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Tuomas Kovanen

Definition of Electric and Magnetic Forces on Riemannian Manifold

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Abstract

The precise relationship of electricity and magnetism to mechanics is becoming a critically important question in modern engineering. Although well known engineering methods of modeling forces in electric and magnetic systems have been adequate for bringing many useful technologies into our everyday life, the further development of these technologies often seems to require more extensive modeling. In the search for better modeling it is relevant to examine the foundations of the classical models.

The classical point charge definitions of electric and magnetic field quantities are not designed for the determination of forces in typical engineering problems. In the approach of this thesis the relationship of electric and magnetic field quantities to forces on macroscopic objects is built in the definitions. As a consequence, the determination of electrostatic and magnetostatic forces on macroscopic objects becomes a clear-cut issue. What makes the approach work is that the system of interacting objects is considered as a whole. This is contrary to the classical approach where an unrealistic test object is used to define electric and magnetic fields as properties of the source object only.

A limitation in the classical notions of electric field intensity and magnetic induction is that they are sufficient to determine forces on only rigid objects. To allow for deformable objects the model needs to be combined with that of continuum mechanics. The notions suggested in this thesis are more general as they allow rigidity to be considered with respect to the test object itself, which may not be rigid in the usual sense.

The reguired generality is obtained by using the mathematics of differential geometry as the framework for the definitions. This has the additional benefit of making clear the mathematical structures required for the definitions. Also, it allows the clear identification of the mathematical objects involved. Both of these outcomes are important for creating efficient and flexible computational codes for modeling.

Preface

I wrote this thesis while working with the electromagnetics research group at Tampere University of Technology. The subject of the thesis originates from scientific and technological interest in clarifying the relationship between electromagnetics and mechanics – a question that boils down to the definition of electromagnetic field quantities.

During my time with the electromagnetics group the head of the group has been Professor Lauri Kettunen. I appreciate Lauri's effort in creating and maintaining circumstances that make scientific work possible in our unit. Lauri has also been the supervisor of this work, and I want to thank him for his teaching and careful guidance along the way. An important advisor of this work has been Timo Tarhasaari. Some time ago, Timo managed to teach me the basics of the mathematics used in this thesis, and since then our discussions have greatly influenced my scientific thinking. I want to thank Timo and Lauri for trusting me with this particular research subject, and for giving me enough freedom and guidance for its investigation.

I am thankful to Maija-Liisa Paasonen and Lasse Söderlund for taking care of numerous administrative tasks so that I have had all the time to focus on this work. I thank Antti Stenvall for providing me with his finite element solver for use in this work. That gave me much freedom in force calculation as it allowed access to complete solution data of the field problem. I thank John Shepherd for the grammatical proof-reading of this thesis. Also, I want to thank the personell of our unit for keeping up a good atmosphere. The various ball games we have played together during the years have been special to me.

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It is not self-evident that one has a decent possibility to acquire an education. I thank my childhood family for giving me this possibility. Finally, I want to thank my wife Birgitta and my daughter Kerttu; I feel as lucky as a man can be for having you in my life.

Contents

List of symbols

Chapter 2

$\mathbb{R}$	Set of all real numbers
V	Finite dimensional vector space
V^*	Dual space of V
δ^i_j	Component of Kronecker tensor
U	Vector subspace of V
W	Transformation matrix between bases of V
$T^r_s(V)$	Vector space of tensors on V of type $\binom{r}{s}$
$\otimes$	Tensor product
$\bigwedge_p(V)$	Vector space of p-covectors on V
$\wedge$	Exterior product
$\mathrm{sgn}(\sigma)$	Signature of permutation σ
$\det(\cdot)$	Determinant
$\bigwedge^p(V)$	Vector space of p-vectors on V
Or	Orientation of V
$TransOr$	Transverse orientation of a vector subspace of V
$[\cdot]$	Equivalence class
g	Inner product on V
$\|\| \cdot \|\|$	Norm on V induced by g
$u\angle v$	Angle between vectors u and v

Chapter 3

Ω	Manifold modeling physical space
o_1, o_2	Usually submanifolds of Ω modeling the interacting objects

$W[o_1, o_2]$	Work done by o_1 on o_2
Ψ_t	1-parameter embedding of objects o_1 and o_2 into Ω
$\mathbb{R}^n$	Set of all n-tuples of real numbers
S	The underlying point set of a manifold
A	Smooth atlas
$\chi_{i,j}$	Charts for a manifold
M	Generic manifold, or manifold with boundary
H^n	Half space of $\mathbb{R}^n$
∂M	Boundary of M
N	Submanifold of M
$T_x M$	Tangent space of M at point x
γ	Path of movement of o_2
F_{12}	Force on object o_2, a covector
F_{21}	Force on object o_1, a covector
$\triangle$	Simplex of $\mathbb{R}^n$
s	Simplex of a manifold with parallelism
$\{s\}$	p-vector of simplex s
$\tilde{\rho}_1$	Charge density of object o_1, a twisted 3-form
$\tilde{\rho}_2$	Charge density of object o_2, a twisted 3-form
$\tilde{\sigma}_1$	Surface charge density of object o_1, a twisted 2-form
$\tilde{\sigma}_2$	Surface charge density of object o_2, a twisted 2-form
$\cup$	Set union
Q_1	Total charge of o_1
Q_2	Total charge of o_2
$\tilde{D}_1$	Electric displacement of object o_1, a twisted 2-form
$\tilde{D}_2$	Electric displacement of object o_2, a twisted 2-form
$[\cdot]_1$	Discontinuity over ∂o_1
$[\cdot]_2$	Discontinuity over ∂o_2
d	Exterior derivative
t_1	Tangential trace to ∂o_1
s_t	Simplex parametrized by t
$\tilde{F}_{12}$	Volume force density of object o_2, a covector-valued 3-form
$\tilde{F}_{21}$	Volume force density of object o_1, a covector-valued 3-form
$\tilde{f}_{12}$	Surface force density of object o_2, a covector-valued 2-form
$\tilde{f}_{21}$	Surface force density of object o_2, a covector-valued 2-form
$T_x^p M$	p-vector space of M at point x

$T_p^x M$	p-covector space of M at point x
$L(T_x^p M; T_1^x M)$	Vector space of linear maps from $T_x^p M$ to $T_1^x M$
$L(T_x^1 M; T_p^x M)$	Vector space of linear maps from $T_x^1 M$ to $T_p^x M$
$\mathcal{G}$	Linear isomorphism between $L(T_x^p M; T_1^x M)$ and $L(T_x^1 M; T_p^x M)$
∇	Connection, usually Levi-Civita
∇_u	Covariant derivative to the direction of u
Γ_{ij}^k	Christoffel symbol
$(\tilde{F}_{12})_i$	Component 3-form of $\tilde{F}_{12}$ in a covector basis
E_1	Electric field intensity of object o_1
E_2	Electric field intensity of object o_2
i_v	Interior product with respect to vector field v
$L(T_x^2 \partial o_2; T_1^x \partial o_2)$	Vector space of linear maps from $T_x^2 \partial o_2$ to $T_1^x \partial o_2$
$L(T_x^2 \partial o_2; T_1^x \Omega)$	Vector space of linear maps from $T_x^2 \partial o_2$ to $T_1^x \Omega$
ϵ_0	Permittivity of vacuum
$\star$	Hodge operator
g	Riemannian metric
$< \cdot, \cdot >$	Inner product of multivectors
$\tilde{T}_x^{n-p} M$	Vector space of twisted (n-p)-vectors at point $x \in M$
$[u, v]$	Lie bracket of vector fields u and v
$\partial / \partial x^i$	Basis vector field of a coordinate chart, partial derivative
$\tilde{T}_{12}$	Stress of object o_2, a covector-valued 2-form
$\tilde{T}_{21}$	Stress of object o_1, a covector-valued 2-form
d_∇	Covariant exterior derivative
$\dot{\wedge}$	Generalized exterior product
o'_{12}	Parallelizable neighbourhood containing o_1 and o_2
E	The sum of E_1 and E_2
$\tilde{D}$	The sum of $\tilde{D}_1$ and $\tilde{D}_2$
t_2^+	Tangential trace to ∂o_2 from outside of o_2
t_2^-	Tangential trace to ∂o_2 from inside of o_2
$\tilde{F}_2, \tilde{F}_{22}$	Covector-valued 3-forms, see (3.23) and (3.24)
$\tilde{f}_2, \tilde{f}_{22}$	Covector-valued 2-forms, see (3.25) and (3.26)
$\tilde{T}_{22}$	Covector-valued 2-form, see (3.27) and (3.28)
$\tilde{F}$	Covector-valued 3-form, see (3.30)
$\tilde{f}$	Covector-valued 2-form, see (3.31)
$\tilde{T}$	Covector-valued 2-form, see (3.32) and (3.33)
$\tilde{J}_1$	Current density of object o_1, a twisted 2-form

$\tilde{J}_2$	Current density of object o_2, a twisted 2-form
$\tilde{j}_1$	Surface current density of object o_1, a twisted 1-form
$\tilde{j}_2$	Surface current density of object o_2, a twisted 1-form
$\tilde{H}_1$	Magnetic field intensity of object o_1, a twisted 1-form
$\tilde{H}_2$	Magnetic field intensity of object o_2, a twisted 1-form
B_1	Magnetic induction of object o_1, a 2-form
B_2	Magnetic induction of object o_2, a 2-form
μ_0	Permeability of vacuum
B	The sum of B_1 and B_2
$\tilde{H}$	The sum of $\tilde{H}_1$ and $\tilde{H}_2$

Chapter 4

I	Real number interval containing 0
U	Open subset of a manifold
ϕ	flow (parametrized transformation on a manifold)
ϕ_t	flow at parameter value t,
$\mathcal{L}_v$	Lie derivative with respect to vector field v
$(\phi_t^{-1})_*$	Pushforward by the inverse of ϕ_t
ϕ_t^*	Pullback by ϕ_t
F_{12}	Force on o_2, a map from Killing vectors to reals
$\tilde{\theta}$	Angle twisted vector field
r	Position vector field
x^i	Euclidean coordinate 0-form
$L(T^p_x M; T^x_q M)$	Vector space of linear maps from $T^p_x M$ to $T^x_q M$
$L(T^q_x M; T^x_p M)$	Vector space of linear maps from $T^q_x M$ to $T^x_p M$
$\mathcal{G}$	Linear isomorphism between $L(T^p_x M; T^x_q M)$ and $L(T^q_x M; T^x_p M)$
$\tilde{T}^x_{n-q} M$	Vector space of twisted (n-q)-covectors at $x \in M$
$L(T^p_x M; \tilde{T}^x_{n-q} M)$	Vector space of linear maps from $T^p_x M$ to $\tilde{T}^x_{n-q} M$
$\star$	Value-side Hodge operator, or the ordinary one
i_v	Value-side interior product, or the ordinary one
$\tilde{\tau}_{12}$	Torque on object o_2, a twisted covector
$\tilde{K}_{12}$	Covector-valued 2-form, see (4.14) and (4.15)
$\tilde{K}_{22}$	Covector-valued 2-form, see (4.21) and (4.22)

Chapter 5

$\tilde{\rho}_1^f, \tilde{\rho}_2^f$	Free charge densities, see (5.6) and (5.8)
$\tilde{\rho}_1^p, \tilde{\rho}_2^p$	Polarization charge densities, see (5.6) and (5.8)
$\tilde{\sigma}_1^f, \tilde{\sigma}_2^f$	Free surface charge densities, see (5.7) and (5.9)
$\tilde{\sigma}_1^p, \tilde{\sigma}_2^p$	Polarization surface charge densities, see (5.7) and (5.9)
$\tilde{P}_1, \tilde{P}_2$	Polarizations of objects o_1 and o_2, see (5.10)-(5.13)
$\tilde{D}'_1, \tilde{D}'_2$	Twisted 2-forms, see (5.14)-(5.15)
$\tilde{D}'$	Sum of $\tilde{D}'_1$ and $\tilde{D}'_2$
$\tilde{P}$	Sum of $\tilde{P}_1$ and $\tilde{P}_2$
$\tilde{\rho}^f$	Sum of $\tilde{\rho}_1^f$ and $\tilde{\rho}_2^f$
$\tilde{\sigma}^f$	Sum of $\tilde{\sigma}_1^f$ and $\tilde{\sigma}_2^f$
χ	Electric susceptibility
$\tilde{P}_r$	Remanent polarization, twisted 2-form
$\tilde{J}_1^f, \tilde{J}_2^f$	Free current densities, see (5.27) and (5.29)
$\tilde{J}_1^m, \tilde{J}_2^m$	Magnetization current densities, see (5.27) and (5.29)
$\tilde{j}_1^f, \tilde{j}_2^f$	Free surface current densities, see (5.28) and (5.30)
$\tilde{j}_1^m, \tilde{j}_2^m$	Magnetization surface current densities, see (5.28) and (5.30)
$\tilde{M}_1, \tilde{M}_2$	Magnetizations of objects o_1 and o_2, see (5.31)-(5.34)
$\tilde{H}'_1, \tilde{H}'_2, \tilde{H}'$	Twisted 1-forms, see subsection 5.2.1
$\tilde{M}$	Sum of $\tilde{M}_1$ and $\tilde{M}_2$
$\tilde{J}^f$	Sum of $\tilde{J}_1^f$ and $\tilde{J}_2^f$
$\tilde{j}^f$	Sum of $\tilde{j}_1^f$ and $\tilde{j}_2^f$
χ_m	Magnetic susceptibility, see (5.40)
$\tilde{M}_r$	Remanent magnetization, twisted 1-form, see (5.40)
$\tilde{\rho}_1^m, \tilde{\rho}_2^m$	Magnetic charge densities of objects o_1 and o_2
$\tilde{\sigma}_1^m, \tilde{\sigma}_2^m$	Magnetic surface charge densities of objects o_1 and o_2
M_1, M_2	Magnetic polarizations of o_1 and o_2, see (5.48)-(5.51)
B'_1, B'_2, B'	2-forms, see subsection 5.2.2
M	Sum of M_1 and M_2
$\tilde{J}$	Sum of $\tilde{J}_1$ and $\tilde{J}_2$
$\tilde{j}$	Sum of $\tilde{j}_1$ and $\tilde{j}_2$
χ'_m	Magnetic susceptibility, see (5.57)
M_r	Remanent magnetic polarization, 2-form, see (5.57)

Chapter 6

I	Value of current, real number
S	Surface whose boundary is loop carrying current I
B_{ext}	Magnetic induction caused by external sources, a 2-form
$\{S\}$	The 2-vector of surface S
$F(v)$	Virtual work done when displacing dipole by vector field v
F_x	Force on a dipole at point x, a covector
v_x	Virtual displacement vector at point x
m	Magnetic dipole moment 2-vector
$\tilde{C}_x$	Couple on a dipole at point x, a twisted covector
$\tilde{F}$	Force density, a covector-valued twisted 3-form
$\tilde{C}$	Couple density, a covector-valued twisted 3-form
$\tilde{m}$	Magnetization, a 2-vector-valued twisted 3-form
$\tilde{V}$	Twisted 3-form related to a local 1-form basis
m	2-vector representing the value-part of $\tilde{m}$
U	Small volume containing dipoles
m_i	Dipole moment 2-vector at ith point in U
v_i	Virtual displacement vector of ith dipole in U
$\tilde{q}_m$	Magnetic charge, a twisted scalar
$\tilde{H}_{ext}$	External magnetic field intensity, a twisted 1-form
C	Path whose boundary points have magnetic charges $\pm\tilde{q}_m$
$\{C\}$	The vector of path C
$\tilde{\mathrm{m}}$	Magnetic dipole moment twisted vector
$\tilde{C}_{12}$	Couple density of object o_2, a covector-valued 3-form
$\tilde{C}_{21}$	Couple density of object o_1, a covector-valued 3-form
$\tilde{C}_2$	Covector-valued twisted 3-form, see (6.31) or (6.46)
$\tilde{C}_{22}$	Covector-valued twisted 3-form, see (6.32) or (6.47)

Chapter 7

d	Separation parameter, non-negative real number
$v_{\|\|}$	Component of vector field v tangent to ∂o_2
$v_{\perp}$	Component of vector field v normal to ∂o_2

n	Unit normal vector field of ∂o_2
n_2	Normal trace to ∂o_2
$\star_s$	Surfacic Hodge operator
$\tilde{M}^{\pm}$	The value of $\tilde{M}$ at ∂o_2 from outside or from inside
$M^{\pm}$	The value of M at ∂o_2 from outside or from inside

Chapter 8

$\boldsymbol{J}$	Cartesian proxy-vector field for current density
$\boldsymbol{B}$	Cartesian proxy-vector field for magnetic induction
$\times$	Cross product
$\nabla\times$	The curl operator in Cartesian coordinates
$\boldsymbol{a}, \boldsymbol{b}$	Arbitrary Cartesian vector fields
$\nabla\cdot$	Divergence in Cartesian coordinates
$\bar{I}$	Identity matrix

Appendix A

e_1, e_2, e_3	Basis vector fields of a Cartesian coordinate chart
$\omega^1, \omega^2, \omega^3$	1-forms of the basis dual to (e_1, e_2, e_3)
$\boldsymbol{E}_1$	Triplet of components of Cartesian proxy for E_1
$\boldsymbol{B}_1$	Triplet of components of Cartesian proxy for $\star B_1$
$\boldsymbol{v}$	Triplet of Cartesian components of v
$\mathrm{d}V$	Cartesian volume form
ρ_1, ρ_2	Functions representing $\tilde{\rho}_1$ and $\tilde{\rho}_2$
n_1, n_2	Outward unit normals of ∂o_1 and ∂o_2
n	Sum of n_1 and n_2
$\boldsymbol{n}$	Triplet of Cartesian components of n
$\mathrm{d}A$	Area form on $\partial o_1 \cup \partial o_2$
σ_1, σ_2	Functions representing $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$
$\boldsymbol{T}$	Matrix representing $\tilde{T}$ in Cartesian basis
$\boldsymbol{E}$	Triplet of components of Cartesian proxy for E
E_1, E_2, E_3	Cartesian components of E

T_i	Component 2-forms of $\tilde{T}$
T_i^j	Entry of matrix $\boldsymbol{T}$
$(\nabla \cdot \boldsymbol{T})_i$	Entry of triplet $\nabla \cdot \boldsymbol{T}$
∂_j	Derivative with respect to the jth Cartesian coordinate
$\boldsymbol{T}_{11}$	Matrix representing $\tilde{T}_{11}$ in Cartesian basis
$\boldsymbol{T}_{22}$	Matrix representing $\tilde{T}_{22}$ in Cartesian basis
$\boldsymbol{P}_1, \boldsymbol{P}_2$	Triplets representing $\tilde{P}_1$ and $\tilde{P}_2$
ρ_1^f, ρ_2^f	Functions representing $\tilde{\rho}_1^f$ and $\tilde{\rho}_2^f$
$\boldsymbol{D}'_1, \boldsymbol{D}'_2, \boldsymbol{D}'$	Triplets representing $\tilde{D}'_1$, $\tilde{D}'_2$ and $\tilde{D}'$
$\boldsymbol{J}_1, \boldsymbol{J}_2$	Triplets representing $\tilde{J}_1$ and $\tilde{J}_2$
$\hat{j}_2$	1-form on Ω whose tangential trace on ∂o_2 is $\tilde{j}_2$
$\boldsymbol{j}_1, \boldsymbol{j}_2$	Triplets representing $\hat{j}_1$ and $\hat{j}_2$
$\boldsymbol{H}_1, \boldsymbol{H}_2$	Triplets representing $\tilde{H}_1$ and $\tilde{H}_2$
ρ_1^m, ρ_2^m	3-forms, or their representative functions
σ_1^m, σ_2^m	2-forms, or their representative functions
$\boldsymbol{M}_1, \boldsymbol{M}_2$	Triplets representing $\tilde{M}_1$ and $\tilde{M}_2$, or M_1 and M_2
$\boldsymbol{J}_1^f, \boldsymbol{J}_2^f$	Triplets representing $\tilde{J}_1^f$ and $\tilde{J}_2^f$
$\boldsymbol{H}'_1, \boldsymbol{H}'_2, \boldsymbol{H}'$	Triplets representing $\tilde{H}'_1$, $\tilde{H}'_2$ and $\tilde{H}'$
$\boldsymbol{B}'_1, \boldsymbol{B}'_2, \boldsymbol{B}'$	Triplets representing B'_1, B'_2 and B'

Appendix B

$\boldsymbol{\theta}$	Cartesian components of $\tilde{\theta}$
$\boldsymbol{r}$	Cartesian components of r
σ_2^f	Function representing $\tilde{\sigma}_2^f$
$\boldsymbol{j}_2^f$	Triplet representing $\tilde{j}_2^f$

Appendix C

$\boldsymbol{M}_2^-$	Triplet representing $\tilde{M}_2^-$, or M_2^-
$\boldsymbol{n}_2$	Triplet of Cartesian components of n_2
$\boldsymbol{M}_{2t}^-$	Tangential component of $\boldsymbol{M}_2^-$

Appendix D

$\mathcal{V}_1, \mathcal{V}_2$	Sets of 3-dimensional submanifolds
$\partial\mathcal{V}_1, \partial\mathcal{V}_2$	Sets of 2-dimensional submanifolds
$FORCE$	Force as a map of specific type, see (D.1)
$STRESS$	Stress as a map of specific type, see (D.1)
$\boldsymbol{T}_{12}$	Matrix representing $\tilde{T}_{12}$ in Cartesian basis
$\boldsymbol{T}_{21}$	Matrix representing $\tilde{T}_{21}$ in Cartesian basis
$\mathcal{N}^*$	Set of 3-dimensional elements of the dual mesh
V	Usually element of $\mathcal{N}^*$
λ	Nodal function of the node dual to V

Chapter 1

Introduction

Modern engineering involves the designing of electromechanical equipment more and more accurately. This trend is driven by economy: often even small improvement in performance or efficiency can result in significant economic benefit. This can be seen in the design of everyday equipment such as electric motors, where efficiency improvement and noise level reduction have received increasing attention in industry. The accurate design of electromechanical equipment, in turn, calls for detailed computational modeling.

Although electricity and magnetism have been the subject of scientific interest for millenia, the modeling of electric and magnetic forces in engineering problems is still, surprisingly, a rather confusing one. Especially, the modeling of electric and magnetic forces in systems where deformations occur has remained an open question. In principle, there should be no problem in determining forces from known electric and magnetic field quantities because these quantities are defined in terms of forces in the first place; the definitions should yield the forces directly. That this does not work in the usual engineering tasks means the classical definitions of electric field intensity and magnetic induction are not tailor-made for such purposes.

To clarify the situation we recall that in the classical definitions the idealized notion of point charge is used. Also, to make the field quantities independent of test charge (observer) the effect of the test charge on the underlying charge and current distributions is removed by a limiting process concerning the magnitude of the test charge. Consequently, the defining equation (Lorentz force law) can be used directly only to determine forces on small (test) particles whose charge magnitudes are also small so that they do not affect the underlying charge and current distributions.

For the determination of electric and magnetic forces in most common engineering applications (such as electromechanical actuators of finite size) electric and magnetic field quantities will be defined here such that they take into account the effect of observer on the underlying charge and current distributions. The observer here is taken to be an interacting object of finite size (such as the rotor of an electric motor). Indeed, in practice we usually want to determine forces on objects that are not infinitely small, and that have a definite influence on the underlying charge and/or current distribution. Once all concepts are properly defined the issue of electric and magnetic forces is no longer a mystery.

We aim at a general framework that is applicable even when the objects are deformable. Currently, the modeling of such systems relies on the classical definition of electric field intensity and magnetic induction. In particular, this means the following.

> Classical modeling relies on the use of a Cartesian rigid body (metric) as a reference.

Consequently, one ends up in a situation where the modeling of deformable objects requires the coupling of electricity or magnetism with the continuum mechanical model. This kind of modeling underlies the present-day technology employing electromechanical equipment. However, the approach involves inherent difficulties such as the ambiguous separation of local force into an electromagnetic part, and a short-range part to be taken into account by mechanical stresses. Here, the classical approach will not be followed.

> In this thesis, we will not restrict ourselves to the use of a Cartesian rigid body as a reference.

Instead, the reference for rigidity will be adapted to the situation.

> In this thesis, the interacting test object itself is used as a reference for rigidity.

The goal is to make known methods for rigid interacting objects applicable also in nontrivial situations involving (Cartesian) deformable objects. This would be a benefit compared to the classical modeling.

To achieve the desired generality the definitions will be given on an abstract level (meaning that we consider the mathematical structures involved). The stage for the definitions is provided by modern mathematics of *differential geometry*. To make different structure layers visible we will begin with a *differentiable manifold* and add more structure when necessary. Eventually we will have need of a metric structure, which leads us to *Riemannian manifold*. This results in an *axiomatic system* where the instance of metric is not specified. In the axiomatic system for electrostatics, for instance, charge will be taken as an additional *primary term*. We will *define* force on a rigid body in terms of the primary terms charge and metric. In constructing the definition we will introduce electric field intensity and electric displacement as auxiliary terms with certain *defining properties*. The definition can be roughly stated as follows.

> *A map taking constant vector fields (virtual displacements) to real numbers (virtual works) on a Riemannian manifold is* an electrostatic rigid-body force on the Riemannian manifold *if there exists a piecewise smooth twisted 3-form (charge density), a twisted 2-form (electric displacement), and a 1-form (electric field intensity), with the electrostatic laws, including a force law, as defining properties.*

This purely mathematical definition contains a clear predicate to test whether a mathematical object satisfying the preliminary condition deserves to be called electrostatic force.

In the above definition we talk of the constancy of vector fields – a notion that is in general defined by using an independent structure. Here this notion will be defined by using metric structure so that constancy of vector fields will be taken relative to metric. Thus, the specification of an instance for metric structure will give an instance for the axiomatic system. To make a connection with observations we make the following definition.

> *An electrostatic rigid-body force on a specific Riemannian manifold is* a physical model *if it correctly predicts virtual works observed under constant virtual displacements.*

The predicate of this physical definition involves observations with virtual displacements that are constant in the specific metric. It provides a test of whether a specific metric and charge fit the modeling of a physical situation. When using the metric provided by a Cartesian coordinate system we get the classical model for electrostatic forces on rigid objects.

By adding mathematical structure layer by layer we will clearly see which structures are forced to us by the laws that govern electrostatic and magnetostatic forces on rigid objects. This question is relevant from both the technical and philosophical points of view. The technical interest in this question arises from the computational modeling of the natural phenomena in question. Computer programs used to model the phenomena imitate the underlying mathematical structures, so to write these programs it is essential to have a comprehensive understanding of the structures involved. On the other hand, a specific question of philosophical interest is whether the laws of electrostatics or magnetostatics really force us to use the structure of Euclidean space (affine space whose accociated vector space has an inner product) in the formulation of these laws. We will see that Euclidean coordinates only need to exist locally in the vicinities of the interacting objects if we want to define torques on the objects. For the definition of forces on rigid objects the vicinities of the objects only need to be parallelizable.

In the given definitions I will focus on the electrostatic and magnetostatic interactions between two material objects in free space. This abstract situation is relevant for many practical engineering applications. The accommodation of materials will be straightforward when charges and currents are understood to include also *equivalent charges* and *equivalent currents*, as referred to in the engineering community. Besides equivalent charges and currents we will also examine the use of *polarization* and *magnetization* as primary terms.

An aspect of the given definitions is that each of the two objects will be provided with its own set of field quantities. Thus, from the point of view of one of the objects the fields of the other object appear as external fields. The external interactions will be taken as a primary notion to emphasize that they constitute what is unambiguous in classical mesoscopic treatments of electric and magnetic forces involving materials. The given definitions will be suitable for modeling the behavior of rigid objects. By adding the fields of the two objects we will obtain total fields, and all expressions involving total fields will be taken as theorems derived from the starting points involving external fields. This is a matter of taste, and the construction may be reversed if preferred. However, there is a benefit by using the external fields, and this is the fact that the fields of one of the objects will be smooth on the material interfaces of the other object.

