Chapter 7: Periodic Output Regulation for Infinite-Dimensional Systems

The second main contribution of the thesis, the theory of output regulation for distributed parameter systems with periodic exosystems, was presented in Chapter 7. The first main result of the chapter stated that the state of the closed-loop system — and in particular its asymptotic behavior — could be described using a periodic mild solution of the associated infinite-dimensional Sylvester differential equation. Using this knowledge we were able to characterize the solvability of the periodic output regulation problem using the solvability of certain constrained Sylvester differential equations. The results presented in this chapter establish that methods similar to the ones familiar from the case of time-invariant exosystems can be succesfully applied in the treatment of the periodic output regulation problem for infinite-dimensional systems.

Chapter 8: Controller Design for Periodic Exosystems

In the final theoretically oriented chapter of this thesis we constructed a static state feedback law and a periodically time-dependent dynamic error feedback controller solving the periodic output regulation problem. The resulting control laws generalized well-known control structures from the theory of finite-dimensional and infinite-dimensional time-invariant systems. We concluded the chapter by studying an example of choosing the parameters of such controllers. In this example we considered steering the output of a scalar delay system to a nonsmooth triangle signal generated by a one-dimensional periodic exosystem. We constructed both a static state feedback law and a periodic error feedback controller solving the periodic output regulation problem.

9.1 Further Comparison of the Two Types of Exosystems

In this section we will compare the strengths and weaknesses of the theories of output regulation related to the two types of exosystems used in this thesis. The control of distributed parameter systems with time-invariant exosystems has been studied extensively in the literature. Already for this reason it comes as no surprise that the theory concerning these types of signal generators — even if infinite-dimensional — is much richer. However, we have already seen that the use of the periodic exosystem has many advantages. First one of these is that, as we demonstrated in Chapter 2, the periodic exosystem is an extremely natural way of generating reference and disturbance signals of the form

$$y_{ref}(t) = y_n(t)t^n + \cdots + y_1(t)t + y_0(t),$$

where $y_j(\cdot)$ are continuous but possibly nonsmooth periodic functions.

For time-invariant signal generators the regulator equations are a well-established set of tools in the study of output regulation. For infinite-dimensional systems they were first systematically used by Byrnes et.al. [6]. However, if we compare the corresponding results in Chapters 3 and 7, we can see that to a remarkable extent the theory for periodic exosystems can be developed in a same way as that for time-invariant exosystems. The main difference between the key results is that the Sylvester operator equations are replaced by the corresponding Sylvester differential equations.

Of the main elements of the theory of output regulation, especially the problem of stabilization has been studied much less for time-dependent systems than in the time-invariant case. Indeed, stabilization of strongly continuous semigroups is a well-known research area, whereas there are very little results available on the corresponding problem for infinite-dimensional nonautonomous systems. In the controller design for periodic output regulation we encounter this problem when using an observer-based error feedback controller. The existence of such a controller requires that we can choose a strongly continuous periodic function $L(\cdot)$ in such a way that the evolution family associated to the family

$$\left(\begin{pmatrix} A & E(t) \\ 0 & S(t) \end{pmatrix} + L(t) \begin{pmatrix} C & F(t) \end{pmatrix}, \mathcal{D}(A) \times W \right) \tag{9.1}$$

of operators on the space $X \times W$ is exponentially stable. In the example we considered in Section 8.4 we already saw that if the original plant is exponentially stable, if we do not consider rejection of disturbances (i.e., $E(t) \equiv 0$), and if the operators $F(t)$ are invertible for all $t \in [0, \tau]$, we can stabilize this system with an operator-valued function $L(\cdot)$ satisfying $L(t) \equiv \begin{pmatrix} 0 & -F(t)^{-1} \end{pmatrix}$. For this choice we have

$$\begin{pmatrix} A & E(t) \\ 0 & S(t) \end{pmatrix} + L(t) \begin{pmatrix} C & F(t) \end{pmatrix} = \begin{pmatrix} A & 0 \\ -F(t)^{-1}C & -I \end{pmatrix}$$

for all $t \in \mathbb{R}$. Since the operators A and $-I$ generate exponentially stable semigroups, the evolution family associated to this family of operators is exponentially stable by Lemma A.2. However, outside these kinds of special cases the availability of results on the stabilization of families of operators such as (9.1) is limited, but some results do exist even for more general inifinite-dimensional nonautonomous systems [34].