Most of the time we will assume that the interacting objects are separated by free space. In these cases it will be seen that in electrostatics the forces and torques on rigid objects may be defined in terms of equivalent charges or in terms of electric polarization. The equivalence of these alternative definitions is a theorem in the axiomatic system. In magnetostatics one can use as primary terms either equivalent currents, equivalent magnetic charges, magnetization, or magnetic polarization. Further theorems are that forces and torques may also be determined from the total fields. Also, the Maxwell's stress tensor method of force calculation will appear as a theorem implied by the used starting points. When the two objects are in contact there will be some difficulties with surface terms, and thus we will be left with unproved conjectures. The question of local forces inside deformable objects will be touched upon, but it seems that one cannot unambiguously define the concept of electric or magnetic force density in materials that will fit for the modeling of deformations.

The organization of this thesis is as follows. Chapter 2 introduces tensors in abstract vector space. Here, emphasis is put on *multivectors* and *multicovectors*, as these antisymmetric tensors fit so naturally to the modeling of electromagnetic phenomena. This chapter is a preliminary to the main content of this thesis, and it is also there to serve the student of electromagnetism in his or her beginning studies. If one is familiar with multivectors and multicovectors it is possible to skip this chapter and start directly from Chapter 3. (A review of twisted multivectors and multicovectors may still be in order.) In Chapter 3 we construct the predicates defining electric and magnetic forces on objects with charge and current distributions. The required differential geometric concepts will be introduced along the way, elucidating the role of each concept in the axiomatic system. Providing the stage for the definitions this general framework will appear as indentions to the main text. Among the concepts encountered are *covector-valued differential forms*, their integration and (covariant) exterior differentiation. In Chapter 4 the concept of force will be generalized to account for rotations of the objects. The description of rotations of the interacting objects by appropriate virtual displacement vector fields will prove useful in our construction. These vector fields will be conveniently characterized by using the *Lie derivative*. In chapters 5 and 6 we will consider forces and torques in the presence of materials, whereas in Chapter 7 situations in which the interacting objects are in contact with each other will be examined. Chapter 8 is devoted to finding direct engineering benefit from the given approach. Here, I will examine magnetic forces by considering simple example arrangements, and suggest a reinterpretation of the heuristic engineering rule that "magnetic field lines tend to shorten themselves to produce forces". This chapter does not build on the earlier chapters, so it is possible to read this before going to the formal treatment of the other chapters. Finally, conclusions are made in Chapter 9.

This thesis contains four appendices. The first three show how the classical vector analysis formulae for force densities, stresses, and torque densities can be obtained from the expressions appearing in the text. Here, Cartesian metric will be used for familiarity, and results given on a Cartesian coordinate basis. Appendix D shows how force densities can be computed from finite element approximations of electric and magnetic field quantities.

Chapter 2

Mathematical preliminaries

A basic tool in modeling is linearization. For example, the notion of force is obtained by linearizing energy, making force a linear operator that takes in displacement vectors to produce small (infinitesimal) changes in energy. Linear operators that take vectors to real numbers are in general called tensors. In later chapters we will make use of linearity locally at the points of the underlying physical space, that is, we will find it necessary to introduce tensor fields. For this purpose we first introduce tensors in vector space.

A bare vector space structure is needed to define tensors. No metric notions, such as an angle between vectors and the length of a vector, are required. These notions will be taken relative to an inner product, which is an additional structure that will be given for a vector space. Being aware of which parts of the analysis call for metric is essential to achieve our technological goal.

2.1 Vectors and covectors

Vector space is the mathematical structure needed for the addition and scalar multiplication of the elements of a set. The elements of vector space are called *vectors*. Formally, a set V is called *vector space* (over real numbers) if there is vector addition $+ : V \times V \to V$ and scalar multiplication $\cdot : \mathbb{R} \times V \to V$ satisfying the following axioms. The vector addition is commutative and associative, that is,

$$u + v = v + u$$
$$(u + v) + w = u + (v + w)$$

for all $u, v, w \in V$. There is a zero element $0 \in V$ satisfying

$$u + 0 = u$$

for all $u \in V$. Also, for all $u \in V$, there is an addition inverse $-u \in V$, such that

$$u + (-u) = 0.$$

Scalar multiplication must distribute with both the vector addition and the addition of real numbers, that is,

$$a \cdot (u + v) = a \cdot u + a \cdot v$$
$$(a + b) \cdot u = a \cdot u + b \cdot u$$

for all $a, b \in \mathbb{R}$ and $u, v \in V$. Scalar multiplication must also be compatible with the multiplication of real numbers, that is,

$$a \cdot (b \cdot u) = (ab) \cdot u$$

for all $a, b \in \mathbb{R}$ and $u \in V$. Finally, we must have

$$1 \cdot u = u$$
$$0 \cdot u = 0$$

for all $u \in V$. From now on the centered dot indicating the scalar multiplication will be left out. The *dimension* of a vector space V is the number of linearly independent vectors that span V. I will consider here only finite dimensional vector spaces. A set of linearly independent vectors that span V form a *basis* for V. Given a basis $(u_1, u_2, \ldots, u_n)$ for V any $v \in V$ may be given as the linear combination

$$v = \sum_{i=1}^{n} v^i u_i,$$

where the v^i's are the *components* of v in the basis. In the following the *summation convention* will be employed by which a summation is implied when an index is repeated on upper and lower levels. Thus, the above may be written as $v = v^i u_i$. One easily thinks of a vector space in terms of a basis. However, there is no intrinsic significance to any basis. In the following one objective will be to develop intuition that is invariant under general change of basis, that is, under general linear transformation of basis vectors.

Given a vector space V we may consider linear maps from V to $\mathbb{R}$. We will use such a map, in particular, to associate a virtual work (real number) to a given virtual displacement vector. We denote as V^* the vector space of linear maps $V \to \mathbb{R}$ with vector addition and scalar multiplication defined for arbitrary $\alpha, \beta \in V^*$ such that

$$(\alpha + \beta)(v) = \alpha(v) + \beta(v),$$
$$(a\alpha)(v) = a\big(\alpha(v)\big)$$

for all $a \in \mathbb{R}$ and $v \in V$. The vector space V^* is called the *dual space* of V, and its elements are called *covectors*. Given a basis $(u_1, u_2, \dots, u_n)$ for V there is the *dual basis* $(\alpha^1, \alpha^2, \dots, \alpha^n)$ for V^* defined by $\alpha^j(u_i) = \delta^j_i$, where $\delta^i_j = 1$ when $i = j$ and $\delta^i_j = 0$ when $i \neq j$. The operation of α^j on arbitrary vector v is given as

$$\alpha^j(v) = \alpha^j(v^i u_i) = v^i \alpha^j(u_i) = v^j$$

so the above really defines the covectors $\alpha^1, \alpha^2, \dots, \alpha^n$. The operation of $\beta \in V^*$ on arbitrary vector $v \in V$ may be given as

$$\beta(v) = \beta(v^i u_i) = v^i \beta(u_i) = \beta(u_i)\alpha^i(v).$$

Thus β is expressed in the dual basis as $\beta = \beta(u_i)\alpha^i = \beta_i \alpha^i$ with the components $\beta_i = \beta(u_i)$. This shows that the α^i's span V^*. To see that $\alpha^1, \alpha^2, \dots, \alpha^n$ are linearly independent we assume that $\beta_i \alpha^i = 0$. By giving the basis vector u_j to this covector we find that $\beta_j = 0$, and the linear independence follows by going through all the indices j. Thus, the dimension of V^* is the same as the dimension of V.

Let V be an n-dimensional vector space and β a covector belonging to V^*. Those vectors of V that satisfy $\beta(u) = 0$ form an (n-1)-dimensional vector subspace U of V. In a basis this equation is written as $\beta_i u^i = 0$ which is clearly an equation for (n-1)-dimensional plane in the u^i-coordinates. In general, those vectors that satisfy $\beta(v) = a$ form an affine subspace of V whose elements are of the form $v+u$, where $u \in U$, see Figure 2.1 left. Clearly β does not make a distinction between vectors that belong to the same affine subspace. This justifies the graphical representation of β as shown in the 2-dimensional case in Figure 2.1 right. Note also that the specification of p covectors $\beta^1, \dots, \beta^p$ yields an (n-p)-dimensional subspace of V composed of vectors v that satisfy $\beta^i(v) = 0$ for all $i = 1, \dots, p$.

The definition of V^* does not require a basis for the underlying vector space V. Thus, for given $v \in V$ and $\beta \in V^*$ the number $\beta(v)$ is independent from the basis used to represent v and β. This means that when the basis is changed the components of v and β must change in a consistent manner to make $\beta(v) = \beta_i v^i$ invariant. Let us take two bases $(u_1, u_2, \dots, u_n)$ and $(w_1, w_2, \dots, w_n)$ for V such that the change of basis is given as $w_j = w^i_j u_i$ for

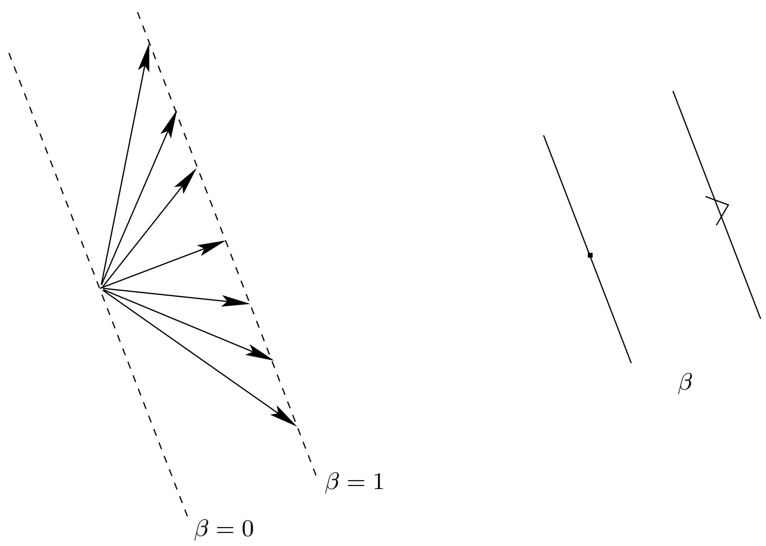


Figure 2.1: Left: vector subspace formed by those vectors v that satisfy $\beta(v) = 0$ (written $\beta = 0$ in the figure), and affine subspace formed by those vectors v that satisfy $\beta(v) = 1$. Some vectors of the affine subspace are also shown. Right: the graphical representation of the covector β. Here V is 2-dimensional.

all $j = 1, \ldots, n$, where the w^i_j's form a matrix with nonvanishing determinant. A vector v may be given in the basis $(w_1, w_2, \ldots, w_n)$ as $v = v^j_w w_j$ with the components v^j_w. By changing the basis we get

$$v = v^j_w w_j = v^j_w (w^i_j u_i) = (v^j_w w^i_j) u_i = v^i_u u_i,$$

so the components of v in the basis $(u_1, u_2, \ldots, u_n)$ are given as $v^i_u = v^j_w w^i_j$. By denoting as v_u and v_w the column vectors containing the v^i_u's and v^j_w's, respectively, and as W the transformation matrix with entries w^i_j (j being the row index and i the column index), we may write this as $v_u = W v_w$. We thus have

$$v_w = W^{-1} v_u \tag{2.1}$$

for the change of basis formula for the components of a vector $v \in V$. This is called *contravariant transformation*. The elements of V are sometimes called *contravariant vectors*. Let us next examine how the components of a covector transform. For this, we denote as $(\alpha^1, \alpha^2, \ldots, \alpha^n)$ the dual basis of $(u_1, u_2, \ldots, u_n)$ as above, and as $(\gamma^1, \gamma^2, \ldots, \gamma^n)$ the dual basis of

$(w_1, w_2, \ldots, w_n)$. A covector β may be given in the basis $(\gamma^1, \gamma^2, \ldots, \gamma^n)$ as $\beta = \beta_i^\gamma \gamma^i$, where the components β_i^γ are given as $\beta_i^\gamma = \beta(w_i)$. By expressing the w_i's in the other basis we get

$$\beta_i^\gamma = \beta(w_i) = \beta(w_i^j u_j) = w_i^j \beta(u_j) = w_i^j \beta_j^\alpha$$

where the β_j^α's are the components of β in the basis $(\alpha^1, \alpha^2, \ldots, \alpha^n)$. By denoting as β^γ and β^α the row vectors containing the β_i^γ's and β_j^α's, respectively, we thus have

$$\beta^\gamma = \beta^\alpha W. \tag{2.2}$$

This is called *covariant transformation*, and the elements of V^* are sometimes called *covariant vectors*. By the transformation rules (2.1) and (2.2) we may write the number $\beta(v)$ as

$$\begin{aligned}
\beta(v) &= \beta_i^\gamma \gamma^i (v_w^j w_j) \\
&= \beta_i^\gamma v_w^i \\
&= \beta^\gamma v_w \\
&= \beta^\alpha W W^{-1} v_u \\
&= \beta^\alpha v_u \\
&= \beta_i^\alpha v_u^i,
\end{aligned}$$

so $\beta(v)$ may be written either as $\beta_i^\gamma v_w^i$ or as $\beta_i^\alpha v_u^i$.

Because V^* is a vector space we may consider its dual space V^{**}. We may associate to each vector $v \in V$ an element $v^{**} \in V^{**}$ by defining $v^{**}(\beta) = \beta(v)$ for all $\beta \in V^*$. This correspondence is linear and bijective so V and V^{**} are isomorphic [1]. In the following we will have some use for this identification, and then we will denote v^{**} simply as v. Because of this duality we will sometimes talk of the *pairing* of a covector and a vector when we mean the map $V^* \times V \to \mathbb{R};\ (\beta, v) \mapsto \beta(v)$.

2.2 Tensors

Given a vector space V and its dual space V^* we may consider multilinear maps that take r covectors and s vectors to real numbers. Such a map is thus of the type $V^* \times \cdots \times V^* \times V \times \cdots \times V \to \mathbb{R}$, where we have r copies of V^* and s copies of V. Multilinearity means that the map is linear in each argument when the other arguments are held fixed. We denote as $T_s^r(V)$

the vector space of such maps with vector addition and scalar multiplication defined for arbitrary $t_1, t_2 \in T^r_s(V)$ such that

$$\begin{aligned}(t_1+t_2)(\beta^1,\dots,\beta^r,v_1,\dots,v_s) &= t_1(\beta^1,\dots,\beta^r,v_1,\dots,v_s)+\\ &\qquad t_2(\beta^1,\dots,\beta^r,v_1,\dots,v_s)\\ (at_1)(\beta^1,\dots,\beta^r,v_1,\dots,v_s) &= at_1(\beta^1,\dots,\beta^r,v_1,\dots,v_s)\end{aligned}$$

for all $\beta^1,\dots,\beta^r \in V^*$ and $v_1,\dots,v_s \in V$. The elements of the vector space $T^r_s(V)$ are called *tensors*, *contravariant* of order r and *covariant* of order s, or simply, *of type* $\binom{r}{s}$. Clearly, we have $T^0_1(V) = V^*$. Also, by the identification of V and V^{**}, we have $T^1_0(V) = V$.

To construct new tensors from old ones we may use the tensor product. For tensors $t_1 \in T^{r_1}_{s_1}(V)$ and $t_2 \in T^{r_2}_{s_2}(V)$ their *tensor product* is the tensor $t_1 \otimes t_2 \in T^{r_1+r_2}_{s_1+s_2}(V)$ defined by

$$\begin{aligned}t_1\otimes t_2(\alpha^1,\dots,\alpha^{r_1},&\beta^1,\dots,\beta^{r_2},u_1,\dots,u_{s_1},v_1,\dots,v_{s_2})\\ &= t_1(\alpha^1,\dots,\alpha^{r_1},u_1,\dots,u_{s_1})t_2(\beta^1,\dots,\beta^{r_2},v_1,\dots,v_{s_2})\end{aligned}$$

for all $\alpha^1,\dots,\alpha^{r_1},\beta^1,\dots,\beta^{r_2} \in V^*$ and $u_1,\dots,u_{s_1},v_1,\dots,v_{s_2} \in V$. From the definition it follows that the tensor product $\otimes$ is associative and bilinear, that is, for tensors t_1, t_2 and t_3 of appropriate type it satisfies

$$\begin{aligned}t_1\otimes(t_2\otimes t_3) &= (t_1\otimes t_2)\otimes t_3,\\ t_1\otimes(t_2+t_3) &= t_1\otimes t_2+t_1\otimes t_3,\end{aligned}$$

and further

$$t_1\otimes(at_2) = (at_1)\otimes t_2 = a(t_1\otimes t_2)$$

for all $a \in \mathbb{R}$.

To find out the dimension of $T^r_s(V)$ we consider the operation of $t \in T^r_s(V)$ on r covectors and s vectors. By using a basis $(u_1,u_2,\dots,u_n)$ for V, and the associated dual basis $(\alpha^1,\alpha^2,\dots,\alpha^n)$, we have

$$\begin{aligned}t(\beta^1,\dots,\beta^r,v_1,\dots,v_s) &= t(\beta^1_{i_1}\alpha^{i_1},\dots,\beta^r_{i_r}\alpha^{i_r},v^{j_1}_1 v_{j_1},\dots,v^{j_s}_s v_{j_s})\\ &= \beta^1_{i_1}\cdots\beta^r_{i_r}v^{j_1}_1\cdots v^{j_s}_s t(\alpha^{i_1},\dots,\alpha^{i_r},v_{j_1},\dots,v_{j_s}).\end{aligned}$$

By further using

$$\begin{aligned}\beta^1_{i_1}\cdots\beta^r_{i_r}v^{j_1}_1\cdots v^{j_s}_s &= \beta^1(u_{i_1})\cdots\beta^r(u_{i_r})\alpha^{j_1}(v_1)\cdots\alpha^{j_s}(v_s)\\ &= u_{i_1}(\beta^1)\cdots u_{i_r}(\beta^r)\alpha^{j_1}(v_1)\cdots\alpha^{j_s}(v_s)\\ &= u_{i_1}\otimes\cdots\otimes u_{i_r}\otimes\alpha^{j_1}\otimes\cdots\otimes\alpha^{j_s}(\beta^1,\dots,\beta^r,v_1,\dots,v_s)\end{aligned}$$

we find that t may be given as

$$t = t(\alpha^{i_1}, \ldots, \alpha^{i_r}, v_{j_1}, \ldots, v_{j_s}) u_{i_1} \otimes \cdots \otimes u_{i_r} \otimes \alpha^{j_1} \otimes \cdots \otimes \alpha^{j_s}. \tag{2.3}$$

There are n^{r+s} terms to sum over in (2.3). The n^{r+s} tensors $u_{i_1} \otimes \cdots \otimes u_{i_r} \otimes \alpha^{j_1} \otimes \cdots \otimes \alpha^{j_s}$ thus span $T^r_s(V)$. To see that these n^{r+s} tensors are linearly independent we assume that $t^{i_1 \cdots i_r}_{j_1 \cdots j_s} u_{i_1} \otimes \cdots \otimes u_{i_r} \otimes \alpha^{j_1} \otimes \cdots \otimes \alpha^{j_s} = 0$, and give $(\alpha^{k_1}, \ldots, \alpha^{k_r}, u_{l_1}, \ldots, u_{l_s})$ to this tensor to find that $t^{k_1 \cdots k_r}_{l_1 \cdots l_s} = 0$. The linear independence follows because this holds for all permutations of the indices $k_1 \cdots k_r, l_1 \cdots l_s$. Thus the dimension of $T^r_s(V)$ is n^{r+s}.

2.3 Multivectors and -covectors

We will have particular use for *antisymmetric* tensors of type $\binom{0}{p}$. Such a tensor takes in p vectors to produce a real number in such a way that when two of the vector arguments are interchanged the sign of the number is changed. This is what is meant by antisymmetry. It implies that when the same vector appears in two different argument places the tensor yields zero. Antisymmetric tensors are also called *skew symmetric* or *alternating*. The antisymmetric tensors of type $\binom{0}{p}$ form a vector subspace of $T^0_p(V)$, denoted as $\bigwedge_p(V)$. The elements of this subspace are called *p-covectors*. An important example is the magnetic induction 2-covector: it takes in a virtual displacement vector and a velocity vector to produce, once multiplied by charge magnitude, the virtual work done on the test charge. The above definition of p-covectors only makes sense when $p > 1$. We define 1-covectors to be just covectors of V^*. Also, we define 0-covectors to be linear mappings of type $\mathbb{R} \to \mathbb{R}$, that is, $\bigwedge_0(V) = \bigwedge_1(\mathbb{R})$. We will often identify 0-covector $a \in \bigwedge_0(V)$ with a real number by setting $a = a(1)$. If a tensor is an element of $\bigwedge_p(V)$ for some p it is called a *multicovector*.

Let us find out, as an example, the dimension of $\bigwedge_2(V)$. By using the covector basis $(\alpha^1, \ldots, \alpha^n)$ we may give any 2-covector $\omega \in \bigwedge_2(V) \subset T^0_2(V)$ as

$$\omega = \omega_{ij} \alpha^i \otimes \alpha^j. \tag{2.4}$$

By antisymmetry we have $\omega(v, w) = -\omega(w, v)$ for all $v, w \in V$. Using this to the vectors of the basis $(u_1, \ldots, u_n)$ whose dual basis is $(\alpha^1, \ldots, \alpha^n)$, we see that $\omega_{ij} = -\omega_{ji}$ for all $i, j = 1, \ldots, n$. Because this holds also when $i = j$ we must have that $\omega_{ii} = 0$ for all $i = 1, \ldots, n$. We may thus write

$$\omega = \sum_{1 \le i < j \le n} \omega_{ij} (\alpha^i \otimes \alpha^j - \alpha^j \otimes \alpha^i). \tag{2.5}$$

The terms $\alpha^i \otimes \alpha^j - \alpha^j \otimes \alpha^i$ are 2-covectors because for vectors v, w we have

$$\begin{aligned}(\alpha^i \otimes \alpha^j - \alpha^j \otimes \alpha^i)(v,w) &= \alpha^i(v)\alpha^j(w) - \alpha^i(w)\alpha^j(v) \\ &= -\big(\alpha^i(w)\alpha^j(v) - \alpha^i(v)\alpha^j(w)\big) \\ &= -(\alpha^i \otimes \alpha^j - \alpha^j \otimes \alpha^i)(w,v).\end{aligned}$$

These 2-covectors span $\bigwedge_2(V)$, and their linear independence is shown in the familiar way. For instance, when $n = 3$ we have three additive terms left in (2.5) compared to the nine terms in (2.4). In general the antisymmetry reduces the number of additive terms from n^2 to $\frac{n!}{2!(n-2)!}$ [1, 2, 3]. Thus, in general, the dimension of $\bigwedge_2(V)$ is $\frac{n!}{2!(n-2)!}$. Similarly, the dimension of $\bigwedge_p(V)$ is $\frac{n!}{p!(n-p)!}$ [1, 2, 3].

We will need to construct new multicovectors from old ones. The tensor product does not qualify for this because the tensor product of a p-covector and a q-covector is a tensor of type $\binom{0}{p+q}$ that is not antisymmetric in all its $p + q$ arguments (although it is antisymmetric in the first p arguments and in the last q arguments). Thus we need an other kind of product to yield a multicovector. With covectors we already know how this can be done. For covectors $\alpha, \beta \in \bigwedge_1(V)$ we define the 2-covector $\alpha \wedge \beta \in \bigwedge_2(V)$ by

$$\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha.$$

Note that (2.5) may now be written as

$$\omega = \sum_{1 \le i < j \le n} \omega_{ij} \alpha^i \wedge \alpha^j. \tag{2.6}$$

The generalization to multicovectors yields an operator that takes a p-covector and a q-covector to a (p+q)-covector. For the definition of this operator we let $P(p, q)$ denote the set of all permutations σ of the index set $\{1, \ldots, p+q\}$ that satisfies $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p + 1) < \cdots < \sigma(q)$. Then, for $\omega \in \bigwedge_p(V)$ and $\eta \in \bigwedge_q(V)$ their *exterior product* is the (p+q)-covector $\omega \wedge \eta \in \bigwedge_{p+q}(V)$ defined by

$$(\omega \wedge \eta)(v_1, \ldots, v_{p+q}) = \sum_{\sigma \in P(p,q)} \operatorname{sgn}(\sigma)\omega(v_{\sigma(1)}, \ldots, v_{\sigma(p)})\eta(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}),$$

where $\operatorname{sgn}(\sigma)$ is the signature of the permutation σ. It is 1 when σ is an even permutation and -1 when σ is an odd permutation. The exterior product $\wedge$ is bilinear and associative like the tensor product [1, 3]. The property that is different from the tensor product is the graded anticommutativity, that is, for $\omega \in \bigwedge_p(V)$ and $\eta \in \bigwedge_q(V)$, we have

$$\omega \wedge \eta = (-1)^{pq}\eta \wedge \omega,$$

as shown in [1, 3]. Similarly to (2.6) we may give a p-covector $\omega \in \bigwedge_p(V)$ by using the covector basis $(\alpha^1, \dots, \alpha^n)$ as

$$\omega = \sum_{1 \le i_1 < \cdots < i_p \le n} \omega_{i_1 \cdots i_p} \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_p} \tag{2.7}$$

as shown in [1, 3]. Finally, from the definition of exterior product it follows that for covectors $\beta^1, \dots, \beta^p$ and vectors $v_1, \dots, v_p$ we have

$$(\beta^1 \wedge \cdots \wedge \beta^p)(v_1, \dots, v_p) = \det\big(\beta^i(v_j)\big), \tag{2.8}$$

the determinant of the matrix with entries $\beta^i(v_j)$, see [3]. In particular, the operation of the n-covector $\alpha^1 \wedge \cdots \wedge \alpha^n$ on n vectors gives the determinant of the components of these vectors in the basis $(u_1, \dots, u_n)$.

We will also need antisymmetric tensors of type $\binom{p}{0}$ called *p-vectors*. These objects are dual to p-covectors, and we will use them to represet, for instance, virtual surface and volume elements with orientation. They form a subspace of $T_0^p(V)$ which we will denote as $\bigwedge^p(V)$. We define 1-vectors to be just vectors of V. We also identify 0-vectors with real numbers so that $\bigwedge^0(V) = \mathbb{R}$. The operation of 0-covector $b \in \bigwedge_0(V)$ on 0-vector $a \in \bigwedge^0(V)$ is thus $b(a) = b(a1) = a(b(1)) = ab \in \mathbb{R}$. If a tensor belongs to $\bigwedge^p(V)$ for some p it is called *multivector*. The exterior product may be defined for multivectors just as it was defined for multicovectors above. Thus, for instance, two vectors $v, w \in V$ may be used to construct a 2-vector

$$v \wedge w = v \otimes w - w \otimes v.$$

Thus we have

$$(v \wedge w)(\beta^1, \beta^2) = v(\beta^1)w(\beta^2) - w(\beta^1)v(\beta^2)$$

for covectors β^1, β^2. More generally, for vectors $v_1, \dots, v_p$ and covectors $\beta^1, \dots, \beta^p$ we have

$$(v_1 \wedge \cdots \wedge v_p)(\beta^1, \dots, \beta^p) = \det\big(v_i(\beta^j)\big) \tag{2.9}$$

in alignment with (2.8). By using (2.8) and (2.9) together with the identification $v_i(\beta^j) = \beta^j(v_i)$ we get

$$\begin{aligned}(\beta^1 \wedge \cdots \wedge \beta^p)(v_1, \dots, v_p) &= \det\big(\beta^i(v_j)\big)\\ &= \det\big(v_j(\beta^i)\big)\\ &= \det\big(v_i(\beta^j)\big)\\ &= (v_1 \wedge \cdots \wedge v_p)(\beta^1, \dots, \beta^p).\end{aligned}$$

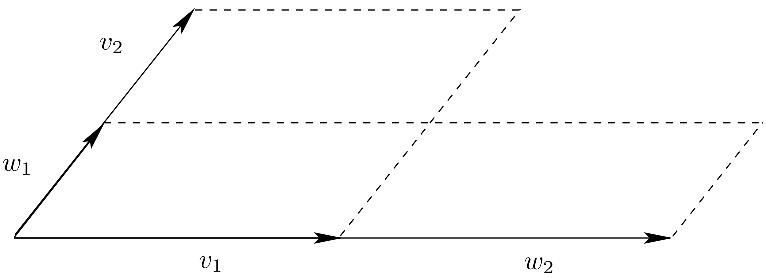


Figure 2.2: Pairs of vectors (w_1, w_2) and (v_1, v_2) with unit area relative two each other.

This shows that $(\beta^1 \wedge \cdots \wedge \beta^p)(v_1, \ldots, v_p)$ depends only on the p-vector $v_1 \wedge \cdots \wedge v_p$ (and not on the p individual vectors $v_1, \ldots, v_p$). That is, if two arrays of vectors $(v_1, \ldots, v_p)$ and $(w_1, \ldots, w_p)$ yield the same p-vector they will be mapped to the same number by a p-covector. So when do $(v_1, \ldots, v_p)$ and $(w_1, \ldots, w_p)$ yield the same p-vector? Before answering this question we introduce two new notions. First, when $(v_1, \ldots, v_p)$ and $(w_1, \ldots, w_p)$ span the same subspace of V we may express each of the w_j's in terms of the v_i's as $w_j = w^i_j v_i$, where the w^i_j's form a matrix with nonvanishing determinant. In this case we define the *p-volume of* $(w_1, \ldots, w_p)$ *relative to* $(v_1, \ldots, v_p)$ as the absolute value of the determinant $\det(w^i_j)$. As an example case consider $p = 2$ and $(w_1, w_2) = (\frac{1}{2}v_2, 2v_1)$. Clearly (w_1, w_2) and (v_1, v_2) have unit 2-volumes (or areas) relative to each other, see Figure 2.2. This follows readily from the above definition. We have

$$\det(w^i_j) = \det \begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{pmatrix} = -1.$$

The second notion that we introduce at this point is about orientation. We say that $(w_1, \ldots, w_p)$ and $(v_1, \ldots, v_p)$ have the *same orientation* if $\det(w^i_j) > 0$, and *opposite orientation* if $\det(w^i_j) < 0$. In the example of Figure 2.2 the two pairs of vectors thus have opposite orientation. Now, to answer the question raised above, we note that the p-vector formed by the w_j's may be given as

$$\begin{aligned} w_1 \wedge \cdots \wedge w_p &= w^{i_1}_1 v_{i_1} \wedge \cdots \wedge w^{i_p}_p v_{i_p} \\ &= w^{i_1}_1 \cdots w^{i_p}_p v_{i_1} \wedge \cdots \wedge v_{i_p} \\ &= w^{i_1}_1 \cdots w^{i_p}_p \operatorname{sgn}(\sigma_{i_1 \ldots i_p}) v_1 \wedge \cdots \wedge v_p, \end{aligned}$$

where $\sigma_{i_1 \ldots i_p}$ permutes $(1, \cdots, p)$ to $(i_1, \cdots, i_p)$. But $w^{i_1}_1 \cdots w^{i_p}_p \operatorname{sgn}(\sigma_{i_1 \ldots i_p})$ is the determinant of the matrix of w^i_j's [3]. We thus have

$$w_1 \wedge \cdots \wedge w_p = \det(w^i_j) v_1 \wedge \cdots \wedge v_p, \tag{2.10}$$

and we see that $(v_1, \ldots, v_p)$ and $(w_1, \ldots, w_p)$ yield the same p-vector provided that they have the same orientation and the same p-volumes relative to each other, that is, $\det(w^i_j) = 1$. Indeed, simple multivectors such as $v_1 \wedge \cdots \wedge v_p$ may be taken as equivalence classes of arrays of vectors spanning the same subspace, two arrays being equivalent if the above determinant condition holds. See [4] for the construction of multivectors from this point of view. This is the intuitive picture of multivectors that we want to keep in mind. Finally, because the operation of p-covector on p vectors depends only on the p-vector formed by the p vectors, p-covectors may be taken as linear maps from p-vectors to real numbers. The operation of $\omega \in \bigwedge_p(V)$ on $v \in \bigwedge^p(V)$ is given by using a basis $(u_1, \ldots, u_n)$ for V, and its dual basis $(\alpha^1, \ldots, \alpha^n)$, as

$$\begin{aligned} \omega(v) &= \sum_{1 \le i_1 < \cdots < i_p \le n} \omega_{i_1 \cdots i_p} \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_p} \Big(\sum_{1 \le j_1 < \cdots < j_p \le n} v^{j_1 \cdots j_p} u_{j_1} \wedge \cdots \wedge u_{j_p} \Big) \\ &= \sum_{1 \le i_1 < \cdots < i_p \le n} \sum_{1 \le j_1 < \cdots < j_p \le n} \omega_{i_1 \cdots i_p} v^{j_1 \cdots j_p} \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_p} (u_{j_1}, \ldots, u_{j_p}). \end{aligned}$$

2.4 Twisted multivectors and -covectors

Let us further develop the concept of orientation introduced above. Given two bases $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ for the vector space V we can say whether they have the same or opposite orientation by calculating the determinant $\det(w^i_j)$. The relation "have the same orientation" is an equivalence relation in the set of all bases for V. The two equivalence classes are the two possible *orientations* of V. Clearly orientation is an additional structure that may be given for a vector space. To orient V one simply selects a basis from one of the two equivalence classes. If a basis belongs to the selected orientation we say it is *positively oriented*, otherwise it is *negatively oriented*. Having specified orientation by the basis $(v_1, \ldots, v_n)$ we may use the covectors $(\alpha^1, \ldots, \alpha^n)$ of the dual basis to test if $(w_1, \ldots, w_n)$ is positively oriented. We have

$$(\alpha^1 \wedge \cdots \wedge \alpha^n)(w_1, \ldots, w_n) = \det(w^i_j),$$

which must be positive for $(w_1, \ldots, w_n)$ to be positively oriented. Note how the antisymmetry of the n-covector $\alpha^1 \wedge \cdots \wedge \alpha^n$ indicates the effect of permuting $(w_1, \ldots, w_n)$. When $(w_1, \ldots, w_n)$ is positively oriented so are its even permutations. Odd permutations, on the other hand, are negatively oriented.

Let us next consider a p-dimensional subspace U of V. As any vector space we may orient U by taking p ordered vectors that form a basis for the space. In many cases, however, the orientation of U is not relevant. For instance, when $n = 3$ and $p = 2$ we may rather want to specify a crossing direction through the plane U. Motivated by this we let U be an (n-1)-dimensional subspace of the n-dimensional vector space V. We select a nonzero covector $\alpha \in V^*$ such that $\alpha(u) = 0$ for all $u \in U$. We say that vectors v and w *are on the same side of* U if $\alpha(v)$ and $\alpha(w)$ have the same sign. This is an equivalence relation in V. The two equivalence classes of vectors are the two *sides* of U. As an example, let V be spanned by (v_1, v_2, v_2), and take U to be the subspace spanned by (v_1, v_2). Further, let $(\alpha^1, \alpha^2, \alpha^3)$ be the dual basis. Now we may use the basis covector α^3 to tell us whether two vectors are on the same side of U. One of the two sides is given by the vectors w that satisfy $\alpha^3(w) = \alpha^3(w^i v_i) = w^3 > 0$. Note that not all of the w's are in the subspace spanned by v_3. Let us next generalize this to p-dimensional subspace U. Now we select $n-p$ covectors $\alpha^1, \ldots, \alpha^{n-p} \in V^*$ each of which satisfies $\alpha^i(u) = 0$ for all $u \in U$. We further require the covectors to be linearly independent so that $\alpha^1 \wedge \cdots \wedge \alpha^{n-p}$ is nonzero. Now, the arrays of vectors $(v_1, \ldots, v_{n-p})$ and $(w_1, \ldots, w_{n-p})$ are taken to be equivalent if $(\alpha^1 \wedge \cdots \wedge \alpha^{n-p})(v_1, \ldots, v_{n-p})$ and $(\alpha^1 \wedge \cdots \wedge \alpha^{n-p})(w_1, \ldots, w_{n-p})$ have the same sign. The two equivalence classes of arrays of vectors are called the *transverse orientations* of U. Clearly transverse orientation is an additional structure that can be given for a subspace of a vector space. For the vector space itself we take its transverse orientation to be just a sign (plus or minus). As an example, let again V be spanned by (v_1, v_2, v_2), but now take U to be the subspace spanned by v_1. We further let $(\alpha^1, \alpha^2, \alpha^3)$ be the dual basis. Now we may use the 2-covector $\alpha^2 \wedge \alpha^3$ to tell us whether pairs of vectors are equivalent. One of the transverse orientations are given by pairs of vectors (w_1, w_2) that satisfy $(\alpha^2 \wedge \alpha^3)(w_1, w_2) = w_1^2 w_2^3 - w_2^2 w_1^3 > 0$.

Multivectors carry in them the notion of orientation. A p-vector of the form $u_1 \wedge \cdots \wedge u_p$ is positively oriented in the orientation specified by $(u_1, \ldots, u_p)$. This is convenient for some purposes. For instance, in the case of a virtual displacement we are really interested in the direction of the displacement in the subspace spanned by the virtual displacement vector. However, in many cases it is the transverse orientation of the subspace spanned by $(u_1, \ldots, u_p)$ that is relevant. For instance, to express the electric current through a virtual surface element we need to take into account to which direction the current passes through the surface element. Let us consider the subspace U spanned by $(u_1, \ldots, u_p)$, and form the equivalence class of arrays of p vectors that has the same p-volume relative to $(u_1, \ldots, u_p)$. We denote this equivalence class as $[(u_1, \ldots, u_p)]$. We associate to $[(u_1, \ldots, u_p)]$

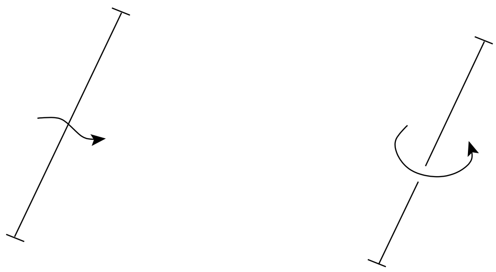


Figure 2.3: 1-volume with transverse orientation in two dimensions (left) and in three dimensions (right).

Figure 2.4: Two representations of the rightmost object of Figure 2.3.

a transverse orientation of U. The object of our interest is thus the *p-volume with transverse orientation* given as the pair $\big([(u_1, \dots, u_p)], TransOr\big)$, where $TransOr$ is the transverse orientation. Figure (2.3) shows the graphical representations of a 1-volume with transverse orientation in 2-dimensional and 3-dimensional vector spaces. Note that we have not yet defined the addition or scalar multiplication of these objects.

In a vector space V we will represent a p-volume with transverse orientation by a pair consisting of a p-vector and an orientation for V. By convention, the p-volume with transverse orientation represented by $(u_1 \wedge \cdots \wedge u_p, Or)$ is $\big([(u_1, \dots, u_p)], TransOr\big)$, where $TransOr$ is determined by an array of vectors $(w_1 \dots, w_{n-p})$ satisfying

$$(w_1 \dots, w_{n-p}, u_1, \dots, u_p) \in Or. \tag{2.11}$$

Note that the negative of $u_1 \wedge \cdots \wedge u_p$ and the orientation opposite to Or represents the same p-volume with transverse orientation, see Figure 2.4 as an example. Addition and scalar multiplication of p-volumes with transverse orientation may be defined in terms of these representatives. For instance, to add $\big([(u_1, \dots, u_p)], TransOr_u\big)$ and $\big([(v_1, \dots, v_p)], TransOr_v\big)$ we first select an orientation Or for V. Then we select arrays of vectors $(u_1, \dots, u_p) \in [(u_1, \dots, u_p)]$ and $(v_1, \dots, v_p) \in [(v_1, \dots, v_p)]$ both of which satisfy (2.11).

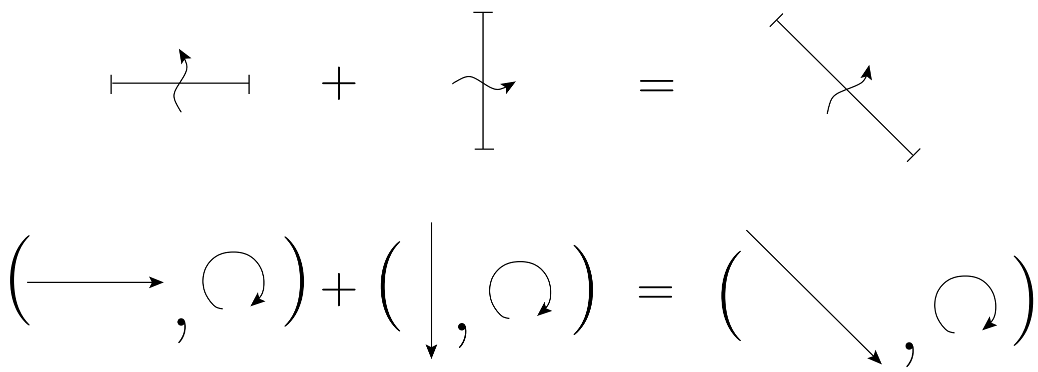


Figure 2.5: Addition of twisted vectors in two dimensions in terms of representative vectors.

Adding together the p-vectors $u_1 \wedge \cdots \wedge u_p$ and $v_1 \wedge \cdots \wedge v_p$ yields another p-vector. Together with the orientation Or this p-vector represents the sum of $\big([(u_1, \ldots, u_p)], TransOr_u\big)$ and $\big([(v_1, \ldots, v_p)], TransOr_v\big)$. See Figure 2.5 for an example where $p = 1$. Note that the addition so defined does not depend on the choice of orientation Or. With the vector space structure understood, these objects are called *twisted p-vectors*. The used graphical representation is from Burke [5].

We could have defined twisted p-vectors directly as equivalence classes of pairs (v, Or) consisting of a p-vector $v \in \bigwedge^p(V)$ and an orientation Or for V, two pairs (v_1, Or_1) and (v_2, Or_2) being equivalent if $v_1 = v_2$ and $Or_1 = Or_2$, or if $v_1 = -v_2$ and $Or_1 = -Or_2$. As above the addition and scalar multiplication are defined in terms of representatives whose orientations coincide.

The objects dual to twisted p-vectors are called *twisted p-covectors*. For instance, current density will be modeled by a twisted 2-covector. It takes in a twisted 2-vector modeling a surface element with crossing direction and gives the current through the element in the specified direction. Similarly, surface current density (current sheet) is modeled by a twisted covector defined on the surface. For a twisted vector modeling a line element with crossing direction it gives the current through the element to the specified direction. This operation is shown in Figure 2.6. The twisted covector has its lines oriented contrary to an ordinary covector whose line has transverse orientation, see Figure 2.1. The graphical representation is from Burke [5].

In a vector space V twisted p-covectors may be formally defined as equivalence classes of pairs (ω, Or) consisting of a p-covector $\omega \in \bigwedge_p(V)$ and an orientation Or for V, two pairs (ω_1, Or_1) and (ω_2, Or_2) being equivalent if

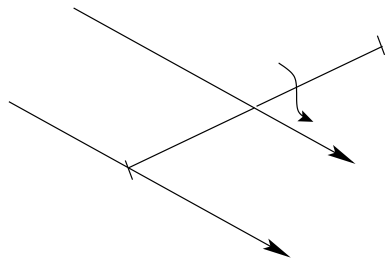


Figure 2.6: The operation of a twisted covector on a twisted vector in two dimensions.

$\omega_1 = \omega_2$ and $Or_1 = Or_2$, or if $\omega_1 = -\omega_2$ and $Or_1 = -Or_2$. Let us denote as $[(\omega, Or)]$ the twisted p-covector containing (ω, Or). In addition to (ω, Or) it contains $(-\omega, -Or)$. The operation of $[(\omega, Or)]$ on a twisted p-vector $[(v, Or)]$ is well defined, that is, independent of the representatives. If we select orientation Or the operation is realized as $\omega(v)$, and if we select the opposite orientation $-Or$ the operation is realized as $-\omega(-v)$, which is the same number.

2.5 Further examples of tensors

There are also tensors that are antisymmetric in some of their arguments but not in all of them. As an example, such a tensor may be used to model the virtual work done under the virtual displacement of a volume element. The tensor takes in one vector representing the virtual displacement and three vectors representing the volume element. It is antisymmetric in the three vectors of the volume element. Another example is the Maxwell's stress tensor. It is naturally a tensor of type $\binom{0}{3}$ that takes in one vector modeling virtual displacement and two vectors modeling virtual surface element. It is antisymmetric in the vectors of the surface element. Let V be three dimensional and t a tensor of $T_3^0(V)$ that is antisymmetric in its last two arguments. We may express t by using a covector basis $(\alpha^1, \alpha^2, \alpha^3)$ as

$$t = t_{ijk}\alpha^i \otimes \alpha^j \otimes \alpha^k.$$

There are $3^3 = 27$ terms to add in this expression. But now $t(u, v, w) = -t(u, w, v)$ for all $u, v, w \in V$. Using this to the basis vectors dual to $\alpha^1, \alpha^2, \alpha^3$ we find $t_{ijk}\alpha^i(u) = -t_{ikj}\alpha^i(u)$ for all $u \in V$. Thus $t_{ijk} = -t_{ikj}$ for all indices i, j, k. This also implies $t_{ijj} = 0$ for all i, j. Using this in the above expression for t we find

$$\begin{aligned} t = &\sum_{1\le j<k\le 3} t_{1jk}\alpha^1 \otimes (\alpha^j \otimes \alpha^k - \alpha^k \otimes \alpha^j) \\ &+ \sum_{1\le j<k\le 3} t_{2jk}\alpha^2 \otimes (\alpha^j \otimes \alpha^k - \alpha^k \otimes \alpha^j) \\ &+ \sum_{1\le j<k\le 3} t_{3jk}\alpha^3 \otimes (\alpha^j \otimes \alpha^k - \alpha^k \otimes \alpha^j), \end{aligned}$$

where we have only $3 \cdot 3 = 9$ terms left to add. This may be written more compactly by using the exterior product and the summation convention as

$$t = \sum_{1\le j<k\le 3} t_{ijk}\alpha^i \otimes (\alpha^j \wedge \alpha^k).$$

That the nine terms $\alpha^i \otimes (\alpha^j \wedge \alpha^k)$ are linearly independent is confirmed in the familiar way. Thus, when V is three dimensional, the dimension of the subspace of $T^0_3(V)$ consisting of tensors that are antisymmetric in the last two arguments is 9. Indeed, the matrix of Maxwell's tensor has nine entries. (In fact, Maxwell's tensor will only be represented by a tensor such as t after an orientation for V has been specified. This is because the 2-vector it takes in is actually a twisted 2-vector.) Note that t may also be given in the form

$$t = \alpha^i \otimes t_i,$$

where the 2-covectors t_i are given as

$$t_i = \sum_{1\le j<k\le 3} t_{ijk}\alpha^j \wedge \alpha^k.$$

2.6 Inner product

Before concluding this chapter let us examine how some of the above constructions appear when an inner product on V is given. An *inner product* on V, denoted here as g, is an element of $T^0_2(V)$ that satisfies the following axioms. First, it is *symmetric*. That is,

$$g(v, w) = g(w, v)$$

for all vectors v, w. Second, it is *positive definite*, meaning that

$$g(v, v) \ge 0$$

for all vectors v, with equality only if $v = 0$. Inner product allows us to define the norm of a vector and angle between two vectors. The *norm* $||v||$ of vector v is defined by

$$||v|| = g(v, v)^{1/2}.$$

The *angle* $u\angle v$ between vectors u and v is defined by

$$||u||||v||\cos(u\angle v) = g(u, v).$$

Vectors u and v are said to be *orthogonal* if $g(u, v) = 0$, and *orthonormal* if in addition they have unit norms.

Inner product is an additional structure that can be given for a vector space. To specify the structure one can select a basis which is to be taken as orthonormal. Let $(e_1, \ldots, e_n)$ denote such a basis so that $g(e_i, e_i) = 1$ for all $i = 1, \ldots, n$, and $g(e_i, e_j) = 0$ if $i \neq j$. The specified inner product may thus be given by using the dual basis $(\alpha^1, \ldots, \alpha^n)$ as

$$g = \sum_{i=1}^{n} \alpha^i \otimes \alpha^i.$$

This is how one constructs a Cartesian coordinate system in practice. One uses a ruler and a square to specify line segments that are to be identified with orthonormal vectors spanning a vector space. One then identifies points with vectors and takes the components of the vectors in the orthonormal basis as coordinates. In an arbitrary covector basis $(\beta^1, \ldots, \beta^n)$ an inner product g is given as

$$g = g_{ij}\beta^i \otimes \beta^j,$$

where the g_{ij}'s form a symmetric positive definite matrix.

An inner product allows the identification of vectors and covectors. Let u be a vector of V. This vector defines a covector $\beta^u \in V^*$ by

$$\beta^u(v) = g(u, v) \tag{2.12}$$

for all $v \in V$. In any basis we have $\beta^u(v) = \beta^u_j v^j$ and $g(u, v) = g_{ij}u^i v^j$ so the components of β^u are given as

$$\beta^u_j = g_{ij}u^i.$$

Note that in an orthonormal basis the components of u and β^u are equal. Now, because the matrix of g_{ij}'s is positive definite it has an inverse whose entries we may denote as g^{ij}. We may thus solve from the above equation the u^i's in terms of the β^u_j's, that is,

$$u^i = g^{ij}\beta^u_j.$$

This yields the conclusion that for a given covector $\beta^u \in V^*$ there exists a unique vector $u \in V$ such that (2.12) holds. However, we do not want to think of the operation of a covector β^u on a vector v in terms of a "proxy-vector" u and an inner product as in (2.12) because this representation depends on the chosen inner product. We rather want to rely on Figure 2.1 with our intuition.

Now we turn to the modeling side from the abstract linear algebra considered so far. So let V be a 3-dimensional vector space and (e_1, e_2, e_3) an orthonormal basis defining a Cartesian coordinate system. Further, let the orientation be specified by (e_1, e_2, e_3), and let $(\alpha^1, \alpha^2, \alpha^3)$ be the dual basis. Evaluating $\alpha^1 \wedge \alpha^2$ on arbitrary vectors $u, v \in V$ we get

$$(\alpha^1 \wedge \alpha^2)(u, v) = u^1 v^2 - u^2 v^1,$$

a component of the cross product of (u^1, u^2, u^3) and (v^1, v^2, v^3). It gives the signed area of the parallelopiped formed by the projections of u and v into the plane spanned by (e_1, e_2). The sign is plus if the projections form a pair of vectors that has the same orientation as (e_1, e_2). Otherwise, the sign is minus. If u and v belong to the subspace spanned by (e_1, e_2) this is plus or minus the area of (u, v). The notion of relative area between pairs of vectors is now changed to the notion of area of a pair of vectors (with respect to an inner product). Similarly, evaluating $\alpha^1 \wedge \alpha^2 \wedge \alpha^3$ on vectors $u, v, w \in V$ gives

$$(\alpha^1 \wedge \alpha^2 \wedge \alpha^3)(u, v, w) = \det \begin{pmatrix} u^1 & v^1 & w^1 \\ u^2 & v^2 & w^2 \\ u^3 & v^3 & w^3 \end{pmatrix}$$

which is the scalar triple product of (u^1, u^2, u^3), (v^1, v^2, v^3) and (w^1, w^2, w^3), and is thus the signed volume of the parallelopiped formed by (u, v, w).

Chapter 3

Forces in terms of charges and currents

Let us begin the study of electric and magnetic forces by examining a system of two interacting objects o_1 and o_2 in a vacuum. A relevant quantity in the modeling of such a system is the *work* done on one of the objects under the evolution of the system. The work done by object o_1 when the center of mass of object o_2 moves along a path is a mapping $W[o_1, o_2]$ that relates a real number to the path of movement. In practice this mapping will be realized by using another quantity that gives contributions to work of infinitesimal displacements of o_2 along its path. For this, let us assume that the evolution of states occurs without fractures of the objects, and express the evolution by a smooth one-parameter embedding that maps the objects o_1 and o_2 into physical space Ω for each parameter value. That is, we assume that an embedding

$$\Psi_t : o_1 \cup o_2 \to \Omega$$

is given for all $t \in T$, where T is a real number interval. This makes sense as the physical space is taken to be a *manifold*, and the two objects are taken to be *manifolds with boundaries*. The images of the objects o_1 and o_2 under the embedding are *submanifolds with boundaries* of Ω and they are denoted as o_1^t and o_2^t.

Manifolds. The idea of a differentiable manifold is to smoothly patch together local entities on which differentiation is defined. The resulting global entity will thus be more general than any individual local entity, and will be useful in a wider range of modelings. An appropriate local entity is the vector space $\mathbb{R}^n$. The derivative of a differentiable mapping of type $\mathbb{R}^n \to \mathbb{R}^m$ is given by the Jacobian matrix. For the patching together of the local entities we assume an underlying point set S, and use bijections $\chi_i : U_i \subset S \to V_i \subset \mathbb{R}^n$, with V_i open, called *charts*. A *smooth atlas* A on S is a collection of charts that satisfies the axioms

(i) a collection of charts of A covers S, that is $\bigcup_{i \in I} U_i = S$ for some index set I (existence of covering),

(ii) the charts of A are compatible, meaning that for any two charts χ_i and χ_j with overlapping domains U_i and U_j their transition map $\chi_j \circ \chi_i^{-1} : \chi_i(U_i \cap U_j) \to \chi_j(U_i \cap U_j)$ together with its inverse is infinitely many times differentiable (smooth compatibility),

(iii) any chart compatible with all overlapping charts of A belongs to A (maximality).

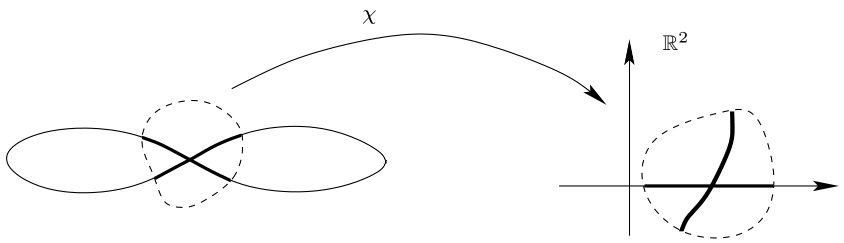
A *smooth manifold* is a pair (S, A). It is *n-dimensional* if the charts of A are mappings to $\mathbb{R}^n$.

By axioms (i) and (ii) the differentiability of a mapping from S to the underlying point set S' of another smooth manifold (S', A') has meaning as this notion does not depend on the charts selected from A and A' to represent the mapping. Axiom (iii) ensures that all compatible charts are allowed (as they should if we are to regard a smooth atlas as a differentiable structure).

Although S was just a point set to begin with it is given the structure of topological space by the smooth atlas. We take $W \subset S$ to be open if for each point $x \in W$ there is a chart in A with domain U such that $x \in U \subset W$ [1, 3]. Thus topological notions such as neighbourhoods of points have meaning on S. If S has the further topological properties of being Hausdorff separable and second countable (containing countable basis of open sets) we call (S, A) a *manifold*. This definition that begins with a point set S (rather than a second countable Hausdorff space) emphasizes the modeling point of view. To specify a smooth atlas for a manifold it is sufficient to specify a collection of charts satisfying axioms (i) and (ii). The existence of a unique smooth atlas containing these charts in then guaranteed [2]. We will often identify a manifold $M = (S, A)$ with its underlying point set S.