The stabilization of the closed-loop system also has its difficulties in the case of the infinite-dimensional exosystem. We already saw in Chapter 5 that since the internal model contained in the robust controller had an infinite number of eigenvalues on the imaginary axis, it was not possible to stabilize the closed-loop system exponentially. Instead, we had to consider strong closed-loop stability. In the case of an infinite-dimensional output space this has to be further weakened into *weak stability* [15]. On the other hand, we did see that it was possible to achieve this type of stability under very reasonable assumptions.

One of the most serious drawbacks of using an infinite-dimensional signal generator is the difficulty of realizing the resulting controller. The internal model principle studied in Chapter 4 shows us that in order for the controlled system to be able to robustly track and reject the signals generated by an infinite-dimensional exosystem, the controller itself must necessarily contain an infinite-dimensional component, namely the internal model of the signal generator. Implementing such a controller is not possible. We can of course use an approximation of the controller, but this differs very little from the result of directly designing a controller to solve the output regulation problem related to a finite-dimensional approximation of our original signal generator. On the other hand, using a periodic exosystem allows us to achieve tracking of infinite-dimensional signals with a finite-dimensional time-dependent controller. Implementing such control laws is in general possible. However, as we saw in the example considered in Section 8.4, it is still possible that in order to do this the parameters of the controller need to be approximated numerically. In our example this was the case with the infinite part of the integral related to the periodic solution of the Sylvester differential equation. The lack of knowledge on the robustness properties of the time-dependent controllers also makes it difficult to estimate the effect of such approximations on the success of the output regulation.

The realizability of controllers is an issue for any control systems attempting to track and reject general periodic signals. In the frequency domain the internal model based controller design for this problem is called *repetitive control* [59, 16, 56]. The frequency domain approach leads to a different set of strategies for approximating the reference and disturbance signals and, in particular, lowering the priority of tracking and rejecting the high frequency content in the signals [56, Sec. 2]. These realistic simplifications to the problem allow design of controllers that are capable of approximate tracking and disturbance rejection with high accuracy, but which are nevertheless implementable in practical applications.

In Section 2.4 we already discussed the differences between the classes of signals generated by the two types of exosystems. We saw that in general the classes of signals generated by infinite-dimensional exosystems are much larger than the ones generated by periodic exosystems. As was already remarked there, using a periodic exosystem can therefore be argued to lead to design of simpler controllers if we are only interested in tracking a narrow class of signals. However, in practice the choice between an infinite-dimensional time-invariant controller and a finite-dimensional time-dependent one makes it difficult to compare the complexities of the control laws.

In the case of the infinite-dimensional exosystem our use of the scale spaces W_α gave us a very concrete and powerful classification for the reference and disturbance signals based on their smoothness properties. The same spaces also helped us in Chapter 3 to further relate the smoothness of the signals to the conditions required for the solvability of the output regulation problem and, later in Chapter 5, to the existence of a controller solving the robust output regulation problem. For periodic signal generators no such classification is necessary because for such exosystems these smoothness properties are largely unaffected by the initial state, and the levels of smoothness of the generated signals can be readily determined from the differentiability properties of the functions $S(\cdot)$ and $F(\cdot)$. However, the effect of the smoothness properties of these parameters of the exosystem on the solvability of the periodic output regulation problem is a topic of further research.

9.2 Robust Output Regulation for Periodic Exosystems

As already mentioned, the main difference between the control structures designed for solving the output regulation problems related to the two types of exosystems is that the one designed for the infinite-dimensional exosystem is guaranteed to achieve tracking of the reference signals regardless of uncertainties in the parameters of the system to be controlled. In Chapter 4 we also saw that for infinite-dimensional exosystems the powerful internal model principle can be used to characterize such robust controllers. The purpose of this section is to discuss the possibilities and main difficulties in studying the robust output regulation for periodic exosystems and the possibility of deriving results similar to the internal model principle for this problem.