For some purposes we need to allow the underlying point set S to have boundary points. These are points that do not have a neighbourhood diffeomorphic to an open set of $\mathbb{R}^n$; instead we require that they have a neighbourhood diffeomorphic to an open set of the half-space $H^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}$. The half-space H^n has the relative topology induced from the open sets of $\mathbb{R}^n$. The open sets of H^n are thus of the form $H^n \cap V$, where V is an open set of $\mathbb{R}^n$. Now, by modifying the above definition of manifold such that the codomains of the charts are open sets of H^n results in the notion of *manifold with boundary*. If M is a manifold with boundary, then a point $x \in M$ is a boundary point if its images under all charts belong to ∂H^n, where $\partial H^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n = 0\}$ is the boundary of H^n. The union of boundary points of M is the boundary of M, denoted as ∂M. If ∂M is empty then M is also a manifold. If M is an n-dimensional manifold with boundary then ∂M is an (n-1)-dimensional manifold whose boundary is empty [1]. Its smooth atlas is obtained by restricting the charts of M to ∂M.

Often one needs to do analysis on subsets of a manifold. In the following we will restrict analysis to subsets that are manifolds themselves – with the differentiable structure obtained from the background manifold. Let M be n-dimensional manifold. If there is for each point of $N \subset M$ a chart of M containing this point in which N looks locally like $\mathbb{R}^p$ for some $p \leq n$, then, for each such chart χ_i with domain U_i, we may form the restriction $\chi_i|U_i \cap N : U_i \cap N \to \mathbb{R}^p$ to obtain an atlas that charts N by open subsets of $\mathbb{R}^p$. With these charts N becomes a p-dimensional manifold. Such a subset is called a *submanifold* of M. In the inset there is an example of a subset of the manifold $\mathbb{R}^2$ that contains a point whose neighbourhood, no matter how small, does not look like $\mathbb{R}$ in the charts of $\mathbb{R}^2$.



This “figure eight” is thus not a submanifold of $\mathbb{R}^2$. An example of a (1-dimensional) submanifold of $\mathbb{R}^2$ is a circle in $\mathbb{R}^2$. Note that a submanifold $N \subset M$ has the relative topology obtained by taking the intersections of the open sets of M with N.

Let us denote as $F[o_1^t, o_2^t]$ the *force covector* at the center of mass of o_2^t. By definition, it gives the *virtual work* done by o_1 on o_2 for a *tangent vector* describing the *virtual displacement* of the center of mass of o_2.

Tangent vectors. Tangent vectors bring in linearity to the analysis on manifolds by taking what is common to parametrized paths through a point on the manifold. Let M be a manifold and $\tau : I \to M$ a smooth map from an open interval $I \subset \mathbb{R}$ to M such that $0 \in I$. If $\tau(0) = x$ we call τ a *trajectory* at x. Two trajectories τ_1 and τ_2 at $x \in M$ are *tangent* at x if their representations in a chart have the same derivatives at the origin, that is, $(\chi \circ \tau_1)'(0) = (\chi \circ \tau_2)'(0)$. The tangency of trajectories at a point is well defined because it does not depend on the used chart (just use another overlapping chart φ, apply the chain rule to $(\varphi \circ \tau_1)'(0) = (\varphi \circ \chi^{-1} \circ \chi \circ \tau_1)'(0)$, and use the tangency in chart χ). It is an equivalence relation in the set of all trajectories at the point. The equivalence class containing τ is denoted as $[\tau]$. The set of all equivalence classes of trajectories at $x \in M$ is denoted as T_xM.

By using a chart χ for M we may represent an equivalence class of trajectories $[\tau] \in T_xM$ by the vector $(\chi \circ \tau)'(0) \in \mathbb{R}^n$ sitting at the point $\chi(x) \in \mathbb{R}^n$. In another overlapping chart φ, then, the representation of $[\tau]$ is given by the chain rule as $J_{\varphi \circ \chi^{-1}}\big(\chi(x)\big)(\chi \circ \tau)'(0)$, where $J_{\varphi \circ \chi^{-1}}\big(\chi(x)\big)$ is the Jacobian matrix of $\varphi \circ \chi^{-1}$ at $\chi(x)$. This vector is attached to the point $\varphi(x)$. This is how the representation of $[\tau]$ changes when we change a chart. The above association of elements of T_xM and vectors of $\mathbb{R}^n$ can be shown to be bijective [1]. Thus, we can make T_xM an n-dimensional vector space by defining the vector space operations for its elements in terms of representative n-tuples by using the vector space structure of $\mathbb{R}^n$. The resulting vector space structure of T_xM is natural in the sense that it does not depend on the used chart [1]. The vector space T_xM is called the *tangent space* of M at x, and its elements are called *tangent vectors*. Note that it does not make (invariant) sense to add tangent vectors from tangent spaces at different points of M.

The force 1-form $F[o_1^T, o_2^T]$ is a covector field on Ω that assigns to each point in the path of movement of o_2 a covector at the point. It may be integrated over oriented 1-dimensional submanifolds of Ω modeling the paths of movement. Its relation to the work done by o_1 when o_2 is moved along a path γ is

$$W[o_1, o_2](\gamma) = \int_\gamma F[o_1^T, o_2^T]. \qquad (3.1)$$

Orientation and integration. The idea in the integration of a p-form (a p-covector field) over a p-dimensional submanifold is to chop the submanifold into small pieces and feed the p-vectors corresponding to these pieces to the p-covector values of the p-form. Adding together the resulting numbers gives the value of the integral. For this process to succeed we must have orientations specified consistently for all tangent spaces of the submanifold. The p-vectors we give to the p-covector values of the p-form are chosen such that they are positively oriented in this orientation system.

Not all manifolds can be given a consistent orientation system. We say that a manifold is *orientable* if we can select an array of smooth vector fields that form a basis for the tangent space at each point of the manifold. Such an array defines the array of smooth 1-forms that constitute the dual basis at each point of the manifold. If $(\alpha^1, \ldots, \alpha^n)$ is an array of such 1-forms we have the nonzero n-form $\alpha^1 \wedge \cdots \wedge \alpha^n$ defined on the manifold. An *orientation* of a manifold is an equivalence class of arrays of basis vector fields, two arrays being equivalent if each one of them is mapped by $\alpha^1 \wedge \cdots \wedge \alpha^n$ to a function that is positive at each point of the manifold.

Let now N be a p-dimensional submanifold of n-dimensional manifold M. We have a *natural inclusion* $i : N \to M$ by which each point of N is considered as a point of M. By using this map we may also consider tangent vectors at points of N as tangent vectors at points of M. The vector $[\tau] \in T_xN$ may be considered as the vector $[i \circ \tau] \in T_{i(x)}M$. We may also restrict a covector at $i(x) \in M$ to a covector at $x \in N$ by allowing it to operate only on vectors of $T_x N$ (considered as vectors of $T_{i(x)} M$). Such a restriction will be used to integrate a p-form defined on M over N.

To chop the submanifold N into pieces we let s be an oriented *p-simplex* on N. It is defined formally as an equivalence class of pairs consisting of a chart for N and an oriented p-simplex in the codomain of the chart (line segment, triangle, tetrahedron, corresponding to p=1,2,3, respectively). Two pairs are taken as equivalent if the change-of-chart map transforms the p-simplices to each other such that the orientations are preserved (Jacobian determinant is positive). We will often identify the p-simplex containing $(\chi, \triangle)$ with the preimage $\chi^{-1}(\triangle)$. Note that this definition requires us to specify a class of charts in which each change-of-chart map has constant Jacobian matrix; it is only the codomains of these specific charts where the preimage $\chi^{-1}(\triangle)$ appears as a p-simplex. (This will not be a serious drawback because eventually this structure will be needed for another reason.) We associate to a p-simplex s containing $(\chi, \triangle)$ a p-vector as follows. We first take an array of tangent vectors $(v_1, \ldots, v_p)$ whose representation in the chart χ is an orientation defining array of edge vectors of the simplex $\triangle$, attached to the barycenter of $\triangle$. The p-vector $v_1 \wedge \cdots \wedge v_p$ contains all oriented parallelopipeds with unit p-volume relative to $(v_1, \ldots, v_p)$. Multipyling it by $1/p!$ we get the wanted notion of an "oriented infinitesimal p-simplex". The *p-vector of* s is defined to be $1/p! v_1 \wedge \cdots \wedge v_p$. It is denoted as $\{s\}$.

A smooth manifold can always be covered by a family of simplices that may overlap with each other only at their boundaries [6]. Thus we may assume that such a *triangulation* is given for the p-dimensional submanifold N. Further, a triangulation can be endlessly refined by subdivision procedures. Let k be the number of p-simplices in the triangulation of N. At any value of k we arrange all the simplices to be positively oriented in the given orientation of N. Then, we define the integral of p-form ω over N as

$$\int_N \omega = \lim_{k \to \infty} \sum_{i=1}^{k} \omega_i(\{s_i\}),$$

where ω_i is the p-covector value of ω at the point of the p-vector $\{s_i\}$. This definition of integration is in accordance with the one given in [4].

Note that by taking the paths of movement as submanifolds (that do not have a preferred parametrization) we assume that the rate of movement is irrelevant. Thus, the evolution of the system may be considered to consist of successive states of static objects. In the following some particular state of the system will be taken into consideration. The force covector $F[o_1^t, o_2^t]$ at such a state will be denoted simply as F_{12}, whereas the submanifolds o_1^t and o_2^t will be denoted as o_1 and o_2.

3.1 Electrostatic forces on objects with charge distributions

Let us first examine situations where the source of the interaction under study may be taken as static distribution of *electric charges*. I will consider the charge distributions of the two objects separately by first introducing *volume charge density* $\tilde{\rho}_1$ *of object* o_1 and *volume charge density* $\tilde{\rho}_2$ *of object* o_2. The physical space and the two objects will be taken to be 3-dimensional so that $\tilde{\rho}_1$ and $\tilde{\rho}_2$ are 3-forms, that is, 3-covector fields. They vanish outside o_1 and o_2, respectively. I emphasize that $\tilde{\rho}_1$ and $\tilde{\rho}_2$ are functions of both o_1 and o_2, although this is not indicated in the notation. This concerns also all other quantities that will be introduced. Then, to take into account charges on the surfaces of the objects, I will use *surface charge density* $\tilde{\sigma}_1$ *of object* o_1 and *surface charge density* $\tilde{\sigma}_2$ *of object* o_2. These are 2-forms that are defined on $\partial o_1 \cup \partial o_2$, and that vanish outside of ∂o_1 and ∂o_2, respectively. For instance, the *total charge* Q_1 of o_1 is given as

$$Q_1 = \int_{o_1} \tilde{\rho}_1 + \int_{\partial o_1} \tilde{\sigma}_1. \tag{3.2}$$

The charge densities are, in fact, twisted differential forms implying that the signs of their integrals do not depend on the orientation of the underlying space manifold. (I assume that the space manifold is orientable so that it will be possible to represent a twisted form by an ordinary one whose sign depends on the selected orientation.) As charge is conserved in a closed system such as o_1 and o_2, the total charges Q_1 and Q_2 are independent of the state of the system.

Integration of twisted forms. The values of a twisted p-form are twisted p-covectors that operate on twisted p-vectors to yield numbers. This means that twisted p-forms are made to be integrated over submanifolds whose tangent spaces can be given a consistent system of transverse orientations. Not all submanifolds are of this kind. We call a p-dimensional submanifold N of n-dimensional manifold M *transverse orientable* if we can select an array $(\alpha^1, \dots, \alpha^{n-p})$ of linearly independent smooth 1-forms each of which satisfies $\alpha^i(u) = 0$ for all tangent vector fields u on N. One of the two *transverse orientations* of N is given by the arrays $(v_1, \dots, v_{n-p})$ of smooth vector fields for which the (n-p)-form $\alpha^1 \wedge \dots \wedge \alpha^{n-p}$ gives a function that is positive everywhere.

Let us assume that M is orientable so we may represent a twisted p-form by a pair consisting of an ordinary p-form ω and an orientation Or for M. Thus, $(-\omega, -Or)$ represents the same twisted p-form. We may also represent the transverse oriented submanifold N, whose transverse orientation contains $(v_1, \dots, v_{n-p})$, by the pair (N, Or), where N has an orientation specified by the array $(u_1, \dots, u_p)$ satisfying

$$(v_1, \dots, v_{n-p}, u_1, \dots, u_p) \in Or.$$

We define the integral of a twisted p-form over submanifold with transverse orientation by using these representatives. We set

$$\int_{(N,Or)} (\omega, Or) = \int_N \omega.$$

By using the representation $(-\omega, -Or)$ for the twisted p-form we get the same number because in the representation $(N, -Or)$ of the transverse oriented submanifold the orientation of N is reversed according to the above rule.

The charge densities of objects o_1 and o_2 may both be represented by an auxiliary quantity. For instance, the quantity related to the charge densities of o_1 is the *electric displacement* $\tilde{D}_1$ *of object* o_1. It is a twisted 2-form whose defining properties are given by using the *exterior derivative* d and *tangential trace* t_1 as

$$\mathrm{d}\tilde{D}_1 = \tilde{\rho}_1, \tag{3.3}$$

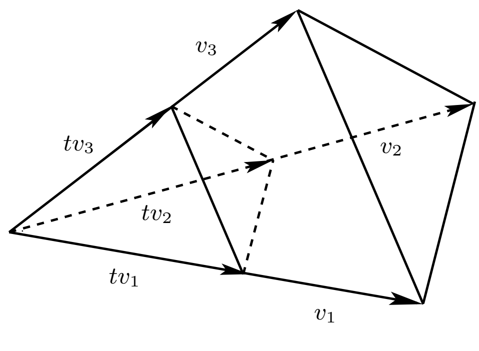
$$[\mathrm{t}_1 \tilde{D}_1]_1 = \tilde{\sigma}_1, \tag{3.4}$$

where $[\mathrm{t}_1\tilde{D}_1]_1$ is the discontinuity of $\mathrm{t}_1\tilde{D}_1$ over ∂o_1 (the value outside minus the value inside). The exterior derivative is restricted to points where $\tilde{D}_1$ is smooth, whereas the tangential traces required to define the above discontinuity restrict $\tilde{D}_1$ to ∂o_1 from the two sides by using the natural inclusion map. The *electric displacement* $\tilde{D}_2$ *of object* o_2 is defined similarly.

Exterior derivative. The exterior derivative of a p-form is a (p+1)-form that measures, when evaluated on the (p+1)-vector of a given (p+1)-simplex, the integral of the p-form over the boundary of the simplex when the simplex is made vanishingly small. Given a p-form ω its *exterior derivative* is the (p+1)-form $\mathrm{d}\omega$ whose value at arbitrary point x of the manifold is defined for vectors $v_1, \ldots, v_{p+1}$ by

$$(\mathrm{d}\omega)_x(v_1, \ldots, v_{p+1}) = \lim_{t \to 0} \frac{(p+1)!}{t^{p+1}} \int\limits_{\partial s_t} \omega,$$

where s_t is the (p+1)-simplex whose (p+1)-vector is $t^{p+1}/(p+1)! v_1 \wedge \cdots \wedge v_{p+1}$. In the inset we show an example where $p = 2$.



The above definition requires an orientation to be specified for ∂s_t. For this we first consider s_t as a (p+1)-dimensional submanifold whose orientation is specified by the array $(v_1, \ldots, v_{p+1})$ (extended to be an array of smooth vector fields on s_t by using a chart). We further give ∂s_t a transverse orientation that consists of all vectors pointing outside from s_t. This determines an orientation for ∂s_t according to the familiar rule, and this is the orientation used in the definition above. This definition of exterior derivative is in accordance with the one given in [4].

By the above definition we have for a 0-form f (that may be identified with a function) the following expression for $(\mathrm{d}f)_x(v)$ in chart χ. By using symbols $\hat{f}$, $\boldsymbol{x}$ and $\boldsymbol{v}$ for $f \circ \chi^{-1}$, $\chi(x)$ and $(\chi \circ \tau)'(0)$, where $\tau \in v$, we have

$$(\mathrm{d}f)_x(v) = \lim_{t \to 0} \frac{\hat{f}(\boldsymbol{x} + t\boldsymbol{v}) - \hat{f}(\boldsymbol{x})}{t}.$$

Thus $(\mathrm{d}f)_x(v)$ is the directional derivative of f to the direction of v.

For twisted forms the exterior derivative is defined in terms of representatives. For the twisted p-form represented by (ω, Or) its exterior derivative is the twisted (p+1)-form represented by $(\mathrm{d}\omega, Or)$.

The defining properties of quantities $\tilde{D}_1$ and $\tilde{D}_2$ are designed to introduce into the model the idea that the total charge of an object may be obtained by integrating its electric displacement over any closed surface surrounding the object. To see this, let us first select an observation surface surrounding object o_1. This surface is the boundary of o_1' that contains object o_1 as its subset. Then, since $\tilde{\rho}_1$ vanish in $o_1' - o_1$, the total charge of object o_1 may be given as

$$Q_1 = \int\limits_{o_1} \mathrm{d}\tilde{D}_1 + \int\limits_{o_1' - o_1} \mathrm{d}\tilde{D}_1 + \int\limits_{\partial o_1} [\mathrm{t}_1 \tilde{D}_1]_1.$$

By further requiring that the intersection of $\partial o_1'$ and ∂o_1 is empty, and by using *Stokes' theorem* to the first two terms we find that the term involving the discontinuity of $\mathrm{t}_1 \tilde{D}_1$ vanishes. We thus have the desired property

$$Q_1 = \int\limits_{\partial o_1'} \tilde{D}_1$$

for all appropriate observation surfaces $\partial o_1'$.

Stokes' theorem. The exterior derivative is defined such that for a p-form ω and a (p+1)-simplex s (whose (p+1)-vector resides at point x) we have the approximate relation

$$(\mathrm{d}\omega)_x(\{s\}) \approx \int\limits_{\partial s} \omega.$$

Thus, when we integrate dω over (p+1)-dimensional submanifold N whose triangulation contains k simplices, we have, by endlessly subdividing the triangulation,

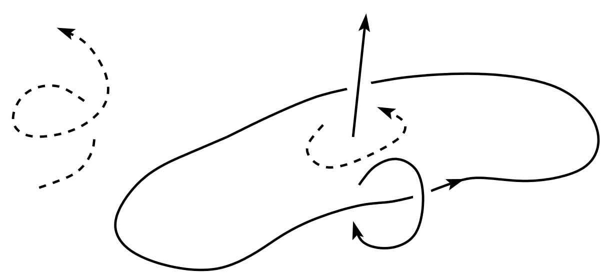
$$\int_N \mathrm{d}\omega = \lim_{k\to\infty} \sum_{i=1}^{k} (\mathrm{d}\omega)_i(\{s_i\}) = \lim_{k\to\infty} \sum_{i=1}^{k} \int_{\partial s_i} \omega = \int_{\partial N} \omega,$$

where the last equality follows because each p-simplex (subset of ∂s_i) that is not a subset of ∂N appears twice, and with opposite orientations. Here ∂N has an orientation determined from the outward pointing transverse orientation for ∂N and the orientation of N by using the familiar rule. The second equality above needs proving. For a rigorous treatment, see [4]. This is *Stokes' theorem*, see also [1, 2, 3, 5].

When Stokes' theorem is applied to twisted forms, the submanifold and its boundary must have transverse orientations. In terms of representatives we have

$$\int_{(N,Or)} \mathrm{d}(\omega, Or) = \int_{(N,Or)} (\mathrm{d}\omega, Or) = \int_N \mathrm{d}\omega = \int_{\partial N} \omega = \int_{(\partial N,Or)} (\omega, Or).$$

In the inset we clarify how the orientations work in case of a 2-dimensional transverse oriented submanifold of a 3-dimensional manifold. The dashed arrows indicate the orientations used to represent the transverse orientations (shown in solid lines).



For a relation between charges and forces we will first express the force on o_2 by using local force densities. More precisely, the force covector F_{12} will be expressed by using *volume force density* $\tilde{F}_{12}$ and *surface force density*

$\tilde{f}_{12}$ as

$$F_{12} = \int\limits_{o_2} \tilde{F}_{12} + \int\limits_{\partial o_2} \tilde{f}_{12}. \tag{3.5}$$

The force F_{21} will be expressed similarly in terms of force densities $\tilde{F}_{21}$ and $\tilde{f}_{21}$. Here, the force densities are taken as *covector-valued differential forms.*

Vector- and covector-valued forms. Covector-valued p-form on a manifold is an object whose value at a point on the manifold is a linear map from p-vectors at the point to covectors at the point. For more precise definition, let us denote as $T^p_x M$ and $T^x_p M$ the p-vector space and p-covector space at $x \in M$, respectively, and consider the vector space of linear maps from $T^p_x M$ to $T^x_1 M$. This vector space is denoted as $L(T^p_x M; T^x_1 M)$. A *covector-valued p-form on* M is a field of such objects defined at the points of M.

There is a dual point of view of covector-valued forms which is obtained by identifying the space $L(T^p_x M; T^x_1 M)$ with $L(T^1_x M; T^x_p M)$. For this, let us consider $\eta_x \in L(T^p_x M; T^x_1 M)$, and notice that for fixed vector $v_x \in T^1_x M$ the map that sends $u_x \in T^p_x M$ to $\eta_x(u_x)(v_x) \in \mathbb{R}$ is a p-covector at x. Let us denote this p-covector as $\mathcal{G}(\eta_x, v_x)$. By definition, it satisfies

$$\mathcal{G}(\eta_x, v_x)(u_x) = \eta_x(u_x)(v_x).$$

Since the p-covector $\mathcal{G}(\eta_x, v_x)$ depends linearly on v_x it is actually the result of the operation of $\mathcal{G}(\eta_x, \cdot) \in L(T^1_x M; T^x_p M)$ on v_x. Thus, for given $\eta_x \in L(T^p_x M; T^x_1 M)$ we find $\mathcal{G}(\eta_x, \cdot) \in L(T^1_x M; T^x_p M)$ by the above procedure. This correspondence is linear and bijective making the two spaces isomorphic [7]. Depending on the situation we will use both of these points of view when working with covector-valued forms.

There is yet another point of view of covector-valued forms which shows that we are dealing with tensor fields of specific type. For this we consider the successive operations of $\eta_x \in L(T^p_x M; T^x_1 M)$ on a p-vector and a vector. By similar arguments as above we find that η_x may be identified with an element of $L(T^1_x M \times T^p_x M; \mathbb{R})$, that is, with a covariant tensor of order $p + 1$ which is antisymmetric in its last p arguments.

A similar construction is behind *vector-valued p-forms on M*. The value of a vector-valued p-form at $x \in M$ is thus an element of $L(T^p_x M; T^1_x M)$.

To clarify the integration above, we first note that $\tilde{F}_{12}$ being a covector-valued 3-form means its operation on a vector field v (by using the dual point of view) yields the 3-form $\mathcal{G}(\tilde{F}_{12}, v)$ that may be integrated over 3-dimensional submanifolds. Similarly, the operation of the covector-valued 2-form $\tilde{f}_{12}$ on a vector field v yields the 2-form $\mathcal{G}(\tilde{f}_{12}, v)$. As was the case with the charge densities these 3- and 2- forms are twisted differential forms. Then, following the idea that the virtual displacement vector represents the displacements of all the points of o_2, we define the integral $\int_{o_2} \tilde{F}_{12} + \int_{\partial o_2} \tilde{f}_{12}$ to be the covector whose value on vector v is given as

$$\Big(\int\limits_{o_2} \tilde{F}_{12} + \int\limits_{\partial o_2} \tilde{f}_{12}\Big)(v) = \int\limits_{o_2} \mathcal{G}(\tilde{F}_{12}, v) + \int\limits_{\partial o_2} \mathcal{G}(\tilde{f}_{12}, v),$$

where v on the right hand side is extended to be a constant vector field. To give meaning to the notion of constant vector field we assume a *connection* on Ω.

Connection and covariant derivative. The concept of manifold does not yet contain the idea of a constant vector field. The additional structure needed is a *connection*. It may be defined as a mapping ∇ that takes two smooth vector fields u and v on the manifold M to a third smooth vector field on M. The resulting vector field is denoted as $\nabla_u v$. For smooth functions f, g on M and smooth vector fields u, v, w on M any connection ∇ satisfies the axioms

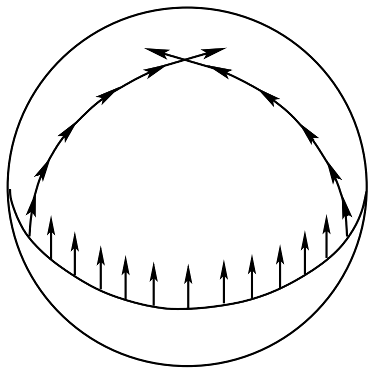
(i) $\nabla_{fu+gv} w = f\nabla_u w + g\nabla_v w$ (function-linearity in the first argument),

(ii) $\nabla_u(fv + gw) = f\nabla_u v + \mathrm{d}f(u)v + g\nabla_u w + \mathrm{d}g(u)w$ (product rule of derivatives and linearity in the second argument).

These two axioms state that ∇_u is a first order derivative of vector fields. The resulting vector field $\nabla_u v$ is called the *covariant derivative of v to the direction of u*.

Covariant derivative generalizes to vector fields the directional derivative of functions. Thus, the latter is sometimes denoted by the same symbol so that $\nabla_u f = \mathrm{d}f(u)$. In particular, as is the case with $\mathrm{d}f(u)$, the value of $\nabla_u v$ at a point depends on the value of u only at this point (and not on its values at neighbouring points). This is implied by axiom (i), and it gives meaning to the covariant derivative of a vector field to the direction of a vector at a point. A vector field being *constant* means its covariant derivative to the direction of all vectors at a point vanishes for all points. This is conveniently expressed as $\nabla v = 0$. Note that ∇ being function-linear in the direction argument, we may view ∇v as a vector-valued 1-form.

In general one cannot define a constant vector field on a manifold. This is elucidated in the inset where we show a failed attempt to draw a constant vector field over a loop on 2-dimensional sphere embedded in 3-dimensional Euclidean space. The example is from Burke [5]. Thus, the idea of general use is that of *locally constant* vector field, that is, one whose first derivatives at a point vanish.



Having defined the covariant derivative for functions and vector fields it may be uniquely extended to all smooth tensor fields on M [1]. In particular, we then have for smooth 1-form ω and smooth vector field v the relation $\nabla_u(\omega(v)) = (\nabla_u\omega)(v) + \omega(\nabla_u v)$ for all smooth vector fields u, and we see what it means for a 1-form to be constant (meaning that $\nabla\omega = 0$). A similar relation holds also for other types of tensor fields [1]. The covariant derivative is a tensor derivation, meaning, in particular, that for p-form ω and q-form η we have $\nabla_u(\omega \wedge \eta) = \nabla_u\omega \wedge \eta + \omega \wedge \nabla_u\eta$ [1].

An expression for $\nabla_u v$ in a chart may be given in terms of basis vector fields $e_1, \ldots, e_n$ of the chart by first expressing u and v as

$u = u^i e_i$ and $v = v^i e_i$. Here, I have made use of the summation convention. Then, by defining smooth functions Γ^k_{ij} on the chart by $\nabla_{e_i} e_j = \Gamma^k_{ij} e_k$, and using axioms (i) and (ii), we have in the chart

$$\nabla_u v = (\mathrm{d}v^k(u) + u^i v^j \Gamma^k_{ij}) e_k.$$

The functions Γ^k_{ij} are called the *Christoffel symbols* of the connection in the given chart. They express how the basis vector fields change to the direction of other basis vector fields, and thus make sure that $\nabla_u v$ does not depend on the basis used to represent u and v. When the manifold may be covered by single chart, the specification of the Christoffel symbols provides a convenient way to specify a connection.

Although a connection provides us enough to say whether the extended vector field in the above definition of integration is constant, we still need to make sure that the assumed extension to a constant vector field is possible. This is guaranteed by requiring that both o_1 and o_2 reside on neighbourhoods that are *parallelizable manifolds*.

Parallelizable manifold. A manifold with connection is called *parallelizable* if there is an array of constant basis vector fields on the manifold. Note that given such a basis the 1-forms of the associated dual basis are also constant. The collection of all constant bases on the manifold is called a *parallelism* of the manifold. Alternatively, parallelism may be taken as a collection of all charts whose change-of-chart maps have constant Jacobian matrices; when restricting to such charts the changing of chart does not affect a vector field appearing constant.

To further clarify the integration of the force densities we first take v to be the constant virtual displacement vector field. The operation of $\tilde{F}_{12}$ and $\tilde{f}_{12}$ on v may be given in terms of basis 1-forms $\omega^1, \omega^2, \omega^3$ as

$$\mathcal{G}(\tilde{F}_{12}, v) = \omega^i(v)(\tilde{F}_{12})_i,$$
$$\mathcal{G}(\tilde{f}_{12}, v) = \omega^i(v)(\tilde{f}_{12})_i,$$

where $(\tilde{F}_{12})_i$ and $(\tilde{f}_{12})_i$ are twisted 3- and 2-forms for $i = 1, 2, 3$. Here we have the summation convention in use again. When the basis 1-forms ω^i are constant the components $\omega^i(v)$ of v are also constant, and the information on the distribution of forces is contained in the components $(\tilde{F}_{12})_i$ and $(\tilde{f}_{12})_i$. The integration may be performed as

$$\int_{o_2} \mathcal{G}(\tilde{F}_{12}, v) + \int_{\partial o_2} \mathcal{G}(\tilde{f}_{12}, v) = \int_{o_2} \omega^i(v)(\tilde{F}_{12})_i + \int_{\partial o_2} \omega^i(v)(\tilde{f}_{12})_i$$
$$= \omega^i(v)\Big(\int_{o_2} (\tilde{F}_{12})_i + \int_{\partial o_2} (\tilde{f}_{12})_i\Big),$$

so in practice we integrate the components $(\tilde{F}_{12})_i$ and $(\tilde{f}_{12})_i$.

Our next goal will be to find a relation between charge densities and force densities. For this, we introduce *electric field intensity* E_1 *of object* o_1. This is a 1-form whose relation to the force densities $\tilde{F}_{12}$ and $\tilde{f}_{12}$ is written by using the exterior product $\wedge$ and the *interior product* i_v as

$$\mathcal{G}(\tilde{F}_{12}, v) = \tilde{\rho}_2 \wedge \mathrm{i}_v E_1, \tag{3.6}$$
$$\mathcal{G}(\tilde{f}_{12}, v) = \tilde{\sigma}_2 \wedge \mathrm{t}_2 \mathrm{i}_v E_1. \tag{3.7}$$

We require (3.6) and (3.7) for all vector fields v so they determine $\tilde{F}_{12}$ and $\tilde{f}_{12}$. Here, I assume that o_1 and o_2 are distinct objects whose boundaries have no common points, so $\mathrm{t}_2 \mathrm{i}_v E_1$ is well defined (E_1 is continuous on ∂o_2).

Interior product. The interior product of a p-form with a vector field is the (p-1)-form obtained by putting the vector field in the first argument place of the p-form. Formally, given a p-form ω its *interior product* with vector field v is the (p-1)-form $\mathrm{i}_v\omega$ defined by

$$\mathrm{i}_v\omega(u_2, \ldots, u_p) = \omega(v, u_2, \ldots, u_p)$$

for all vector fields $u_2, \ldots, u_p$. In addition to being linear in both of its arguments the interior product satisfies $\mathrm{i}_{fv}\omega = f\mathrm{i}_v\omega$ for all functions f. Thus $\mathrm{i}_v\omega$ may be viewed as the result of a covector-valued (p-1)-form operating on the vector field v. An important property of interior product is that for p-form ω and q-form η it satisfies

$$\mathrm{i}_v(\omega \wedge \eta) = \mathrm{i}_v\omega \wedge \eta + (-1)^p \omega \wedge \mathrm{i}_v\eta,$$

as proved in [1, 3]. Together with the linearity property this states that i_v is an antiderivation. The antiderivation property will be used repeatedly when deriving basis representations in the appendices.

Note that according to (3.7) the 2-form $\mathcal{G}(\tilde{f}_{12}, v)$ takes in only tangent vectors of ∂o_2 whereas the vector field v is not a tangent vector field on ∂o_2. Thus, although the two-form part of $\tilde{f}_{12}$ is defined on ∂o_2, the covector-values of $\tilde{f}_{12}$ are located at the points of the space manifold wherein ∂o_2 is embedded. This means $\tilde{f}_{12}$ is a covector-valued form only in an extended sense: its value at $x \in \partial o_2$ is not an element of $L\big(T_x^2 \partial o_2; T_1^x \partial o_2\big)$ but an element of $L\big(T_x^2 \partial o_2; T_1^x \Omega\big)$ (where in $T_1^x \Omega$ we mean the image point of x under the natural inclusion map).

The defining properties of E_1 may be given by using the exterior derivative and tangential trace as

$$\mathrm{d}E_1 = 0, \tag{3.8}$$

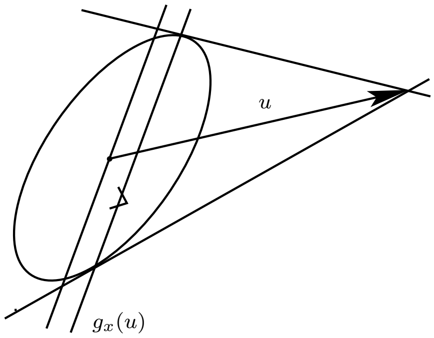
$$[\mathrm{t}_1 E_1]_1 = 0, \tag{3.9}$$

where the exterior derivative is restricted to points where E_1 is smooth. Finally, the last of the required defining properties of $(E_1, \tilde{D}_1)$ is given by using the *Hodge operator* $\star$ as

$$\tilde{D}_1 = \epsilon_0 \star E_1, \tag{3.10}$$

where ϵ_0 is the permittivity of vacuum. The relations (3.3)-(3.10) constitute an abstraction of Coulomb's force law concerning the interaction of static point charges (see [8, 9]). As a consequense of the Hodge operator the relation between $(\tilde{\rho}_1, \tilde{\sigma}_1)$ and E_1, defined by (3.3), (3.4), (3.8), (3.9), and (3.10), requires a *Riemannian metric* on Ω. The *electric field intensity* E_2 *of object* o_2 is defined similarly.

Riemannian metric and Hodge operator. To give meaning to metric notions (lengths, angles, etc) on a manifold an additional structure is needed. For this we need an *inner product* for each tangent space of the manifold. A *Riemannian metric* g on a manifold M is a smooth field of inner products on M. A pair (M, g) is called a *Riemannian manifold*. Note that a Riemannian metric may be taken as a covector-valued 1-form. This is elucidated in the inset where we represent the value of g at a point by an ellipse showing the trajectory formed by the tip of all unit vectors at the point. The parallel lines with the arrowhead on one of them represent the covector $g_x(u_x)$. The method of visualization is from Burke [5].



With a Riemannian metric each tangent space of a manifold becomes an inner product space giving meaning to the *norm of vector* and *angle between vectors at a point*. Also, the *length of a curve* may be defined by taking a unit vector field u tangent to the curve, and integrating the 1-form $g(u)$ over the curve. Finally, the *distance between any two points* is defined (on a connected manifold) by taking the infimum of lengths of curves connecting the points. The topology induced by this distance (metric) coincides with the underlying manifold topology [1, 2].

A Riemannian metric gives rise to an important operator of multivector fields and differential forms. To obtain it we first note that for each point $x \in M$ the inner product on $T^1_x M$ induces an inner product on the p-vector spaces $T^p_x M$. The inner product of p-vectors $u_1 \wedge \cdots \wedge u_p$ and $v_1 \wedge \cdots \wedge v_p$ is given by the determinant of the matrix with entries $g(u_i, v_j)$ [4, 11]. Then, we take a p-vector $u_x \in T^p_x M$ and consider the linear map that sends an (n-p)-vector $v_x \in T^{(n-p)}_x M$ to the n-vector $u_x \wedge v_x \in T^n_x M$. Once a unit n-vector $\sigma_x \in T^n_x M$ has been selected for the basis of $T^n_x M$ ($T^n_x M$ is 1-dimensional) this map may be identified with a real-valued linear map on $T^{(n-p)}_x M$. It then follows from Riesz's representation theorem that there exists a unique (n-p)-vector $\hat{u}_x \in T^{(n-p)}_x M$, such that

$$u_x \wedge v_x =< \hat{u}_x, v_x > \sigma_x \quad \text{for all } v_x \in T^{(n-p)}_x M,$$

where $< \cdot, \cdot >$ denotes the inner product of $T_x^{(n-p)}M$. The sign of $\hat{u}_x$ depends on the sign of σ_x, but the twisted (n-p)-vector represented by $(\hat{u}_x, \sigma_x)$ (or $(-\hat{u}_x, -\sigma_x)$) is uniquely determined by u_x. This twisted (n-p)-vector is denoted as $\star u_x$, and the operator thus obtained is the *Hodge operator of multivectors*. We also note that the (n-p)-vector $\hat{u}_x$ is uniquely determined by the twisted p-vector represented by (u_x, σ_x), so we obtain the *Hodge operator of twisted multivectors*. These operators may be used to define the *Hodge operator of p-covectors* (twisted or not). For instance, the *Hodge operator of a p-covector* $\omega_x \in T_p^x M$ is defined such that

$$(\star\omega_x)(\tilde{v}_x) = \omega_x(\star\tilde{v}_x) \quad \text{for all } \tilde{v}_x \in \tilde{T}_x^{(n-p)}M,$$

where $\tilde{T}_x^{(n-p)}M$ is the space of twisted (n-p)-vectors at $x \in M$. Finally, the *Hodge operator of p-vector fields and p-forms* may be defined in a pointwise fashion [4].

Now we are in a position where the electric fields $(E_1, \tilde{D}_1)$ and $(E_2, \tilde{D}_2)$ may be determined if the charge densities $(\tilde{\rho}_1, \tilde{\sigma}_1)$ and $(\tilde{\rho}_2, \tilde{\sigma}_2)$, and the metric tensor g, are given beforehand. Also, the electric field intensities E_1 and E_2 are related to the force densities $(\tilde{F}_{12}, \tilde{f}_{12})$ and $(\tilde{F}_{21}, \tilde{f}_{21})$, and finally, once also a connection is given, they determine the total forces F_{12} and F_{21} by integration. By taking the objects o_1 and o_2 to be rigid these two quantities will be observable. By calling an object rigid it is meant that the distances between every two points of the object remain the same under the evolution of the system. To make the constant virtual displacement vector field to agree with this notion of rigidity we relate our connection to the used metric in a specific way. This is achieved by requiring that the connection is metric compatible, and that it is in a specific relation to the *Lie bracket*. The proof that these properties guarantee a constant virtual displacement not to distort distances will have to wait until the beginning of the next chapter.

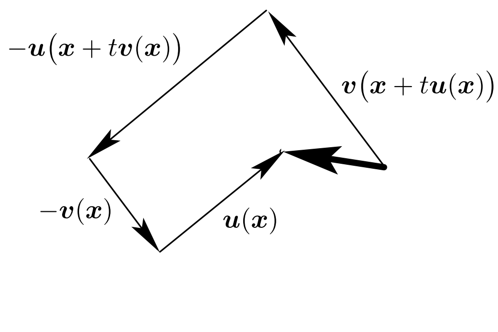
Lie bracket. The Lie bracket arises from considering vector fields as differential operators on functions. Given a smooth vector field u on M, we define a first order differential operator on smooth functions on M, denoted with the same symbol u, such that its value $u(f)$ for a function f is given as $u(f) = \mathrm{d}f(u)$. The smooth functions on M have the product of functions defined in a pointwise fashion. That the operator u satisfies the Leibniz derivation rule with respect to this product ensures it is a differential operator of first order [1, 5]. Besides the above observation that vector fields may be considered as such operators, it turns out that the converse is also true [1, 2, 5]. Furthermore, this correspondence between vector fields and first order differential operators (considered as vector spaces) is linear and bijective, making the two isomorphic [1, 2]. In particular, the basis vector fields of a coordinate chart correspond to partial derivatives with respect to the coordinates.

By using the above isomorphism, we find that a pair of vector fields may be used to obtain yet another vector field. For this, we note that the operator $u \circ v - v \circ u$ is a first order differential operator as the involved second derivatives are canceled. This may be verified by using a coordinate chart [1, 2, 5]. The vector field corresponding to this new differential operator is the *Lie bracket of* u *and* v, denoted as $[u, v]$.

By the above definition of $[u, v]$ we may determine its representation in the basis of a coordinate chart. We denote the coordinate functions as $x^1, \ldots, x^n$, and take into account that the basis vector fields $e_1, \ldots, e_n$ correspond to partial derivatives $\partial/\partial x^1, \ldots, \partial/\partial x^n$, to get

$$[u, v] = (u^j \frac{\partial v^i}{\partial x^j} - v^j \frac{\partial u^i}{\partial x^j}) \frac{\partial}{\partial x^i},$$

where the summation convention is in act. Thus, the Lie bracket $[u, v]$ is represented in the chart by the derivative of v to the direction of u minus the derivative of u to the direction of v. This is elucidated in the inset where the value of $[u, v]$ at a point is represented by the bold arrow divided by t in the limit $t \to 0$. Bold



symbols denote the arrays of components of vectors (whose base points are adjusted for visual effect). In particular, for the basis vector fields $e_1, \dots, e_n$ of a coordinate chart, we have $[e_i, e_j] = 0$ for all indices i, j.

Taken together, the above requirements for our connection means we use the *Levi-Civita connection* determined by the metric.

Levi-Civita connection. On a Riemannian manifold symmetric and metric compatible connection is uniquely specified by the metric tensor. This is called the *Levi-Civita connection*. Given smooth vector fields u, v, w on a Riemannian manifold (M, g) the Levi-Civita connection ∇ satisfies

(i) $\nabla_u v - \nabla_v u = [u, v]$ (symmetry),

(ii) $\nabla_u(g(v, w)) = g(\nabla_u v, w) + g(v, \nabla_u w)$ (metric compatibility).

The existence and uniqueness of ∇ is guaranteed by the Fundamental theorem of Riemannian geometry [1, 2].

Applying property (i) to basis vector fields $e_1, \dots, e_n$ of a coordinate chart, and taking into account that $[e_i, e_j] = 0$ for all i, j, we see that this property is equivalent in the coordinate chart to the symmetry of Christoffel symbols Γ^k_{ij} in the lower indices. The property (ii) expresses the relation between connection and metric, which appears in a coordinate chart as a relation between Christoffel symbols and the components $g(e_i, e_j)$ of the metric tensor. By a property of the covariant derivative this condition is equivalent to the metric tensor being constant, that is, $\nabla g = 0$.

I emphasize a difference in the above definitions to classical textbook approaches (such as that in [10]). This can be seen most directly by taking one of the objects, say o_2, to be arbitrarily small, meaning that E_1 is constant on o_2. Then, from (3.2), (3.5), (3.6) and (3.7), we have $F_{12} = \mathcal{Q}_2 E_1$, with E_1 taken as a covector at the point of o_2. Now, the covector E_1 is determined by the charge distribution of object o_1, *the effect of object o_2 on the distribution being taken into account.* Since we allow for o_2 to affect the charge distribution behind E_1 the total charge of o_2 needs not to be (infinitely) small in order to calculate the force on it by the above formula. This is contrary to the electric field intensity of classical textbooks from which the effect of the test charge (object o_2 in this case) is removed by a limiting process concerning the magnitude of the test charge. Thus, E_1 is not the electric field intensity of classical textbooks. We call it the electric field intensity of object

o_1, although a more descriptive name would be "the electric field intensity of object o_1 in the presence of object o_2".

To examine whether the above theory has Newton's law of action and reaction we first note that since $\tilde{F}_{21}$ vanishes on o_2, and since $\tilde{f}_{21}$ vanishes on ∂o_2, the force on o_2 is given as

$$F_{12} = \int_{o_2} (\tilde{F}_{12} + \tilde{F}_{21}) + \int_{\partial o_2} (\tilde{f}_{12} + \tilde{f}_{21}).$$

To transfer the integration above to integration over o_1 we first require parallelizability from a sufficiently large neighbourhood containing both of the objects o_1 and o_2. Then we express $\tilde{F}_{12}$ and $\tilde{F}_{21}$ as derivatives of other covector-valued forms defined at least on this neighbourhood. That this will be possible is guaranteed by the fact that the neighbourhood is parallelizable and 3-dimensional: the components of $\tilde{F}_{12}$ and $\tilde{F}_{21}$ in a constant covector basis are 3-forms that may be expressed as exterior derivatives of 2-forms. Accordingly, we introduce the *stress* $\tilde{T}_{12}$ as a covector-valued twisted 2-form. It is defined by using the *covariant exterior derivative* d_∇ such that

$$\mathrm{d}_\nabla \tilde{T}_{12} = \tilde{F}_{12}, \tag{3.11}$$

$$[\mathrm{t}_2 \tilde{T}_{12}]_2 = \tilde{f}_{12}, \tag{3.12}$$

where the derivative is restricted to points where $\tilde{T}_{12}$ is smooth, and the tangential trace operates only on the 2-form part of $\tilde{T}_{12}$. The stress $\tilde{T}_{21}$ is defined similarly.

Covariant exterior derivative. Covariant exterior derivative extends the ordinary exterior derivative to the exterior derivative of covector-valued differential forms. It makes use of a connection to apply the ordinary exterior derivative covariantly (independently of basis representation) to the differential form -part of the covector-valued differential form. To define it we first need a generalization of the exterior product of differential forms [12, 13]. For covector-valued p-form η and vector-valued q-form ν their *exterior product* $\eta \dot{\wedge} \nu$ is the (p+q)-form obtained by using the pairing of covector and vector values in the ordinary exterior product (instead of the pointwise multiplication of real values). For instance, in the case of a covector-valued 1-form η and a vector-valued 1-form ν we have

$$\begin{aligned}\eta\dot{\wedge}\nu(u_1 \wedge u_2) &= \eta(u_1)\big(\nu(u_2)\big) - \eta(u_2)\big(\nu(u_1)\big) \\ &= -\Big(\eta(u_2)\big(\nu(u_1)\big) - \eta(u_1)\big(\nu(u_2)\big)\Big) \\ &= -\nu\dot{\wedge}\eta(u_1 \wedge u_2)\end{aligned}$$

for arbitrary vector fields u_1, u_2. Now, the *covariant exterior derivative of a covector-valued p-form* η is the covector-valued (p+1)-form $\mathrm{d}_\nabla \eta$, defined by

$$\mathcal{G}(\mathrm{d}_\nabla \eta, u) = \mathrm{d}\big(\mathcal{G}(\eta, u)\big) - \nabla u \dot{\wedge} \eta$$

for all vector fields u [12, 13]. Here, we consider ∇u as a vector-valued 1-form. On the right hand side, the second term is there to cancel the derivatives of u contained in the first term.

A local representation of $\mathcal{G}(\mathrm{d}_\nabla \eta, u)$ may be obtained by using basis vector fields $e_1, \ldots, e_n$ of a chart and the associated dual basis 1-forms $\omega^1, \ldots, \omega^n$. By expressing $\mathcal{G}(\eta, u)$ in this basis as $\omega^i(u)\eta_i$ we get

$$\begin{aligned}\mathrm{d}\big(\mathcal{G}(\eta, u)\big) &= \mathrm{d}(\omega^i(u)\eta_i) \\ &= \mathrm{d}u^i \wedge \eta_i + u^i \mathrm{d}\eta_i,\end{aligned}$$

where we have used the product rule of derivative that applies to d. To evaluate the term $\nabla u \dot{\wedge} \eta$ it is convenient to use the tensor point of view of ∇u and η. Thus we identify the covector-valued p-form η with $\omega^i \otimes \eta_i$ and the vector-valued 1-form ∇u with $e_k \otimes \alpha^k$, where $\alpha^k = \mathrm{d}u^k + \Gamma^k_{ij} u^j \omega^i$. By using the above definition of $\dot{\wedge}$, with $P(1,p)$ the set of all permutations σ of the index set $\{1, \ldots, 1+p\}$ satisfying $\sigma(2) < \cdots < \sigma(1+p)$, we have

$$\begin{aligned}&(e_k \otimes \alpha^k)\dot{\wedge}(\omega^i \otimes \eta_i)(u_1, \ldots, u_{1+p}) \\ &= \sum_{\sigma \in P(1,p)} \operatorname{sgn}(\sigma)\eta_i(u_{\sigma(2)}, \ldots, u_{\sigma(1+p)})\omega^i(\alpha^k(u_{\sigma(1)})e_k) \\ &= \sum_{\sigma \in P(1,p)} \operatorname{sgn}(\sigma)\omega^i(e_k)\alpha^k(u_{\sigma(1)})\eta_i(u_{\sigma(2)}, \ldots, u_{\sigma(1+p)}) \\ &= \omega^i(e_k) \sum_{\sigma \in P(1,p)} \operatorname{sgn}(\sigma)\alpha^k(u_{\sigma(1)})\eta_i(u_{\sigma(2)}, \ldots, u_{\sigma(1+p)}) \\ &= \omega^i(e_k)\alpha^k \wedge \eta_i(u_1, \ldots, u_{1+p})\end{aligned}$$

for arbitrary vector fields $u_1, \dots, u_{1+p}$. Thus, we get

$$\begin{aligned}\nabla u \dot{\wedge} \eta &= \omega^i(e_k)\alpha^k \wedge \eta_i \\ &= \alpha^k \wedge \eta_k \\ &= (\mathrm{d}u^k + \Gamma^k_{ij} u^j \omega^i) \wedge \eta_k.\end{aligned}$$

This yields for the covariant exterior derivative the basis representation

$$\mathcal{G}(\mathrm{d}_\nabla \eta, u) = \omega^i(u)(\mathrm{d}\eta_i - \Gamma^k_{ji}\omega^j \wedge \eta_k),$$

and we see the Christoffel symbols appearing to ensure that the derivative does not depend on the used basis. In a constant basis this is just the exterior derivative operating on the components η_i. Note that in the general Riemannian case the covariant exterior derivative that uses the Levi-Civita connection depends on the metric of the manifold.

Relations (3.11) and (3.12) may be compared to (3.3) and (3.4). By using an observation surface $\partial o_2'$ surrounding o_2 (similar to $\partial o_1'$ used before, see Figure 3.1) we may write

$$F_{12} = \int\limits_{o_2} \mathrm{d}_\nabla(\tilde{T}_{12} + \tilde{T}_{21}) + \int\limits_{o_2' - o_2} \mathrm{d}_\nabla(\tilde{T}_{12} + \tilde{T}_{21}) + \int\limits_{\partial o_2} [\mathrm{t}_2(\tilde{T}_{12} + \tilde{T}_{21})]_2,$$

where we have made use of the linearity of covariant exterior derivative and the trace operator. As the integrands operate on only constant vector fields, Stokes' theorem may be used to the first two terms. We find that the terms involving the discontinuity of $\mathrm{t}_2(\tilde{T}_{12} + \tilde{T}_{21})$ cancel out. We get

$$F_{12} = \int\limits_{\partial o_2'} (\tilde{T}_{12} + \tilde{T}_{21}).$$

Stokes' theorem for covector-valued forms. The definitions of integration and exterior differentiation of covector-valued forms are compatible in the sense of Stokes' theorem. To see this, we take a smooth covector-valued p-form η, and consider the integral

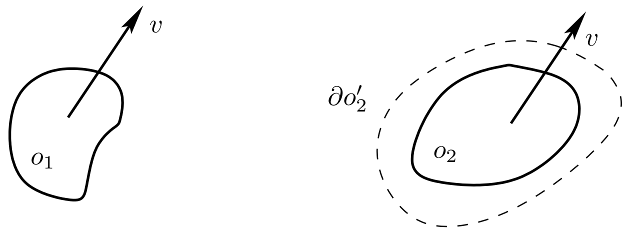


Figure 3.1: An example of valid observation surface $\partial o_2'$ to obtain Newton's law of action and reaction. Since a neighbourhood containing o_1 and o_2 is parallelizable, we may compare forces on the two objects by taking parallel test displacements such as v.

of $\mathrm{d}_\nabla \eta$ over (p+1)-dimensional submanifold N. By the definition of integration, we have

$$\Big(\int_N \mathrm{d}_\nabla \eta\Big)(u) = \int_N \mathcal{G}(\mathrm{d}_\nabla \eta, u),$$

where u is taken as a vector on the left hand side and as a constant vector field on the right hand side. By the definition of covariant exterior derivative, we have

$$\int_N \mathcal{G}(\mathrm{d}_\nabla \eta, u) = \int_N \Big(\mathrm{d}\big(\mathcal{G}(\eta, u)\big) - \nabla u \dot{\wedge} \eta\Big) = \int_N \mathrm{d}\big(\mathcal{G}(\eta, u)\big),$$

where the last equality follows as u is constant. Then, by Stokes' theorem for (ordinary) differential forms, we have

$$\int_N \mathrm{d}\big(\mathcal{G}(\eta, u)\big) = \int_{\partial N} \mathcal{G}(\eta, u),$$

and, finally, by the definition of integration

$$\int_{\partial N} \mathcal{G}(\eta, u) = \Big(\int_{\partial N} \eta\Big)(u),$$

where u is again taken as a vector on the right hand side. Since u is arbitrary, we have the Stokes' theorem $\int_N \mathrm{d}_\nabla \eta = \int_{\partial N} \eta$.

Finally, by selecting the above discussed parallelizable neighbourhood, denoted as o_{12}', to be large enough so that the integral of $\tilde{T}_{12} + \tilde{T}_{21}$ over $\partial o_{12}'$

vanishes (the vanishing of fields at infinity), we get

$$F_{12} = \int\limits_{\partial o_2'} (\tilde{T}_{12} + \tilde{T}_{21}) = - \int\limits_{\partial(o_{12}' - o_2')} (\tilde{T}_{12} + \tilde{T}_{21}) = -F_{21}, \tag{3.13}$$

where the final equality is obtained by using similar arguments as earlier, but in reverse order (using the fact that $\partial(o_{12}' - o_2')$ is a valid observation surface for o_1). By the equality in (3.13) of covectors at different points we mean the equality of the numbers they yield for parallel vectors at the points, see Figure 3.1. Note that this result also makes use of the symmetry of $\tilde{T}_{12} + \tilde{T}_{21}$ in the subscript indices.

The weak point of the above theory is that it assumes that the charge densities are known beforehand. Thus, our task of constructing a relation between charges and forces is not yet complete: the charge densities corresponding to given total charges still need to be determined. In an attempt to deal with this difficulty, let us define E and $\tilde{D}$ by

$$E = E_1 + E_2,$$
$$\tilde{D} = \tilde{D}_1 + \tilde{D}_2,$$

and call the pair $(E, \tilde{D})$ the *total electric field*. Note that we do not take as starting points any additional defining properties of $(E, \tilde{D})$. Instead, they are implied by the defining properties of $(E_1, \tilde{D}_1)$ and $(E_2, \tilde{D}_2)$.