Our first task is obviously to find a suitable formulation for the robust output regulation problem for periodic exosystems. In particular, determining which operators are allowed to be perturbed and fixing the suitable classes of perturbations is not only essential to the usefulness of the resulting control law, but also crucial to the strength of the theory related to the problem. In the definition of the robust output regulation problem in Section 3.5 the control law was required to achieve tracking of the reference signals regardless of the perturbations in the operators A, B, C, D, E, and F of the system provided that the closed-loop remained strongly stable. In particular — since the operators E and F do not appear in the system operator A_e of the closed-loop system — the tracking of the reference signals was to be achieved for arbitrary operators $E' \in \mathcal{L}(W,X)$ and $F' \in \mathcal{L}(W,Y)$. However, when using a periodic exosystem these two parameters are no longer linear operators, but instead operator-valued functions

$$E(\cdot) \in C_\tau(\mathbb{R}, \mathcal{L}(W,X)), \qquad F(\cdot) \in C_\tau(\mathbb{R}, \mathcal{L}(W,Y)).$$

The operators E and F, and the operator-valued functions $E(\cdot)$ and $F(\cdot)$ consist of the output operators and functions of the signal generators. Roughly stated this means that allowing perturbations in these operators requires the robust controller to be able to track a larger class of signals than the one generated by the nominal exosystem.

If we want to keep an eye out for results similar to the internal model principle for periodic signal generators, we will inevitably face difficulties in finding a convenient class of perturbations in the statement of the robust output regulation problem. This is because ultimately the internal model principle studied in Chapter 4 was a consequence of the robustness of the controller with respect to perturbations in precisely the operators E and F. Indeed, the proof of Theorem 4.4 establishing the equivalence of conditional robustness of the controller and the internal model structure relied almost solely on the fact that we were able to choose these two operators in an appropriate manner. In the case of the periodic exosystem this kind of situation could be achieved by — and only by — allowing *all* possible perturbations of the operator-valued functions $E(\cdot) \in C_\tau(\mathbb{R}, \mathcal{L}(W,X))$ and $F(\cdot) \in C_\tau(\mathbb{R}, \mathcal{L}(W,Y))$. If this is done, however, the concept of internal model structure can be defined for periodic error feedback controllers with parameters $\mathcal{G}_1(\cdot)$, $\mathcal{G}_2(\cdot)$, and $K(\cdot)$ by replacing the condition in Definition 4.3 with

$$\dot{\Gamma}(t) + \Gamma(t)S(t) = \mathcal{G}_1(t)\Gamma(t) + \mathcal{G}_2(t)\Delta(t) \quad \Rightarrow \quad \Delta(t) \equiv 0.$$

For this class of admissible perturbations and for this definition of the internal model structure it is further possible to generalize Theorem 4.4 to show that the internal model structure is equivalent to the condition

$$\dot{\Sigma}(t) + \Sigma(t)S(t) = A_e(t)\Sigma(t) + B_e(t) \qquad \Rightarrow \qquad C_e(t)\Sigma(t) + D_e(t) \equiv 0,$$

which in turn can be seen as a generalization of the conditional robustness for periodic exosystems. For a stabilizing controller this condition is further equivalent to solving the robust output regulation problem. Analogously to the case of infinite-dimensional exosystems, this last equivalence follows directly from Theorem 7.1.

Unfortunately, allowing such large classes of perturbations of the functions $E(\cdot)$ and $F(\cdot)$ obviously undermines one of the greatest advantages we gained from using a nonautonomous exosystem: The ability to consider smaller classes of signals. If we allow arbitrary perturbations to the functions $E(\cdot)$ and $F(\cdot)$, we are essentially requiring that the controlled system must be able to track and reject arbitrary continuous τ-periodic signals. This, in turn, is again reflected in the resulting controller as complexity which is unnecessary for the purposes of tracking and rejecting the signals we were originally interested in.

Based on these observations it would be more sensible to instead state the robust output regulation problem in such a way that only the linear operators A, B, C and D directly related to the plant are allowed to be perturbed. Considering a problem defined in this way would probably not lead to results as general and elegant as the internal model principle, but it might nevertheless be possible to find conditions for controllers to achieve tracking of the reference signals regardless of admissible perturbations to these parameters. Conditions related to the robustness with respect to perturbations only in these operators would also be new for distributed parameter systems with time-invariant exosystems.