An example case in which the above move is useful is obtained by assuming that $(E, \tilde{D})$ and the charge densities vanish inside of o_1 and o_2 (ideal conductors). In this case the surface charge densities $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ may be determined for the given total charges Q_1 and Q_2. To examine this in more detail let us deduce the defining properties of $(E, \tilde{D})$ from those of $(E_1, \tilde{D}_1)$ and $(E_2, \tilde{D}_2)$. This is done with a convenient notation if we restrict E and $\tilde{D}$ outside of o_1 and o_2 where they are supported. Then, by the linearity of exterior derivative and Hodge operator, we have

$$\mathrm{d}E = 0, \tag{3.14}$$

$$\mathrm{d}\tilde{D} = 0, \tag{3.15}$$

$$\tilde{D} = \epsilon_0 \star E. \tag{3.16}$$

In addition, we have

$$\int\limits_{\partial o_1} \tilde{D} = Q_1, \tag{3.17}$$

$$\int\limits_{\partial o_2} \tilde{D} = Q_2, \tag{3.18}$$

and further

$$\mathrm{t}_1 E = 0, \tag{3.19}$$

$$\mathrm{t}_2 E = 0, \tag{3.20}$$

which follows by the linearity of tangential traces. Once $(E, \tilde{D})$ is determined from the boundary value problem defined by (3.14)-(3.20) the surface charge densities are given according to (3.4) as

$$\tilde{\sigma}_1 = \mathrm{t}_1 \tilde{D}, \tag{3.21}$$

$$\tilde{\sigma}_2 = \mathrm{t}_2 \tilde{D}. \tag{3.22}$$

Finally, by using $\tilde{\sigma}_1$ (resp. $\tilde{\sigma}_2$) as the source we may solve the pair $(E_1, \tilde{D}_1)$ (resp. $(E_2, \tilde{D}_2)$) from another boundary value problem, and determine the surface force density $\tilde{f}_{12}$ (resp. $\tilde{f}_{21}$). This yields the force covector F_{12} (resp. F_{21}) by integration.

The above discussion raises a practical question of whether one could avoid the task of solving $(E_1, \tilde{D}_1)$ (or $(E_2, \tilde{D}_2)$) and obtain forces directly from $(E, \tilde{D})$. To answer this, we first note that by (3.6) the decomposition of E_1 as $E - E_2$ implies a decomposition of $\tilde{F}_{12}$ as

$$\tilde{F}_{12} = \tilde{F}_2 - \tilde{F}_{22},$$

where $\tilde{F}_2$ and $\tilde{F}_{22}$ are defined by

$$\mathcal{G}(\tilde{F}_2, v) = \tilde{\rho}_2 \wedge \mathrm{i}_v E, \tag{3.23}$$

$$\mathcal{G}(\tilde{F}_{22}, v) = \tilde{\rho}_2 \wedge \mathrm{i}_v E_2, \tag{3.24}$$

for all vector fields v. This follows from the linearity of interior product and the bilinearity of exterior product. In the decomposition of $\tilde{f}_{12}$ we run into the problem that E and E_2 may not be continuous at the points of ∂o_2. To overcome this problem we first express $\mathrm{t}_2 \mathrm{i}_v E_1$ by using the average value $(\mathrm{t}_2 \mathrm{i}_v E_1)^{av} = (\mathrm{t}_2^+ \mathrm{i}_v E_1 + \mathrm{t}_2^- \mathrm{i}_v E_1)/2$, where t_2^+ and t_2^- restrict $\mathrm{i}_v E_1$ to ∂o_2 from the two sides (in the same way as with the terms $[\mathrm{t}_2 E_2]_2$ and $[\mathrm{t}_2 \tilde{D}_2]_2$). Since v and E_1 are continuous at the points of ∂o_2 the average $(\mathrm{t}_2 \mathrm{i}_v E_1)^{av}$ may be calculated by just using the values of v and E_1 on ∂o_2. Thus the average $(\mathrm{t}_2 \mathrm{i}_v E_1)^{av}$ coincides with $\mathrm{t}_2 \mathrm{i}_v E_1$. Averages are needed only for the decomposition of $\mathrm{t}_2 \mathrm{i}_v E_1$ on ∂o_2, which now results in the decomposition of $\tilde{f}_{12}$ as

$$\tilde{f}_{12} = \tilde{f}_2 - \tilde{f}_{22},$$

where $\tilde{f}_2$ and $\tilde{f}_{22}$ are defined by

$$\mathcal{G}(\tilde{f}_2, v) = \tilde{\sigma}_2 \wedge (\mathrm{t}_2 \mathrm{i}_v E)^{av}, \tag{3.25}$$

$$\mathcal{G}(\tilde{f}_{22}, v) = \tilde{\sigma}_2 \wedge (\mathrm{t}_2 \mathrm{i}_v E_2)^{av}, \tag{3.26}$$

for all vector fields v. Here the terms $(\mathrm{t}_2 \mathrm{i}_v E)^{av}$ and $(\mathrm{t}_2 \mathrm{i}_v E_2)^{av}$ should be understood in such a way that they require the values of v only at points of ∂o_2 (and not on the two sides wherein E and E_2 are evaluated). The force on o_2 may be written as

$$\begin{aligned} F_{12} &= \int_{o_2} \tilde{F}_{12} + \int_{\partial o_2} \tilde{f}_{12} \\ &= \int_{o_2} (\tilde{F}_2 - \tilde{F}_{22}) + \int_{\partial o_2} (\tilde{f}_2 - \tilde{f}_{22}) \\ &= \int_{o_2} \tilde{F}_2 + \int_{\partial o_2} \tilde{f}_2 - \Big(\int_{o_2} \tilde{F}_{22} + \int_{\partial o_2} \tilde{f}_{22} \Big). \end{aligned}$$

To see that the term in parenthesis vanishes, and thus may be called the *self-force* of o_2, we define a covector-valued twisted 2-form $\tilde{T}_{22}$, such that

$$\mathrm{d}_\nabla \tilde{T}_{22} = \tilde{F}_{22}, \tag{3.27}$$

$$[\mathrm{t}_2 \tilde{T}_{22}]_2 = \tilde{f}_{22}, \tag{3.28}$$

and proceed with the integration as before to get

$$\int_{o_2} \tilde{F}_{22} + \int_{\partial o_2} \tilde{f}_{22} = \int_{\partial o_2'} \tilde{T}_{22}.$$

The right hand side of this may be given as

$$\int_{\partial o_2'} \tilde{T}_{22} = - \int_{\partial (o_{12}' - o_2')} \tilde{T}_{22} = - \int_{o_{12}' - o_2'} \mathrm{d}_\nabla \tilde{T}_{22} = - \int_{o_{12}' - o_2'} \tilde{F}_{22} = 0,$$

since $\tilde{F}_{22}$ vanishes outside of o_2. Note that the vanishing of self-force requires the use of a parallelizable neighbourhood o_{12}' whose boundary gives no contribution to the integral of $\tilde{T}_{22}$. We thus have

$$F_{12} = \int_{o_2} \tilde{F}_2 + \int_{\partial o_2} \tilde{f}_2, \tag{3.29}$$

so that the force on o_2 may be evaluated directly from $(E, \tilde{D})$. This process may further be simplified by first defining covector-valued twisted 3- and 2-forms $\tilde{F}$ and $\tilde{f}$ by

$$\tilde{F} = \tilde{F}_1 + \tilde{F}_2, \tag{3.30}$$

$$\tilde{f} = \tilde{f}_1 + \tilde{f}_2, \tag{3.31}$$

and a covector-valued twisted 2-form $\tilde{T}$, such that

$$\mathrm{d}_\nabla \tilde{T} = \tilde{F}, \tag{3.32}$$

$$[\mathrm{t}\tilde{T}] = \tilde{f}, \tag{3.33}$$

where the tangential trace of (the 2-form part of) $\tilde{T}$ and its discontinuity are defined on $\partial o_1 \cup \partial o_2$. Then, by taking into account that $\tilde{F}_1$ and $\tilde{f}_1$ vanish outside of o_1 and ∂o_1, respectively, it follows from (3.29)-(3.33) (by using the familiar integration argument involving the observation surface $\partial o_2'$) that

$$F_{12} = \int_{\partial o_2'} \tilde{T}. \tag{3.34}$$

In my view the natural way to determine forces is in terms of the force densities $\tilde{F}_{12} + \tilde{F}_{21}$ and $\tilde{f}_{12} + \tilde{f}_{21}$, or in terms of the stress $\tilde{T}_{12} + \tilde{T}_{21}$. In terms of these quantities we have

$$\begin{aligned} F_{12} &= \int_{o_2} (\tilde{F}_{12} + \tilde{F}_{21}) + \int_{\partial o_2} (\tilde{f}_{12} + \tilde{f}_{21}) \\ &= \int_{\partial o_2'} (\tilde{T}_{12} + \tilde{T}_{21}). \end{aligned}$$

The force F_{21} may be determined from these same quantities. Basis representations of the force densities and the stress are given in section A.1.1.

3.2 Magnetostatic forces on objects with current distributions

A situation different from the previous one arises when the sources of the interaction may be taken to be stationary *electric currents*. The construction of the theory is similar to the previous situation, and it begins with the introduction of mathematical objects suitable for the modeling of stationary current distributions. For instance, the currents inside of o_1 are modeled by *current density* $\tilde{J}_1$ *of object* o_1. This is a twisted 2-form that vanishes outside of o_1. Its integration over 2-dimensional transverse oriented submanifolds yields total currents through the surfaces modeled by the submanifolds. Surface currents on ∂o_1 are modeled by *surface current density* $\tilde{j}_1$ *of object* o_1. This is a twisted 1-form on $\partial o_1 \cup \partial o_2$ that vanishes outside of ∂o_1. The currents inside of o_2 and on its boundary are modeled, respectively, by *current density* $\tilde{J}_2$ *of object* o_2 and *surface current density* $\tilde{j}_2$ *of object* o_2. As in the electric case these quantities are functions of both o_1 and o_2.

It is a property of stationary currents that the current through any surface may be determined from the boundary of the surface. Accordingly, we may introduce *magnetic field intensity* $\tilde{H}_1$ *of object* o_1 as a twisted 1-form with the defining properties

$$\mathrm{d}\tilde{H}_1 = \tilde{J}_1, \tag{3.35}$$

$$[\mathrm{t}_1\tilde{H}_1]_1 = \tilde{j}_1. \tag{3.36}$$

Magnetic field intensity $\tilde{H}_2$ *of object* o_2 is defined similarly. Our next steps in the construction of a relation between currents and forces consist of expressing the force F_{12} as

$$F_{12} = \int_{o_2} \tilde{F}_{12} + \int_{\partial o_2} \tilde{f}_{12}, \tag{3.37}$$

and introducing quantities directly related to the force densities $\tilde{F}_{12}$ and $\tilde{f}_{12}$. For instance, *magnetic induction* B_1 *of object* o_1 is a 2-form whose relation to $\tilde{F}_{12}$ and $\tilde{f}_{12}$ is given as

$$\mathcal{G}(\tilde{F}_{12}, v) = \tilde{J}_2 \wedge \mathrm{i}_v B_1, \tag{3.38}$$

$$\mathcal{G}(\tilde{f}_{12}, v) = \tilde{j}_2 \wedge \mathrm{t}_2\mathrm{i}_v B_1, \tag{3.39}$$

for all vector fields v. The defining properties of B_1 are

$$\mathrm{d}B_1 = 0, \tag{3.40}$$

$$[\mathrm{t}_1 B_1]_1 = 0, \tag{3.41}$$

and the last of the required defining properties of $(B_1, \tilde{H}_1)$ is

$$B_1 = \mu_0 \star \tilde{H}_1, \tag{3.42}$$

where μ_0 is the permeability of vacuum. The operators in (3.40)-(3.42) are defined in the same way as before. The relations (3.35)-(3.42) constitute an abstraction of the experimental force law concerning the interaction of current carrying wires derived by Ampère in 1820 (see [8, 14]). Similar properties concern the *magnetic induction* B_2 *of object* o_2. Having constructed relations between $(\tilde{J}_1, \tilde{j}_1)$ and B_1, and between $(\tilde{J}_2, \tilde{j}_2)$ and B_2, we may determine B_1 (resp. B_2) once the current densities $(\tilde{J}_1, \tilde{j}_1)$ (resp. $(\tilde{J}_2, \tilde{j}_2)$) are given beforehand. Magnetic inductions B_1 and B_2 are further related to the force densities $(\tilde{F}_{12}, \tilde{f}_{12})$ and $(\tilde{F}_{21}, \tilde{f}_{21})$ which yield F_{12} and F_{21} by integration. The analysis of Newton's third law given in section 3.1 carries over to the present situation.

To deal with the problem of not knowing the current densities $(\tilde{J}_1, \tilde{j}_1)$ and $(\tilde{J}_2, \tilde{j}_2)$ we follow the familiar strategy and define the *total magnetic field* $(B, \tilde{H})$ by

$$B = B_1 + B_2,$$
$$\tilde{H} = \tilde{H}_1 + \tilde{H}_2.$$

The defining properties of $(B, \tilde{H})$ are implied by the defining properties of $(B_1, \tilde{H}_1)$ and $(B_2, \tilde{H}_2)$, and by the above decomposition.

An example case where the total magnetic field is useful is when o_1 and o_2 are two infinitely long parallel cylinders that carry current only on their surfaces, and the surface currents are directed along the cylinders. Thus the current densities and the magnetic field vanish inside of the objects. Such a situation may be reduced to a 2-dimensional problem that is mathematically the same as the electrostatic example of section 3.1. Accordingly, the specification of total currents through the cylinders is sufficient to determine the pair $(B, \tilde{H})$. When $(B, \tilde{H})$ is known we get the surface current densities as

$$\tilde{j}_1 = \mathrm{t}_1 \tilde{H}, \tag{3.43}$$
$$\tilde{j}_2 = \mathrm{t}_2 \tilde{H}, \tag{3.44}$$

which may further be used separately as sources to solve for the fields $(B_1, \tilde{H}_1)$ and $(B_2, \tilde{H}_2)$. Finally, we may determine the surface force densities $\tilde{f}_{12}$ and $\tilde{f}_{21}$ that yield total forces by integration.

Regarding the practical question of determining forces directly from $(B, \tilde{H})$, we note that the analysis of section 3.1 may be repeated with obvious changes. However, as in the electric case the natural way to determine forces is in terms of the force densities $\tilde{F}_{12} + \tilde{F}_{21}$ and $\tilde{f}_{12} + \tilde{f}_{21}$, or in terms of the stress $\tilde{T}_{12} + \tilde{T}_{21}$. Basis representations of the force densities and the stress are given in section A.2.1.

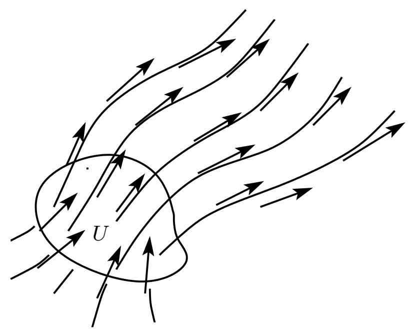
Chapter 4

Torques in terms of charges and currents

In the previous chapter we considered situations in which the displacements of the objects were described by constant virtual displacement vector fields. Here, we will allow also non-constant virtual displacement vector fields with the restriction that they should be compatible with the notion of rigidity of the objects. This means the allowable virtual displacement vector fields will be infinitesimal descriptions of transformations of space that preserve metric notions such as distances of points. Such transformations are taken here as *flows* of the corresponding vector fields.

> **Flows.** A flow on a manifold is a parametrized transformation of points defined on an open subset of the manifold. By denoting as U an open subset of manifold M, and as I a real number interval containing 0, a *flow* on M is a smooth map $\phi : I \times U \to M; (t, x) \mapsto \phi_t(x)$, with $\phi_0 : U \to U$ the identity, that satisfies the conditions
>
> (i) for each parameter $t \in I$ the map $\phi_t : U \to M$ is a diffeomorphism onto the open set $\phi_t(U)$,
>
> (ii) $\phi_{t+s}(x) = \phi_t\big(\phi_s(x)\big)$ whenever $t, s, t{+}s \in I$ and $x, \phi_s(x) \in U$.
>
> Condition (i) implies, in particular, that by using ϕ_t we may transfer trajectories on U to trajectories on $\phi_t(U)$ and vice versa. Because tangent vectors are defined by using trajectories we will be able to transfer also tangent vectors to both of the above "directions". Condition (ii) merely states that ϕ itself does not depend on the parameter, that is, the flow is "time-independent" [1, 3].

Given a smooth vector field v on M, and a point $x \in M$, we may always find a flow $\phi : I \times U \to M$, with $x \in U$, that is described infinitesimally by v. By this we mean that the vector field is tangent to the trajectories $\phi(y) : I \to M$ corresponding to each $y \in U$, or more precisely, the velocity vectors of these trajectories coincide with the values of v. This is elucidated in the inset, wherein the solid lines represent some of the trajectories $\phi(y)$.



If we have two such flows with different domains these flows will agree on the intersection of their domains [1]. We are thus entitled to talk about the *flow of a vector field* at $x \in M$.

So in case of a rigid object the flow of a virtual displacement vector field v must not change metric notions. This statement is conveniently expressed by requiring that the *Lie derivative* of the metric tensor g with respect to v vanishes, that is, $\mathcal{L}_v g = 0$.

Lie derivative. The Lie derivative describes the rate of change of a tensor when the tensor is transfered on a manifold by the flow of a vector field. When the tensor to be transfered is a function we may just use the ordinary directional derivative that compares the values of the function at neighboring points. In case of a vector we may realize the idea as follows. We assume that there is a vector field u on a manifold M, and consider its flow ϕ at $x \in M$. We further denote the point $\phi_t(x)$ as x_t. We may transfer a vector $[\tau] \in T^1_{x_t}M$ to the tangent space $T^1_x M$ by using the inverse ϕ_t^{-1}. The result is the vector $[\phi_t^{-1} \circ \tau] \in T^1_x M$. This linear map from $T^1_{x_t}M$ to $T^1_x M$ is called the push-forward by ϕ_t^{-1}, and is denoted as $(\phi_t^{-1})_*$. Let us next define the Lie derivative of vector field v with respect to u. The result will be a vector field, denoted as $\mathcal{L}_u v$. Note that for each $t \in I$ we can map the vector $v_{x_t} \in T^1_{x_t}M$ to the tangent space $T^1_x M$ by using $(\phi_t^{-1})_*$ wherein it may be compared to v_x. Let us define a map

$$c : I \to T^1_x M; \quad c(t) = (\phi_t^{-1})_*\big(v_{x_t}\big).$$

This map is a trajectory at v_x in the tangent space $T^1_x M$ whose values consist of vectors we have pushed against the flow from points of $\phi_I(x)$. The tangent vector containing this trajectory is the rate of change we are after. Thus, we define

$$(\mathcal{L}_u v)_x = \lim_{t\to 0} \frac{(\phi_t^{-1})_*(v_{x_t}) - v_x}{t}.$$

It can be shown that the Lie derivative defined above coincides with the Lie bracket, that is $\mathcal{L}_u v = [u, v]$, see [2, 3]. Thus, from the coordinate representation of the Lie bracket, we observe that the value of $\mathcal{L}_u v$ at a point depends on the derivative of u at the point (and not only on its value at the point). This is reflected to the fact that $\mathcal{L}_u v$ is not function-linear in u.

The above procedure for transferring a vector may be used also to transfer p-vectors and p-covectors along the flow of a vector field. For a simple p-vector (that is of the form $u_1 \wedge \cdots \wedge u_p$) we just use the push-forward map to the constituent vectors. This extends to arbitrary p-vectors by the linearity of the push-forward map because any p-vector may be expressed as a linear combination of simple p-vectors. To deal with p-covectors we push vectors to the opposite direction than above, that is, we push them to the direction of the flow. To see how this happens, let us consider a p-covector $\omega \in T^{x_t}_p M$. We may define a p-covector in $T^x_p M$ whose value on arbitrary p-vector in $T^p_x M$ is obtained by first pushing the p-vector to $T^p_{x_t} M$ by $(\phi_t)_*$ and then giving it to the p-covector ω. This linear map from $T^{x_t}_p M$ to $T^x_p M$ is called the pull-back by ϕ_t, and is denoted as ϕ_t^*. The above view of the Lie derivative may now be extended for an arbitrary p-form ω as follows. We first use ϕ_t^* to pull back the p-covectors ω_{x_t} corresponding to all values of t, and then take the velocity vector of the resulting curve in the p-covector space $T^x_p M$. The Lie derivative of ω at $x \in M$ is defined as

$$(\mathcal{L}_u \omega)_x = \lim_{t\to 0} \frac{\phi_t^*(\omega_{x_t}) - \omega_x}{t}.$$

The Lie derivative may be defined in the above manner for arbitrary tensors fields [1, 2]. In particular, on a Riemannian manifold we may take the Lie derivative of the metric tensor g as

$$(\mathcal{L}_u g)_x = \lim_{t \to 0} \frac{\phi_t^*(g_{x_t}) - g_x}{t},$$

where the pull-back $\phi_t^*(g_{x_t})$ is defined just as with differential forms.

The Lie derivative is a tensor derivation just like the covariant derivative. This means, in particular, that the two have the same behavior with exterior product [1]. An example of this behavior is the relation $\mathcal{L}_u(v \wedge w) = \mathcal{L}_u v \wedge w + v \wedge \mathcal{L}_u w$, where u, v, w are smooth vector fields on M. The two derivatives also have the same behavior with interior product [1]. For instance, we have for a smooth 1-form ω the relation $\mathcal{L}_u(\omega(v)) = (\mathcal{L}_u \omega)(v) + \omega(\mathcal{L}_u v)$. Note that the Lie derivative does not require metric or connection for its definition.

Vector fields that satisfy $\mathcal{L}_v g = 0$, and thus describe the infinitesimal displacements of rigid objects, are called *Killing vector fields*.

Killing vector fields. On a Riemannian manifold (M, g) continuous symmetries of g are described infinitesimally by vector fields. These vector fields satisfy the equation $\mathcal{L}_u g = 0$, and they are called *Killing vector fields*. From the definition of $\mathcal{L}_u g$ it is seen that the flow $\phi : I \times U \to M$ of u preserves the inner product, that is, for each $x \in U$ we have

$$g_{\phi_t(x)}\big((\phi_t)_* u, (\phi_t)_* v\big) = g_x(u, v)$$

for all $u, v \in T_x^1 M$. This implies that all metric notions are preserved under ϕ_t [1].

The condition of Killing vector fields may be expressed in terms of the Levi-Civita connection. It is a property of the Lie derivative that

$$\mathcal{L}_u(g(v, w)) = (\mathcal{L}_u g)(v, w) + g(\mathcal{L}_u v, w) + g(v, \mathcal{L}_u w),$$

so the condition $\mathcal{L}_u g = 0$ is equivalent to

$$\mathcal{L}_u(g(v, w)) = g(\mathcal{L}_u v, w) + g(v, \mathcal{L}_u w)$$

for all smooth vector fields v and w. The left-hand side $\mathcal{L}_u(g(v,w))$ is just the directional derivative of $g(v,w)$, whereas on the right hand side the arguments $\mathcal{L}_u v$ and $\mathcal{L}_u w$ of g are the Lie brackets $[u,v]$ and $[u,w]$. By using the metric compatibility and symmetry of the Levi-Civita connection the condition may be written as

$$g(\nabla_u v, w) + g(v, \nabla_u w) = g(\nabla_u v - \nabla_v u, w) + g(v, \nabla_u w - \nabla_w u)$$

for all smooth vector fields v and w. This yields to the condition of Killing vector field

$$g(\nabla_v u, w) + g(v, \nabla_w u) = 0$$

for all smooth vector fields v and w. It is thus observed that a constant vector field is a Killing vector field.

As already claimed in the previous chapter, constant virtual displacement vector fields do not distort distances provided that the constancy is defined with respect to the Levi-Civita connection of the used metric. To obtain other allowable vector fields we need to solve the Killing equation $\mathcal{L}_v g = 0$. We want there to be Killing vector fields that describe rotation, so we assume that both o_1 and o_2 reside on neighbourhoods that are *Euclidean manifolds*.

Euclidean manifold. A Riemannian manifold is called *Euclidean manifold* if it may be covered by Euclidean coordinate chart. A Euclidean coordinate chart is one whose basis vector fields are constant and orthonormal (constant according to the Levi-Civita connection). The metric tensor of Euclidean manifold is called *Euclidean metric*, and its Levi-Civita connection is called *Euclidean connection*. Clearly, in Euclidean coordinates the Christoffel symbols of Euclidean connection vanish, and the components of Euclidean metric form an identity matrix.

A change from one Euclidean coordinate chart to another always consists of a rotation and a translation of the coordinate axes. This is reflected to the fact that on Euclidean manifold there are both translational and rotational Killing vector fields. By using Euclidean coordinate chart with coordinate 0-forms x^1, x^2, x^3, basis vector fields $\partial/\partial x^1, \partial/\partial x^2, \partial/\partial x^3$, and dual basis 1-forms $\mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3$ we may express the Euclidean metric g by using the tensor product $\otimes$ as

$$g = \sum_{i=1}^{3} \mathrm{d}x^i \otimes \mathrm{d}x^i.$$

By further expressing the vector field v in components as $v^i\partial/\partial x^i$ we may write $\mathcal{L}_v g$ as

$$\mathcal{L}_v g = \mathcal{L}_{v^i\partial/\partial x^i}(\sum_{i=1}^{3} \mathrm{d}x^i \otimes \mathrm{d}x^i).$$

By using the linearity of $\mathcal{L}_v$ and a product property of the Lie derivative, we have

$$\begin{aligned}\mathcal{L}_v g &= \sum_{i=1}^{3} \mathcal{L}_{v^j\partial/\partial x^j}(\mathrm{d}x^i \otimes \mathrm{d}x^i)\\ &= \sum_{i=1}^{3}(\mathcal{L}_{v^j\partial/\partial x^j}\mathrm{d}x^i \otimes \mathrm{d}x^i + \mathrm{d}x^i \otimes \mathcal{L}_{v^j\partial/\partial x^j}\mathrm{d}x^i).\end{aligned}$$

The Lie derivative is not function-linear in the vector-argument, but satisfies $\mathcal{L}_{v^j\partial/\partial x^j}\mathrm{d}x^i = v^j\mathcal{L}_{\partial/\partial x^j}\mathrm{d}x^i + \mathrm{d}v^j \wedge \mathrm{i}_{\partial/\partial x^j}\mathrm{d}x^i$. Using this, and taking into account that

$$\begin{aligned}\mathcal{L}_{\partial/\partial x^j}\mathrm{d}x^i &= 0\\ \mathrm{i}_{\partial/\partial x^j}\mathrm{d}x^i &= \delta^i_j,\end{aligned}$$

for $i,j = 1,2,3$, we get

$$\mathcal{L}_v g = \sum_{i=1}^{3}(\mathrm{d}v^i \otimes \mathrm{d}x^i + \mathrm{d}x^i \otimes \mathrm{d}v^i).$$

To see that the terms $\mathcal{L}_{\partial/\partial x^j}\mathrm{d}x^i$ vanish, as claimed above, it is sufficient to verify that the values of these 1-forms on arbitrary basis vector field $\partial/\partial x^k$ vanish. By a property of the Lie derivative we have $(\mathcal{L}_{\partial/\partial x^j}\mathrm{d}x^i)(\partial/\partial x^k) = \mathcal{L}_{\partial/\partial x^j}\big(\mathrm{d}x^i(\partial/\partial x^k)\big) - \mathrm{d}x^i(\mathcal{L}_{\partial/\partial x^j}\partial/\partial x^k)$, and this vanishes because $\mathrm{d}x^i(\partial/\partial x^k)$ is the constant function δ^i_k, and because $\partial/\partial x^1, \partial/\partial x^2, \partial/\partial x^3$ is a coordinate basis so that $\mathcal{L}_{\partial/\partial x^j}\partial/\partial x^k = [\partial/\partial x^j, \partial/\partial x^k] = 0$. Finally, once the $\mathrm{d}v^i$'s are given in components as $\partial v^i/\partial x^j \mathrm{d}x^j$, we note that the components of $\mathcal{L}_v g$ are given by

$$(\mathcal{L}_v g)_{ij} = \frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i}$$

for $i,j = 1,2,3$. These components must vanish for $v = v^i\partial/\partial x^i$ to be a Killing vector field. Clearly, the matrix of $(\mathcal{L}_v g)_{ij}$'s is symmetric. There are six equations for the unknowns v^1, v^2, v^3. We see that both the translations

$$\frac{\partial}{\partial x^1}, \quad \frac{\partial}{\partial x^2}, \quad \frac{\partial}{\partial x^3}$$

and the rotations

$$x^2\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^2}, \quad x^3\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^3}, \quad x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1}$$

are solutions. Further, any solution is given as a linear combination of the translations and rotations above.