9.3 Other Topics for Further Research

One of the main topics of further research related to the robust output regulation for infinite-dimensional exosystems is the robustness of strong stability of strongly continuous semigroups. Finding classes of perturbations preserving the strong stability of the closed-loop system is of course instrumental in characterizing the perturbations under which the control structure is still capable of tracking the given reference signals. The preservation of strong stability of semigroups is an acknowledged open problem for which there exist no general results, but it has been studied in certain special cases. In particular, classes finite-rank of perturbations preserving strong stability have been presented in [38] for strongly stable Riesz-spectral operators whose spectra approach the imaginary axis only asymptotically. This is precisely the kind of situation we encountered in Section 5.3, where we stabilized the internal model with an infinite number of eigenvalues on the imaginary axis using pole placement. Also our particular problem concerning the preservation of the strong stability of a closed-loop system consisting of a plant and a dynamic error feedback controller was considered in [38, Sec. 6]. However, in this reference the perturbation was applied only to the copy of the operator S in the system operator $\mathscr{G}_1$ of the controller. The theory of robust output regulation presented in this thesis does not allow perturbations to this operator. Unfortunately, it can also be verified that it is not possible to successfully apply the results presented in the above reference to perturbations in any of the other operators appearing in the system operator A_e of the closed-loop system.

The further research topics also include the solvability of the Sylvester differential equation. Weakening the conditions for the existence of a unique periodic mild solution to this equation would allow us to in turn weaken the conditions for the solvability of the periodic output regulation problem. The conditions we have presented in this thesis are otherwise very minimal, but it would be worthwhile to study the possibility of replacing the requirement of the exponential stability of the closed-loop system with a weaker type of stability. The exponential stability of the evolution family $U_e(t,s)$ was only required to ensure that the operator $\Sigma_\infty(0)$ defined by

$$\Sigma_\infty(0)v = \int_{-\infty}^{0} U_e(0,s)B_e(s)U_S(s,0)v\,ds, \qquad v \in W$$

is a well-defined bounded linear operator, but the convergence of the integrals does not necessarily require stability of exponential type. For example, uniform and sufficiently fast polynomial decays of the integrands would be enough to guarantee that the operator $\Sigma_\infty(0)$ is bounded. One possible approach would be to consider a stability type similar to the *polynomial stability* [3] of strongly continuous semigroups. However, even defining such a concept for strongly continuous evolution families poses certain difficulties due to the fact that the evolution family $U_e(t,s)$ and the operators $A_e(t)$ do not in general commute.

As was already mentioned in Section 9.1, the problem of stabilization of nonautonomous systems has not been studied much in the literature. Further research on this topic is necessary for us to be able to use the periodic error feedback controller outside those special cases where we can determine the stability of the family (9.1) of operators using some particular structure or additional assumptions.

Appendix A

Selected Results From Functional Analysis

The following auxiliary results concerning operator matrices and strongly continuous evolution families are needed in the development of the theory presented in this thesis. The first result is a well-known fact concerning spectral properties of unbounded triangular operator matrices. The second one provides a bound for the growth of a strongly continuous evolution family related to a family of unbounded triangular operator matrices.

Lemma A.1. *Let X_1 and X_2 be Banach spaces and assume that $A_1 : \mathscr{D}(A_1) \subset X_1 \to X_1$, $A_2 : \mathscr{D}(A_2) \subset X_2 \to X_2$, and $A_{12} \in \mathscr{L}(X_2, X_1)$. The spectrum of the triangular block operator matrix $A : \mathscr{D}(A) \subset X_1 \times X_2 \to X_1 \times X_2$,*

$$A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix},$$

with domain $\mathscr{D}(A) = \mathscr{D}(A_1) \times \mathscr{D}(A_2)$ satisfies $\sigma(A) \subset \sigma(A_1) \cup \sigma(A_2)$.

Proof. It is straightforward to verify that if $\lambda \in \rho(A_1) \cap \rho(A_2)$, then the operator

$$\begin{pmatrix} R(\lambda, A_1) & R(\lambda, A_1)A_{12}R(\lambda, A_2) \\ 0 & R(\lambda, A_2) \end{pmatrix} \in \mathscr{L}(X_1 \times X_2)$$

is the bounded inverse of $\lambda I - A$ on $X_1 \times X_2$. This concludes that $\rho(A_1) \cap \rho(A_2) \subset \rho(A)$. □

Lemma A.2. *Assume that there exist exponentially bounded evolution families $U_1(t,s)$ and $U_2(t,s)$ associated to the families $(A_1(t), \mathscr{D}(A_2(t)))$ and $(A_2(t), \mathscr{D}(A_2(t)))$ of unbounded operators on Banach spaces X_1 and X_2, respectively. If $A_{12}(\cdot) \in C_\tau(\mathbb{R}, \mathscr{L}_s(X_2, X_1))$, then there exists a unique exponentially bounded evolution family $U(t,s)$ associated to the family $(A(t), \mathscr{D}(A(t)))$ of operators defined by*

$$A(t) = \begin{pmatrix} A_1(t) & A_{12}(t) \\ 0 & A_2(t) \end{pmatrix}, \qquad \mathscr{D}(A(t)) = \mathscr{D}(A_1(t)) \times \mathscr{D}(A_2(t)) \tag{A.1}$$

for all $t \in \mathbb{R}$. *This evolution family satisfies*

$$U(t,s) = \begin{pmatrix} U_1(t,s) & U_{12}(t,s) \\ 0 & U_2(t,s) \end{pmatrix}, \qquad U_{12}(t,s)x = \int_s^t U_1(t,r)A_{12}(r)U_2(r,s)x\,dr$$

for all $t \geq s$ *and* $x \in X_2$. *If* $M_1, M_2 \geq 1$ *and* $\omega_1, \omega_2 \in \mathbb{R}$ *are such that for all* $t \geq s$ *we have*

$$\|U_1(t,s)\| \leq M_1 e^{\omega_1(t-s)}, \qquad \text{and} \qquad \|U_2(t,s)\| \leq M_2 e^{\omega_2(t-s)},$$

then for any $\omega > \max\{\omega_1, \omega_2\}$ *there exists a constant* $M \geq 1$ *such that*

$$\|U(t,s)\| \leq M e^{\omega(t-s)}$$

for all $t \geq s$. *If* $\omega_1 \neq \omega_2$, *we can choose* $\omega = \max\{\omega_1, \omega_2\}$.

Proof. It is straightforward to verify that $U(t,s)$ is a strongly continuous evolution family. The fact that it is an evolution family associated to the operators (A.1) can be shown by considering $U(t,s)$ as a perturbation of the evolution family associated to the family

$$\left(\begin{pmatrix} A_1(t) & 0 \\ 0 & A_2(t) \end{pmatrix}, \mathcal{D}(A_1(t)) \times \mathcal{D}(A_2(t)) \right)$$

of operators [10, Thm. VI.9.19].

It remains to verify the exponential bound. It is clearly sufficient to show that the term $U_{12}(t,s)$ satisfies such an estimate. To this end, assume first that $\omega_1 \neq \omega_2$ and denote $\omega = \max\{\omega_1, \omega_2\}$. For any $x \in X_2$ with $\|x\| = 1$ and for all $t \geq s$ we have

$$\begin{aligned} \|U_{12}(t,s)x\| &\leq \int_s^t \|U_1(t,r)A_{12}(r)U_2(r,s)x\|dr \leq M_1M_2\|A_{12}\|_\infty \int_s^t e^{\omega_1(t-r)}e^{\omega_2(r-s)}dr \\ &= \frac{M_1M_2\|A_{12}\|_\infty}{|\omega_2 - \omega_1|} \cdot \left| e^{\omega_2(t-s)} - e^{\omega_1(t-s)} \right| \leq \frac{2M_1M_2\|A_{12}\|_\infty}{|\omega_2 - \omega_1|} \cdot e^{\omega(t-s)}. \end{aligned}$$

On the other hand, if $\omega_1 = \omega_2$, then for any $x \in X_2$ with $\|x\| = 1$ and for all $t \geq s$ we have

$$\begin{aligned} \|U_{12}(t,s)x\| &\leq \int_s^t \|U_1(t,r)A_{12}(r)U_2(r,s)x\|dr \leq M_1M_2\|A_{12}\|_\infty \int_s^t e^{\omega_1(t-r)}e^{\omega_2(r-s)}dr \\ &= M_1M_2\|A_{12}\|_\infty (t-s)e^{\omega_1(t-s)}. \end{aligned}$$

If $\omega > \omega_1$, we can clearly choose $M \geq 1$ in such a way that $\|U_{12}(t,s)\| \leq Me^{\omega(t-s)}$ for all $t \geq s$. □

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