Now we are ready to generalize the concept of force used in the previous chapter. Instead of considering the force F_{12} as a covector we take it as a map that takes specific Killing vector fields to real numbers. Because in general the virtual displacement of o_2 consists of a virtual translation, and of a virtual rotation about its center of mass, the domain of this map consists of vector fields v that are linear combinations of a constant vector field and a rotational vector field with respect to the center of mass of o_2. The map will be realized by using the force densities such that the virtual work done on o_2 by o_1 is

$$F_{12}(v) = \int\limits_{o_2} \mathcal{G}(\tilde{F}_{12}, v) + \int\limits_{\partial o_2} \mathcal{G}(\tilde{f}_{12}, v). \tag{4.1}$$

The force F_{21} is generalized similarly. In case of a constant vector field this will reduce to the familiar concept of force used in the previous chapter. The integration in (4.1) does not concern the force densities as covector-valued forms. This means that those results of the previous chapters that are based on the Stokes' theorem of covector-valued forms do not generalize automatically. These results are the law of action and reaction, and the possibility to determine forces from total fields. To obtain similar results in the present case we will express a virtual rotation by a single vector. The rest of the information in the rotational virtual displacement vector field will be included in a covector that maps the virtual rotation vector to virtual work.

Let us consider the virtual rotation of object o_2 about an axis containing its center of mass so that v is a rotational vector field with respect to the center of mass of o_2. By using Euclidean coordinates, with the origin at the center of mass, we have

$$v = \theta^1(x^2\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^2}) + \theta^2(x^3\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^3}) + \theta^3(x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1}), \tag{4.2}$$

where $\theta^1, \theta^2, \theta^3$ are constants. By using the unit 3-vector field $\partial/\partial x^1 \wedge \partial/\partial x^2 \wedge \partial/\partial x^3$ to specify orientation, we have

$$\frac{\partial}{\partial x^1} = \star(\frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3}),$$
$$\frac{\partial}{\partial x^2} = \star(\frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^1}),$$
$$\frac{\partial}{\partial x^3} = \star(\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}),$$

and we observe that (4.2) may be given as

$$v = \star(\tilde{\theta} \wedge r), \tag{4.3}$$

where

$$\tilde{\theta} = \theta^i \frac{\partial}{\partial x^i}, \tag{4.4}$$
$$r = x^i \frac{\partial}{\partial x^i}. \tag{4.5}$$

When the opposite orientation is used, the sign of $\tilde{\theta}$ is changed, and this sign change is compensated by the change of signs of the basis vector fields $\partial/\partial x^1, \partial/\partial x^2, \partial/\partial x^3$ used in (4.2). Thus, the rotation may also be described by using the twisted 2-vector field $\tilde{\theta} \wedge r$ composed of the twisted angle vector field $\tilde{\theta}$ and the position vector field r. Note that $\tilde{\theta}$ is a constant vector field. Expression (4.3) suggests that we may give $\mathcal{G}(\tilde{F}_{12}, v)$, for instance, by using a *2-covector-valued differential form.*

Multivector- and multicovector-valued forms. A q-covector-valued p-form is an object whose value at a point on the manifold is a linear map from p-vectors at the point to q-covectors at the point. It is a generalization of a covector-valued p-form in the sense that the values are multicovectors of arbitrary degree. Let us denote as $L(T^p_x M; T^x_q M)$ the vector space of linear maps from the p-vector space $T^p_x M$ to the q-covector space $T^x_q M$. A *q-covector-valued p-form* is a field of such objects defined at the points of M.

The linear isomorphism between $L(T^p_x M; T^x_1 M)$ and $L(T^1_x M; T^x_p M)$ generalizes in a straightforward way to the present case. Thus, for each $\eta_x \in L(T^p_x M; T^x_q M)$ we find $\mathcal{G}(\eta_x, \cdot) \in L(T^q_x M; T^x_p M)$, such that

$$\mathcal{G}(\eta_x, v_x)(u_x) = \eta_x(u_x)(v_x)$$

for all $u_x \in T^p_x M$, and for all $v_x \in T^q_x M$. We may also identify these objects with covariant tensors of the order $q+p$ that are antisymmetric in the first q arguments and in the last p arguments.

The generalization is performed similarly for vector-valued forms. Thus the value of a *q-vector valued p-form* at $x \in M$ is an element of the vector space $L(T^p_x M; T^q_x M)$.

Using (4.3) and the *value-side Hodge operator*, we obtain

$$\begin{aligned}\mathcal{G}(\tilde{F}_{12}, v) &= \mathcal{G}\big(\tilde{F}_{12}, \star(\tilde{\theta} \wedge r)\big) \\ &= \mathcal{G}(\star\tilde{F}_{12}, \tilde{\theta} \wedge r),\end{aligned}$$

where $\star\tilde{F}_{12}$ is a 2-covector-valued twisted 3-form whose 2-covector values are also twisted. In the following I will not always say explicitly whether the values of a multicovector-valued form are twisted or not.

Value-side Hodge operator. A Hodge operator may be defined for q-covector-valued p-forms in two different ways. The definition depends on whether we want it to operate on the q-covector-values or the p-form-part of the object. Here we need the version that operates on the q-covector-values. Let us consider $\eta_x \in L(T^p_x M; T^x_q M)$ so we have $\mathcal{G}(\eta_x, \cdot) \in L(T^q_x M; T^x_p M)$. When the manifold M is n-dimensional, we define an element $\star\eta_x \in L(T^p_x M; \tilde{T}^x_{n-q} M)$ by defining its dual $\mathcal{G}(\star\eta_x, \cdot) \in L(\tilde{T}^{n-q}_x M; T^x_p M)$. This is done by specifying the operation of $\mathcal{G}(\star\eta_x, \cdot)$ on arbitrary twisted (n-q)-vector $\tilde{v}_x \in \tilde{T}^{n-q}_x M$. That is, we define

$$\mathcal{G}(\star\eta_x, \tilde{v}_x) = \mathcal{G}(\eta_x, \star\tilde{v}_x)$$

for all $\tilde{v}_x \in \tilde{T}^{n-q}_x M$. This defines an element of $L(\tilde{T}^{n-q}_x M; T^x_p M)$ because the Hodge operator for multivectors is linear. The Hodge operator $\star$ defined above maps q-covector-valued p-forms to (n-q)-covector valued p-forms with the (n-q)-covector-values twisted. It is a linear isomorphism between the spaces $L(T^p_x M; T^x_q M)$ and $L(T^p_x M; \tilde{T}^x_{n-q} M)$.

Finally, we use the *value-side interior product* to write

$$\begin{aligned}\mathcal{G}(\star\tilde{F}_{12}, \tilde{\theta}\wedge r) &= \mathcal{G}(\star\tilde{F}_{12}, -r\wedge\tilde{\theta})\\ &= \mathcal{G}(-\mathrm{i}_r\star\tilde{F}_{12}, \tilde{\theta}),\end{aligned}$$

where we have the covector-valued twisted 3-form $-\mathrm{i}_r\star\tilde{F}_{12}$ operating on the constant vector field $\tilde{\theta}$.

Value-side interior product. The interior product may be defined for multicovector-valued forms by letting it operate either on the multicovector-values or on the differential form -part of the object. For the version that performs with the multicovector-values, let us consider $\eta_x \in L(T_x^p M; T_q^x M)$ and $v_x \in T_x^1 M$. We define the element $\mathrm{i}_{v_x}\eta_x \in L(T_x^p M; T_{q-1}^x M)$ by defining its dual $\mathcal{G}(\mathrm{i}_{v_x}\eta_x, \cdot) \in L(T_x^{q-1}M; T_p^x M)$. This is done by specifying the operation of $\mathcal{G}(\mathrm{i}_{v_x}\eta_x, \cdot)$ on an arbitrary (q-1)-vector $u_x \in T_x^{q-1}M$. That is, we define

$$\mathcal{G}(\mathrm{i}_{v_x}\eta_x, u_x) = \mathcal{G}(\eta_x, v_x\wedge u_x)$$

for all $u_x \in T_x^{q-1}M$. This defines an element of $L(T_x^{q-1}M; T_p^x M)$ because the exterior product is bilinear. It follows from the definition, in particular, that the value-side interior product, as a map from $L(T_x^p M; T_q^x M)$ to $L(T_x^p M; T_{q-1}^x M)$, is linear.

Once the above steps are taken also with the surface term $\mathcal{G}(\tilde{f}_{12}, v)$, resulting in

$$\begin{aligned}\mathcal{G}(\tilde{f}_{12}, v) &= \mathcal{G}\big(\tilde{f}_{12}, \star(\tilde{\theta}\wedge r)\big)\\ &= \mathcal{G}(\star\tilde{f}_{12}, \tilde{\theta}\wedge r)\\ &= \mathcal{G}(-\mathrm{i}_r\star\tilde{f}_{12}, \tilde{\theta}),\end{aligned}$$

we may express the virtual work done on o_2 under the virtual rotation as

$$\begin{aligned}F_{12}(v) &= \int_{o_2}\mathcal{G}(\tilde{F}_{12}, v) + \int_{\partial o_2}\mathcal{G}(\tilde{f}_{12}, v)\\ &= \int_{o_2}\mathcal{G}(-\mathrm{i}_r\star\tilde{F}_{12}, \tilde{\theta}) + \int_{\partial o_2}\mathcal{G}(-\mathrm{i}_r\star\tilde{f}_{12}, \tilde{\theta})\\ &= \Big(\int_{o_2}(-\mathrm{i}_r\star\tilde{F}_{12}) + \int_{\partial o_2}(-\mathrm{i}_r\star\tilde{f}_{12})\Big)(\tilde{\theta}),\end{aligned}$$

where, on the last row, we have a twisted covector and a twisted vector at the center of mass of o_2. Thus, if we define the *torque* $\tilde{\tau}_{12}$ as the twisted covector

$$\tilde{\tau}_{12} = \int_{o_2} (-\mathrm{i}_r \star \tilde{F}_{12}) + \int_{\partial o_2} (-\mathrm{i}_r \star \tilde{f}_{12}), \tag{4.6}$$

we have

$$F_{12}(v) = \tilde{\tau}_{12}(\tilde{\theta}). \tag{4.7}$$

The quantities $-\mathrm{i}_r \star \tilde{F}_{12}$ and $-\mathrm{i}_r \star \tilde{f}_{12}$ thus have interpretation as volume and surface density of torque about the center of mass of o_2.

To obtain some familiarity with the above construction let us derive a Euclidean coordinate representation for $-\mathrm{i}_r \star \tilde{F}_{12}$. We use the coordinate basis $(\partial/\partial x^1, \partial/\partial x^2, \partial/\partial x^3)$ to specify orientation. By using the dual basis $(\mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3)$ we have

$$\begin{aligned} \mathcal{G}(-\mathrm{i}_r \star \tilde{F}_{12}, \tilde{\theta}) &= \mathcal{G}\big(\tilde{F}_{12}, \star(\tilde{\theta} \wedge r)\big) \\ &= \mathrm{d}x^i\big(\star(\tilde{\theta} \wedge r)\big)(\tilde{F}_{12})_i. \end{aligned}$$

The twisted 3-forms $(\tilde{F}_{12})_i$ may be represented by using functions F_i as $F_i \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$. We get

$$\begin{aligned} \mathcal{G}(-\mathrm{i}_r \star \tilde{F}_{12}, \tilde{\theta}) &= (F_i \mathrm{d}x^i)\big(\star(\tilde{\theta} \wedge r)\big)\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\ &= \big(-\mathrm{i}_r \star (F_i \mathrm{d}x^i)(\tilde{\theta})\big)\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3. \end{aligned}$$

By using the definition of Hodge operator, we have

$$\star(F_i \mathrm{d}x^i) = F_1 \mathrm{d}x^2 \wedge \mathrm{d}x^3 + F_2 \mathrm{d}x^3 \wedge \mathrm{d}x^1 + F_3 \mathrm{d}x^1 \wedge \mathrm{d}x^2.$$

Then, by using the antiderivation property of the interior product, and reassembling the terms, we get

$$-\mathrm{i}_r \star (F_i \mathrm{d}x^i) = (x^2 F_3 - x^3 F_2)\mathrm{d}x^1 + (x^3 F_1 - x^1 F_3)\mathrm{d}x^2 + (x^1 F_2 - x^2 F_1)\mathrm{d}x^3.$$

Finally, we may express $\mathcal{G}(-\mathrm{i}_r \star \tilde{F}_{12}, \tilde{\theta})$ as

$$\begin{aligned} \mathcal{G}(-\mathrm{i}_r \star \tilde{F}_{12}, \tilde{\theta}) = {} & \mathrm{d}x^1(\tilde{\theta})(x^2 F_3 - x^3 F_2)\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\ & + \mathrm{d}x^2(\tilde{\theta})(x^3 F_1 - x^1 F_3)\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\ & + \mathrm{d}x^3(\tilde{\theta})(x^1 F_2 - x^2 F_1)\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3. \end{aligned}$$

Thus, because the $\mathrm{d}x^i(\tilde{\theta})$'s are constant, we integrate the component 3-forms $(x^2 F_3 - x^3 F_2)\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$, etc.

4.1 Torques in electrostatics

When the sources of the interaction are taken as charges, we have $-\mathrm{i}_r \star \tilde{F}_{12}$ given from (3.6) as

$$\begin{aligned} \mathcal{G}(-\mathrm{i}_r \star \tilde{F}_{12}, \tilde{\theta}) &= \mathcal{G}\big(\tilde{F}_{12}, \star(\tilde{\theta} \wedge r)\big) \\ &= \tilde{\rho}_2 \wedge \mathrm{i}_{\star(\tilde{\theta}\wedge r)} E_1 \\ &= \tilde{\rho}_2 \wedge E_1(\star(\tilde{\theta} \wedge r)) \\ &= \tilde{\rho}_2 \wedge \mathrm{i}_r \mathrm{i}_{\tilde{\theta}} \star E_1 \end{aligned} \tag{4.8}$$

for all vector fields $\tilde{\theta}$. The surface term $-\mathrm{i}_r \star \tilde{f}_{12}$ is given similarly from (3.7), that is,

$$\begin{aligned} \mathcal{G}(-\mathrm{i}_r \star \tilde{f}_{12}, \tilde{\theta}) &= \mathcal{G}\big(\tilde{f}_{12}, \star(\tilde{\theta} \wedge r)\big) \\ &= \tilde{\sigma}_2 \wedge \mathrm{t}_2 \mathrm{i}_{\star(\tilde{\theta}\wedge r)} E_1 \\ &= \tilde{\sigma}_2 \wedge \mathrm{t}_2\big(E_1(\star(\tilde{\theta} \wedge r))\big) \\ &= \tilde{\sigma}_2 \wedge \mathrm{t}_2 \mathrm{i}_r \mathrm{i}_{\tilde{\theta}} \star E_1 \end{aligned} \tag{4.9}$$

for all vector fields $\tilde{\theta}$. The virtual work done on o_2 is thus

$$\tilde{\tau}_{12}(\tilde{\theta}) = \int\limits_{o_2} \tilde{\rho}_2 \wedge \mathrm{i}_r \mathrm{i}_{\tilde{\theta}} \star E_1 + \int\limits_{\partial o_2} \tilde{\sigma}_2 \wedge \mathrm{i}_r \mathrm{i}_{\tilde{\theta}} \star E_1, \tag{4.10}$$

where $\tilde{\theta}$ is taken as a vector on the left hand side, and as a constant vector field on the right hand side. For basis representations of $-\mathrm{i}_r \star \tilde{F}_{12}$ and $-\mathrm{i}_r \star \tilde{f}_{12}$ as defined in (4.8) and (4.9), see section B.1.1.

4.2 Torques in magnetostatics

When the sources of interaction are taken as currents, we have $-\mathrm{i}_r \star \tilde{F}_{12}$ given from (3.38) as

$$\begin{aligned} \mathcal{G}(-\mathrm{i}_r \star \tilde{F}_{12}, \tilde{\theta}) &= \mathcal{G}\big(\tilde{F}_{12}, \star(\tilde{\theta} \wedge r)\big) \\ &= \tilde{J}_2 \wedge \mathrm{i}_{\star(\tilde{\theta}\wedge r)} B_1 \end{aligned} \tag{4.11}$$

for all vector fields $\tilde{\theta}$. Similarly, the surface term $-\mathrm{i}_r \star \tilde{f}_{12}$ is given from (3.39) as

$$\begin{aligned} \mathcal{G}(-\mathrm{i}_r \star \tilde{f}_{12}, \tilde{\theta}) &= \mathcal{G}\big(\tilde{f}_{12}, \star(\tilde{\theta} \wedge r)\big) \\ &= \tilde{j}_2 \wedge \mathrm{t}_2 \mathrm{i}_{\star(\tilde{\theta}\wedge r)} B_1 \end{aligned} \tag{4.12}$$

for all vector fields $\tilde{\theta}$. The virtual work done on o_2 is

$$\tilde{\tau}_{12}(\tilde{\theta}) = \int\limits_{o_2} \tilde{J}_2 \wedge \mathrm{i}_{\star(\tilde{\theta}\wedge r)} B_1 + \int\limits_{\partial o_2} \tilde{j}_2 \wedge \mathrm{i}_{\star(\tilde{\theta}\wedge r)} B_1, \tag{4.13}$$

where $\tilde{\theta}$ is taken as a vector on the left hand side, and as a constant vector field on the right hand side. For basis representations of $-\mathrm{i}_r \star \tilde{F}_{12}$ and $-\mathrm{i}_r \star \tilde{f}_{12}$ as defined in (4.11) and (4.12), see section B.2.1.

4.3 The law of action and reaction

Having introduced the concept of torque we obtain the law of action and reaction by similar calculation as was done in section 3.1. The following applies to both electric and magnetic cases. Let us assume that the virtual rotation of each of the objects takes place about an axis containing its center of mass. We denote as v_1 and v_2 the rotational virtual displacement vector fields of objects o_1 and o_2, and as r_1 and r_2 the associated position vector fields, so that

$$v_1 = \star(\tilde{\theta} \wedge r_1),$$
$$v_2 = \star(\tilde{\theta} \wedge r_2).$$

Since $\tilde{F}_{21}$ vanishes on o_2, and since $\tilde{f}_{21}$ vanishes on ∂o_2, we may write the torque on o_2 as

$$\tilde{\tau}_{12} = \int\limits_{o_2} (-\mathrm{i}_{r_2} \star \tilde{F}_{12} - \mathrm{i}_{r_1} \star \tilde{F}_{21}) + \int\limits_{\partial o_2} (-\mathrm{i}_{r_2} \star \tilde{f}_{12} - \mathrm{i}_{r_1} \star \tilde{f}_{21}).$$

To transfer the integration above to an integration over o_1 we assume a large enough Euclidean (and thus parallelizable) neighbourhood containing both of the objects o_1 and o_2, and express the volume torque densities $\mathrm{i}_{r_2} \star \tilde{F}_{12}$ and $\mathrm{i}_{r_1} \star \tilde{F}_{21}$ in terms of covector valued twisted 2-forms by using the covariant exterior derivative. For instance, we define a covector-valued twisted 2-form $\tilde{K}_{12}$ such that

$$\mathrm{d}_\nabla \tilde{K}_{12} = -\mathrm{i}_{r_2} \star \tilde{F}_{12}, \tag{4.14}$$
$$[\mathrm{t}_2 \tilde{K}_{12}]_2 = -\mathrm{i}_{r_2} \star \tilde{f}_{12}. \tag{4.15}$$

Compare this to the definition of the stress $\tilde{T}_{12}$. A covector-valued twisted 2-form $\tilde{K}_{21}$ is defined similarly.

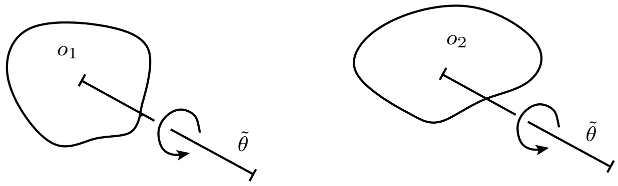


Figure 4.1: The comparison of torques on objects o_1 and o_2 is performed by using parallel twisted vectors indicating the virtual displacement in rotation angle.

Let us go through the calculation given in section 3.1. By using the familiar observation surface $\partial o_2'$ we may express the torque on o_2 as

$$\tilde{\tau}_{12} = \int\limits_{o_2} \mathrm{d}_\nabla(\tilde{K}_{12} + \tilde{K}_{21}) + \int\limits_{o_2' - o_2} \mathrm{d}_\nabla(\tilde{K}_{12} + \tilde{K}_{21}) + \int\limits_{\partial o_2} [\mathrm{t}_2(\tilde{K}_{12} + \tilde{K}_{21})]_2,$$

and by the Stokes' theorem for covector-valued forms we get

$$\tilde{\tau}_{12} = \int\limits_{\partial o_2'} (\tilde{K}_{12} + \tilde{K}_{21}).$$

Letting o_{12}' be the Euclidean neighbourhood containing o_1 and o_2, and taking the integral of $\tilde{K}_{12} + \tilde{K}_{21}$ over $\partial o_{12}'$ to be zero, we get

$$\tilde{\tau}_{12} = \int\limits_{\partial o_2'} (\tilde{K}_{12} + \tilde{K}_{21}) = - \int\limits_{\partial(o_{12}' - o_2')} (\tilde{K}_{12} + \tilde{K}_{21}) = -\tilde{\tau}_{21}, \tag{4.16}$$

where the final equality makes use of the fact that $\partial(o_{12}' - o_2')$ is a valid observation surface for o_1. By the equality in (4.16) of twisted covectors at different points we mean the equality of the numbers they yield for parallel twisted vectors at the points, see Figure 4.1.

4.4 Torques from the total fields

Torques may also be determined directly from the total fields. Let us focus on magnetostatics. From (3.38), the decomposition of B_1 as $B - B_2$ results in the decomposition

$$\tilde{F}_{12} = \tilde{F}_2 - \tilde{F}_{22},$$

where $\tilde{F}_2$ and $\tilde{F}_{22}$ are defined by

$$\mathcal{G}(\tilde{F}_2, v) = \tilde{J}_2 \wedge \mathrm{i}_v B, \tag{4.17}$$

$$\mathcal{G}(\tilde{F}_{22}, v) = \tilde{J}_2 \wedge \mathrm{i}_v B_2, \tag{4.18}$$

for all smooth vector fields v. For the decomposition of the surface term $\tilde{f}_{12}$ we use average values as we did with electric surface forces in section 3.1. This yields, from (3.39), to the decomposition of $\tilde{f}_{12}$ as

$$\tilde{f}_{12} = \tilde{f}_2 - \tilde{f}_{22},$$

where $\tilde{f}_2$ and $\tilde{f}_{22}$ are defined by

$$\mathcal{G}(\tilde{f}_2, v) = \tilde{j}_2 \wedge (\mathrm{t}_2 \mathrm{i}_v B)^{av}, \tag{4.19}$$

$$\mathcal{G}(\tilde{f}_{22}, v) = \tilde{j}_2 \wedge (\mathrm{t}_2 \mathrm{i}_v B_2)^{av}, \tag{4.20}$$

for all vector fields v. The torque on o_2 may now be written as

$$\begin{aligned} \tilde{\tau}_{12} &= \int_{o_2} (-\mathrm{i}_r \star \tilde{F}_{12}) + \int_{\partial o_2} (-\mathrm{i}_r \star \tilde{f}_{12}) \\ &= \int_{o_2} \big(-\mathrm{i}_r \star (\tilde{F}_2 - \tilde{F}_{22}) \big) + \int_{\partial o_2} \big(-\mathrm{i}_r \star (\tilde{f}_2 - \tilde{f}_{22}) \big). \end{aligned}$$

By using the linearity of the value-side Hodge operator and interior product, and reassembling the terms, we get

$$\tilde{\tau}_{12} = \int_{o_2} (-\mathrm{i}_r \star \tilde{F}_2) + \int_{\partial o_2} (-\mathrm{i}_r \star \tilde{f}_2) - \Big(\int_{o_2} (-\mathrm{i}_r \star \tilde{F}_{22}) + \int_{\partial o_2} (-\mathrm{i}_r \star \tilde{f}_{22}) \Big).$$

To see that the term in parenthesis vanishes, and thus may be called the *self-torque* of o_2, we define a covector-valued twisted 2-form $\tilde{K}_{22}$ such that

$$\mathrm{d}_\nabla \tilde{K}_{22} = -\mathrm{i}_r \star \tilde{F}_{22}, \tag{4.21}$$

$$[\mathrm{t}_2 \tilde{K}_{22}]_2 = -\mathrm{i}_r \star \tilde{f}_{22}, \tag{4.22}$$

and proceed with the integration to get

$$\int_{o_2} (-\mathrm{i}_r \star \tilde{F}_{22}) + \int_{\partial o_2} (-\mathrm{i}_r \star \tilde{f}_{22}) = \int_{\partial o_2'} \tilde{K}_{22}.$$

The right hand side may be given as

$$\int\limits_{\partial o'_2} \tilde{K}_{22} = - \int\limits_{\partial(o'_{12}-o'_2)} \tilde{K}_{22} = - \int\limits_{o'_{12}-o'_2} \mathrm{d}_\nabla \tilde{K}_{22} = - \int\limits_{o'_{12}-o'_2} (-\mathrm{i}_r \star \tilde{F}_{22}) = 0,$$

since $\tilde{F}_{22}$ vanishes outside of o_2. Here, we need again the Euclidean neighbourhood o'_{12} whose boundary does not contribute to the integration. We thus have

$$\tilde{\tau}_{12} = \int\limits_{o_2} (-\mathrm{i}_r \star \tilde{F}_2) + \int\limits_{\partial o_2} (-\mathrm{i}_r \star \tilde{f}_2), \tag{4.23}$$

so the torque on o_2 may be evaluated directly from $(B, \tilde{H})$.

Chapter 5

Forces and torques in terms of equivalent charges and equivalent currents

Here, we examine how the presense of materials affects the discussion of forces and torques given in the previous chapters.

5.1 Forces and torques in electrostatics

To model the interaction between dielectric objects a relevant question is whether the theory of previous chapters is applicable, that is, whether dielectric objects may be considered as objects with charge distributions. To examine this, let us pull together the starting points of the above theory. We have

$$\mathrm{d}E_1 = 0, \tag{5.1}$$

$$\mathrm{d}\tilde{D}_1 = \tilde{\rho}_1, \tag{5.2}$$

$$\tilde{D}_1 = \epsilon_0 \star E_1, \tag{5.3}$$

and

$$[\mathrm{t}_1 E_1]_1 = 0, \tag{5.4}$$

$$[\mathrm{t}_1 \tilde{D}_1]_1 = \tilde{\sigma}_1, \tag{5.5}$$

and finally, the force densities $\tilde{F}_{12}$ and $\tilde{f}_{12}$ are determined by

$$\mathcal{G}(\tilde{F}_{12}, v) = \tilde{\rho}_2 \wedge \mathrm{i}_v E_1,$$

$$\mathcal{G}(\tilde{f}_{12}, v) = \tilde{\sigma}_2 \wedge \mathrm{t}_2 \mathrm{i}_v E_1,$$

for all vector fields v. The rest of the starting points are obtained by reversing the roles of o_1 and o_2. Our aim here is to hold on to these starting points while including the effect of dielectric materials in $(\tilde{\rho}_1, \tilde{\sigma}_1)$ and $(\tilde{\rho}_2, \tilde{\sigma}_2)$. As in the previous section the main difficulty is the determination of $(\tilde{\rho}_1, \tilde{\sigma}_1)$ and $(\tilde{\rho}_2, \tilde{\sigma}_2)$. To proceed with this objective, we first suppose that a modeling decision has been made to separate charges into two different types called *free charges* and *polarization charges*. By this it is meant that the charge densities are decomposed as

$$\tilde{\rho}_1 = \tilde{\rho}_1^f + \tilde{\rho}_1^p, \tag{5.6}$$

$$\tilde{\sigma}_1 = \tilde{\sigma}_1^f + \tilde{\sigma}_1^p, \tag{5.7}$$

and

$$\tilde{\rho}_2 = \tilde{\rho}_2^f + \tilde{\rho}_2^p, \tag{5.8}$$

$$\tilde{\sigma}_2 = \tilde{\sigma}_2^f + \tilde{\sigma}_2^p, \tag{5.9}$$

where the superscripts refer to the two types of charges. Then, we express the polarization charge densities $(\tilde{\rho}_1^p, \tilde{\sigma}_1^p)$ and $(\tilde{\rho}_2^p, \tilde{\sigma}_2^p)$ by using *electric polarization* $\tilde{P}_1$ *of object* o_1 and *electric polarization* $\tilde{P}_2$ *of object* o_2. These are twisted 2-forms defined such that

$$-\mathrm{d}\tilde{P}_1 = \tilde{\rho}_1^p, \tag{5.10}$$

$$-[\mathrm{t}_1\tilde{P}_1]_1 = \tilde{\sigma}_1^p, \tag{5.11}$$

and

$$-\mathrm{d}\tilde{P}_2 = \tilde{\rho}_2^p, \tag{5.12}$$

$$-[\mathrm{t}_2\tilde{P}_2]_2 = \tilde{\sigma}_2^p, \tag{5.13}$$

the minus sign being traditional. We also require that $\tilde{P}_1$ and $\tilde{P}_2$ vanish outside of o_1 and o_2, respectively, implying that the total polarization charges of the objects vanish. Relations (5.6)-(5.13) together with this requirement are taken here as additional starting points of the theory. To see how they help us determine $(\tilde{\rho}_1, \tilde{\sigma}_1)$ and $(\tilde{\rho}_2, \tilde{\sigma}_2)$, let us first define $\tilde{D}_1'$ and $\tilde{D}_2'$ by

$$\tilde{D}_1' = \tilde{D}_1 + \tilde{P}_1, \tag{5.14}$$

$$\tilde{D}_2' = \tilde{D}_2 + \tilde{P}_2, \tag{5.15}$$

and then consider the following implications of our starting points:

$$\mathrm{d}E_1 = 0, \tag{5.16}$$

$$\mathrm{d}\tilde{D}_1' = \tilde{\rho}_1^f, \tag{5.17}$$

$$\tilde{D}_1' = \epsilon_0 \star E_1 + \tilde{P}_1, \tag{5.18}$$

and

$$[\mathrm{t}_1 E_1]_1 = 0, \tag{5.19}$$
$$[\mathrm{t}_1 \tilde{D}'_1]_1 = \tilde{\sigma}^f_1. \tag{5.20}$$

Note that according to the usual naming convention, $\tilde{D}'_1$ would be called the electric displacement of object o_1. Our reservation of this name for $\tilde{D}_1$ in the present theory of dielectrics is justified by the role of polarization charges in the theory. Similar implications concern the quantities related to o_2. Then, we define $\tilde{D}'$, $\tilde{P}$, $\tilde{\rho}^f$, and $\tilde{\sigma}^f$ by

$$\tilde{D}' = \tilde{D}'_1 + \tilde{D}'_2,$$
$$\tilde{P} = \tilde{P}_1 + \tilde{P}_2,$$
$$\tilde{\rho}^f = \tilde{\rho}^f_1 + \tilde{\rho}^f_2,$$
$$\tilde{\sigma}^f = \tilde{\sigma}^f_1 + \tilde{\sigma}^f_2,$$

to write down the implications we want, that is,

$$\mathrm{d}E = 0, \tag{5.21}$$
$$\mathrm{d}\tilde{D}' = \tilde{\rho}^f, \tag{5.22}$$
$$\tilde{D}' = \epsilon_0 \star E + \tilde{P}, \tag{5.23}$$

and

$$[\mathrm{t}E] = 0, \tag{5.24}$$
$$[\mathrm{t}\tilde{D}'] = \tilde{\sigma}^f. \tag{5.25}$$

Finally, we take as an additional starting point a relation between $\tilde{P}$ and E. This material dependent *constitutive law* is obtained by making the model agree with experiments, and, consequently, the building up of polarization charges is taken to be a material property.

To see that the above strategy has practical value let us consider an example case of ideal dielectrics in which the free charge densities $\tilde{\rho}^f$ and $\tilde{\sigma}^f$ vanish and the objects have known remanent polarization. We use the constitutive law

$$\tilde{P} = \chi \star E + \tilde{P}_r, \tag{5.26}$$

where χ is the electric susceptibility and $\tilde{P}_r$ is the remanent polarization given beforehand. Having determined $(E, \tilde{D}')$ from the boundary value problem

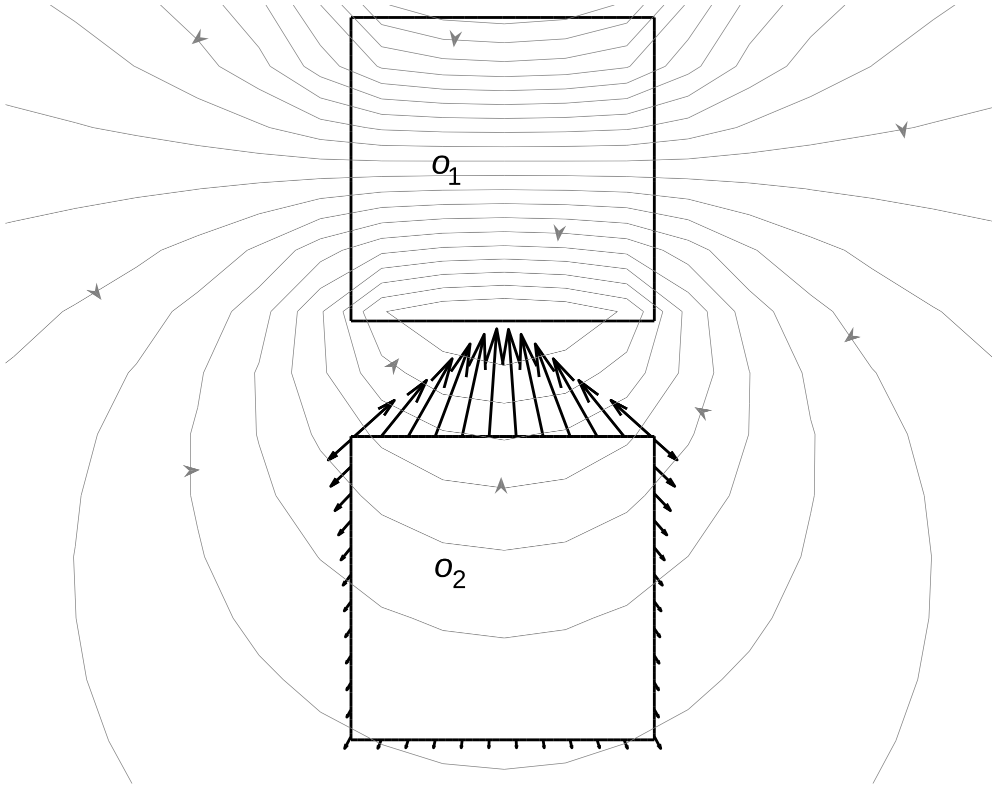


Figure 5.1: A system of dielectric objects. Object o_1 has a nonvanishing upward directed remanent polarization, and both of the objects are characterized by a constant electric susceptibility. The arrows show the vector representation of the surface force density $\tilde{f}_{12}$ on object o_2, and the gray lines with arrowheads visualize the electric field intensity E_1 of object o_1 (equivalue lines of the potential of E_1, the arrowheads indicating the direction of the field).

defined by (5.21)-(5.26), we obtain $\tilde{P}$ from (5.23) so that we know $\tilde{P}_1$ and $\tilde{P}_2$. Then, by using $\tilde{P}_1$ as the source in the boundary value problem defined by (5.16) - (5.20), we may solve for $(E_1, \tilde{D}'_1)$. Finally, by determining the charge densities $\tilde{\rho}_2$ and $\tilde{\sigma}_2$ from (5.8)-(5.9) and (5.12)-(5.13), we obtain the force densities $\tilde{F}_{12}$ and $\tilde{f}_{12}$. A similar procedure yields $\tilde{F}_{21}$ and $\tilde{f}_{21}$. Surface force density $\tilde{f}_{12}$ in a 2-dimensional example geometry is visualized in Figure 5.1. The used finite element approximation method, where fields are elementwise constant, implies that volume force densities cannot be computed directly by this strategy. This is because their computation requires the computation of the exterior derivative of $\tilde{P}$ whose approximation is elementwise constant. We will return to this problem later.

5.2 Forces and torques in magnetostatics

Based on earlier discussion, an obvious strategy for the modeling of interactions between magnetic objects is to examine whether they may be considered as objects with current distributions. Another possibility is to use distributions of *magnetic charges* to describe the magnetic objects, see [8, 15]. In the following subsections we discuss these two approaches in connection with our example situation.

5.2.1 Electric current approach

That magnetic objects are taken as objects with current distributions means we hold on to the starting points of section 3.2, and try to include the effect of magnetic materials in $(\tilde{J}_1, \tilde{j}_1)$ and $(\tilde{J}_2, \tilde{j}_2)$. By a modeling decision to decompose the current densities into *free currents* and *magnetization currents* we have

$$\tilde{J}_1 = \tilde{J}_1^f + \tilde{J}_1^m, \tag{5.27}$$
$$\tilde{j}_1 = \tilde{j}_1^f + \tilde{j}_1^m, \tag{5.28}$$

and

$$\tilde{J}_2 = \tilde{J}_2^f + \tilde{J}_2^m, \tag{5.29}$$
$$\tilde{j}_2 = \tilde{j}_2^f + \tilde{j}_2^m, \tag{5.30}$$

where the superscripts refer to the types of currents. Following familiar lines we represent the magnetization current densities by auxiliary quantities. *Magnetization* $\tilde{M}_1$ *of object* o_1 and *magnetization* $\tilde{M}_2$ *of object* o_2 are twisted 1-forms defined such that

$$\mathrm{d}\tilde{M}_1 = \tilde{J}_1^m, \tag{5.31}$$
$$[\mathrm{t}_1 \tilde{M}_1]_1 = \tilde{j}_1^m, \tag{5.32}$$

and

$$\mathrm{d}\tilde{M}_2 = \tilde{J}_2^m, \tag{5.33}$$
$$[\mathrm{t}_2 \tilde{M}_2]_2 = \tilde{j}_2^m. \tag{5.34}$$

We further require that $\tilde{M}_1$ and $\tilde{M}_2$ vanish outside of o_1 and o_2, respectively, implying that the total magnetization current vanishes through any surface whose boundary resides on the boundary of one of the objects. These quantities will be of use in the determination of the magnetization current densities $(\tilde{J}_1^m, \tilde{j}_1^m)$ and $(\tilde{J}_2^m, \tilde{j}_2^m)$. In view of this, we first set

$$\begin{aligned}
\tilde{H}'_1 &= \tilde{H}_1 - \tilde{M}_1, \\
\tilde{H}'_2 &= \tilde{H}_2 - \tilde{M}_2, \\
\tilde{H}' &= \tilde{H}'_1 + \tilde{H}'_2, \\
\tilde{M} &= \tilde{M}_1 + \tilde{M}_2, \\
\tilde{J}^f &= \tilde{J}_1^f + \tilde{J}_2^f, \\
\tilde{j}^f &= \tilde{j}_1^f + \tilde{j}_2^f,
\end{aligned}$$

and then note that our starting points imply

$$\mathrm{d}B = 0, \tag{5.35}$$

$$\mathrm{d}\tilde{H}' = \tilde{J}^f, \tag{5.36}$$

$$B = \mu_0 \star (\tilde{H}' + \tilde{M}), \tag{5.37}$$

and

$$[\mathrm{t}B] = 0, \tag{5.38}$$

$$[\mathrm{t}\tilde{H}'] = \tilde{j}^f. \tag{5.39}$$

By further taking as an additional starting point an experimental *constitutive law* between $\tilde{M}$ and B, we arrive at a model that will be of practical value. To elucidate this, let us consider an example case where free currents vanish inside of o_1 and o_2 and on their surfaces. We use the constitutive law

$$\tilde{M} = \chi_m \star B + \tilde{M}_r, \tag{5.40}$$

where $\tilde{M}_r$ is the remanent magnetization given beforehand, and χ_m is called here the magnetic susceptibility (departing in units from the traditional magnetic susceptibility). Having arrived at a similar situation as in the case of dielectric objects, we may use the familiar solution strategy. That is, we first determine $(B, \tilde{H}')$ from (5.35) - (5.40) and then obtain $\tilde{M}$ from (5.37). By knowing $\tilde{M}_1$ and $\tilde{M}_2$ we may use them separately as sources to obtain the fields $(B_1, \tilde{H}'_1)$ and $(B_2, \tilde{H}'_2)$, and finally, we may determine the magnetization current densities from (5.31) - (5.34). Then we have all that is required to find the force densities according to (3.38) and (3.39) yielding the total

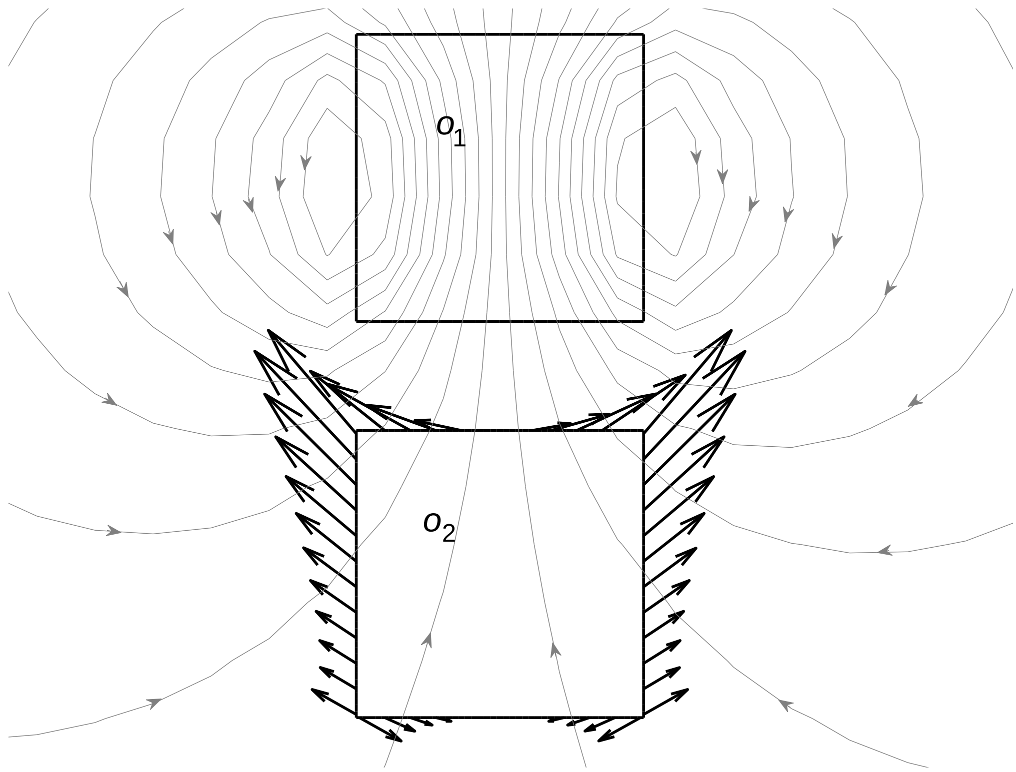


Figure 5.2: A system of magnetic objects. Object o_1 has a nonvanishing upward directed remanent magnetization, and both of the objects are characterized by a constant magnetic susceptibility. The arrows show the vector representation of the surface force density $\tilde{f}_{12}$ on object o_2, and the gray lines with arrowheads visualize the magnetic induction B_1 of object o_1 (field lines of B_1, the arrowheads indicating the direction of the field).

forces and torques by integration. Surface force density $\tilde{f}_{12}$ in the familiar example geometry is shown in Figure 5.2. As before, I will postpone considering the computational problem with volume force densities arising from the used finite element approximation method.

5.2.2 Magnetic charge approach

In the alternative approach to magnetism it is supposed that forces on magnetic materials have a similar character to the electric forces described in sections 3.1 and 5.1. This means magnetic charges are used to describe the objects' behavior. The distribution of magnetic charges inside of object o_1, for instance, is modeled by *magnetic charge density* ρ_1^m *of object* o_1. The difference to the electric charge density is that this is an ordinary (as opposed to twisted) 3-form supported in o_1. In the same way as in the electric case, the distribution of magnetic charges on the objects' surfaces is taken into account by *magnetic surface charge densities* σ_1^m *and* σ_2^m *of objects* o_1 *and* o_2. These are 2-forms supported in $\partial o_1 \cup \partial o_2$ that vanish outside of ∂o_1 and ∂o_2, respectively.

Contrary to the previous subsection, this time the theory of section (3.2) is not used as such. Instead, the defining properties (3.38) - (3.41) are accommodated to the present idea. For instance, the defining properties of $(B_1, \tilde{H}_1)$ become

$$\mathrm{d}B_1 = \rho_1^m, \tag{5.41}$$
$$\mathrm{d}\tilde{H}_1 = \tilde{J}_1, \tag{5.42}$$
$$B_1 = \mu_0 \star \tilde{H}_1, \tag{5.43}$$

and

$$[\mathrm{t}_1 B_1]_1 = \sigma_1^m, \tag{5.44}$$
$$[\mathrm{t}_1 \tilde{H}_1]_1 = \tilde{j}_1, \tag{5.45}$$

and finally, the force densities are determined by

$$\mathcal{G}(\tilde{F}_{12}, v) = \tilde{J}_2 \wedge \mathrm{i}_v B_1 + \rho_2^m \wedge \mathrm{i}_v \tilde{H}_1, \tag{5.46}$$
$$\mathcal{G}(\tilde{f}_{12}, v) = \tilde{j}_2 \wedge \mathrm{t}_2 \mathrm{i}_v B_1 + \sigma_2^m \wedge \mathrm{t}_2 \mathrm{i}_v \tilde{H}_1, \tag{5.47}$$

for all vector fields v. Similar properties concern $(B_2, \tilde{H}_2)$. For basis representations of the force densities $\tilde{F}_{12} + \tilde{F}_{21}$ and $\tilde{f}_{12} + \tilde{f}_{21}$, and the stress $\tilde{T}_{12} + \tilde{T}_{21}$, see section A.2.2. To deal with the difficulty of determining the magnetic charge densities we use exactly the same strategy that was used for determining the polarization charge densities in section 5.1. That is, we define the *magnetic polarization* M_1 *of object* o_1 and the *magnetic polarization* M_2 *of object* o_2 such that

$$-\mathrm{d}M_1 = \rho_1^m, \tag{5.48}$$
$$-[\mathrm{t}_1 M_1]_1 = \sigma_1^m, \tag{5.49}$$

and

$$-\mathrm{d}M_2 = \rho_2^m, \tag{5.50}$$
$$-[\mathrm{t}_2 M_2]_2 = \sigma_2^m, \tag{5.51}$$

and such that they vanish outside of o_1 and o_2, respectively, to yield zero total magnetic charges of the objects. Then, by setting

$$\begin{aligned} B_1' &= B_1 + M_1, \\ B_2' &= B_2 + M_2, \\ B' &= B_1' + B_2', \\ M &= M_1 + M_2, \\ \tilde{J} &= \tilde{J}_1 + \tilde{J}_2, \\ \tilde{j} &= \tilde{j}_1 + \tilde{j}_2, \end{aligned}$$

we find that the used starting points imply

$$\mathrm{d}B' = 0, \tag{5.52}$$

$$\mathrm{d}\tilde{H} = \tilde{J}, \tag{5.53}$$

$$B' = \mu_0 \star \tilde{H} + M, \tag{5.54}$$

and

$$[\mathrm{t}B'] = 0, \tag{5.55}$$

$$[\mathrm{t}\tilde{H}] = \tilde{j}. \tag{5.56}$$

Note that B' is usually called the magnetic induction, whereas in the present modeling the name is reserved for the unprimed quantities. Finally, by adding to the theory a proper *constitutive law* between M and $\tilde{H}$ yields a useful model of magnetic interactions.

To demonstrate the above model let us consider an example case in which the current densities vanish, and employ the constitutive law

$$M = \chi_m' \star \tilde{H} + M_r, \tag{5.57}$$

where M_r is the given remanent magnetic polarization, and χ_m' is called here magnetic susceptibility (departing in units from the traditional magnetic susceptiblity). The problem may be solved by using the familiar solution strategy, leading to the force densities of the same form as in the dielectric case. Surface force density $\tilde{f}_{12}$ of the example case is shown in Figure 5.3.

5.2.3 Relation between the two approaches

The two alternative approaches to magnetism yield the same integrated total forces F_{12} and F_{21} once we set

$$M_1 = \mu_0 \star \tilde{M}_1, \tag{5.58}$$

$$M_2 = \mu_0 \star \tilde{M}_2, \tag{5.59}$$

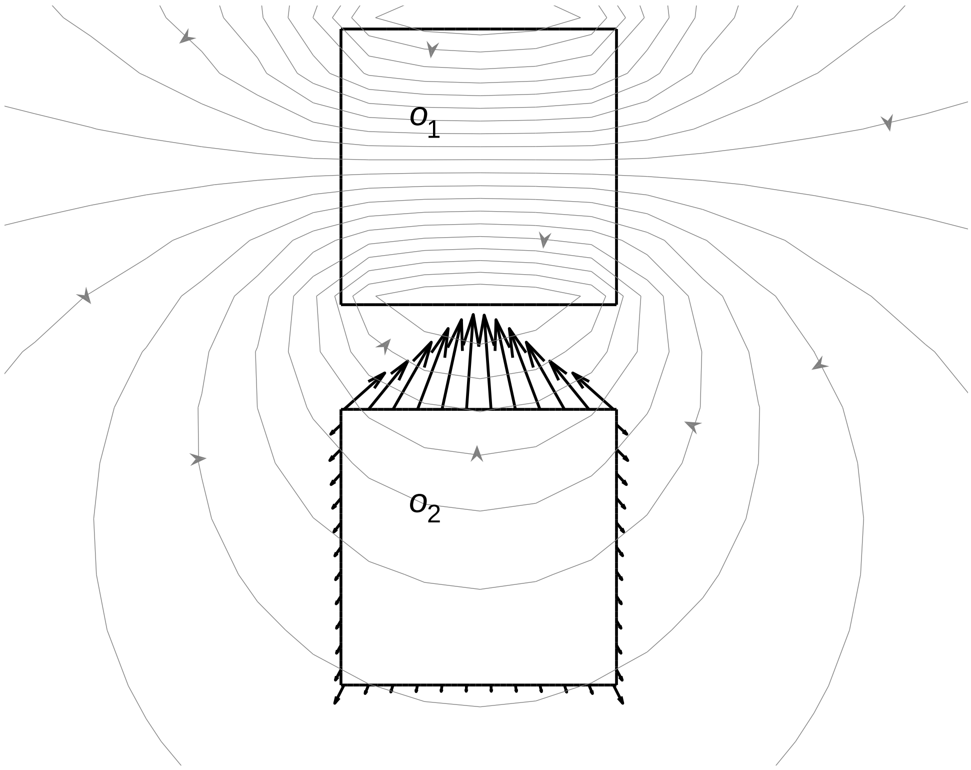


Figure 5.3: A system of magnetic objects. Object o_1 has a nonvanishing upward directed remanent magnetic polarization, and both of the objects are characterized by a constant magnetic susceptibility. The arrows show the vector representation of the surface force density $\tilde{f}_{12}$ on object o_2, and the gray lines with arrowheads visualize the magnetic field intensity $\tilde{H}_1$ of object o_1 (equivalue lines of the potential of $\tilde{H}_1$, the arrowheads indicating the direction of the field).

and further

$$\tilde{J}_1 = \tilde{J}_1{}^f, \tag{5.60}$$

$$\tilde{J}_2 = \tilde{J}_2{}^f, \tag{5.61}$$

$$\tilde{j}_1 = \tilde{j}_1{}^f, \tag{5.62}$$

$$\tilde{j}_2 = \tilde{j}_2{}^f. \tag{5.63}$$

To verify this, let us begin with the electric current model of magnetic materials, and express the virtual work $F_{12}(v)$ as

$$F_{12}(v) = \int_{o_2} \tilde{J}_2^f \wedge \mathrm{i}_v B_1 + \int_{\partial o_2} \tilde{j}_2^f \wedge \mathrm{i}_v B_1 + \int_{o_2} \mathrm{d}\tilde{M}_2 \wedge \mathrm{i}_v B_1 + \int_{\partial o_2} [\mathrm{t}_2 \tilde{M}_2]_2 \wedge \mathrm{i}_v B_1, \tag{5.64}$$

where v is a Killing vector field describing the virtual displacements of the points of o_2. Focusing on the last two terms, and using the property of exterior derivative that says $\mathrm{d}(\tilde{M}_2 \wedge \mathrm{i}_v B_1) = \mathrm{d}\tilde{M}_2 \wedge \mathrm{i}_v B_1 - \tilde{M}_2 \wedge \mathrm{d}\mathrm{i}_v B_1$, we get

$$\int_{o_2} \mathrm{d}\tilde{M}_2 \wedge \mathrm{i}_v B_1 + \int_{\partial o_2} [\mathrm{t}_2 \tilde{M}_2]_2 \wedge \mathrm{i}_v B_1 = \int_{o_2} \tilde{M}_2 \wedge \mathrm{d}\mathrm{i}_v B_1 + \int_{o_2} \mathrm{d}(\tilde{M}_2 \wedge \mathrm{i}_v B_1) + \int_{\partial o_2} [\mathrm{t}_2 \tilde{M}_2]_2 \wedge \mathrm{i}_v B_1. \tag{5.65}$$

Exterior derivative (continued). The exterior derivative is an *antiderivation*, that is, for a p-form ω and a q-form η it satisfies

$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^p \omega \wedge \mathrm{d}\eta,$$

see [1, 3].

Next, we use the familiar integration argument (in which integration is performed over o_2' and Stokes' theorem is applied) to the second term on the right hand side of (5.65), and take into account that $\tilde{M}_2$ vanishes outside of o_2. Then, because $\mathrm{i}_v B_1$ is smooth on ∂o_2, it follows that the surface term in (5.65) is canceled. We are left with

$$F_{12}(v) = \int_{o_2} \tilde{J}_2^f \wedge \mathrm{i}_v B_1 + \int_{\partial o_2} \tilde{j}_2^f \wedge \mathrm{i}_v B_1 + \int_{o_2} \tilde{M}_2 \wedge \mathrm{d}\mathrm{i}_v B_1. \tag{5.66}$$

Thus, the contribution of magnetization currents to total force may be considered as the result of partial integration applied to the rightmost term of (5.66).

On the other hand, according to the magnetic charge model the virtual work done on o_2 is

$$F_{12}(v) = \int_{o_2} \tilde{J}_2 \wedge \mathrm{i}_v B_1 + \int_{\partial o_2} \tilde{j}_2 \wedge \mathrm{i}_v B_1 - \int_{o_2} \mathrm{d}M_2 \wedge \mathrm{i}_v \tilde{H}_1 - \int_{\partial o_2} [\mathrm{t}_2 M_2]_2 \wedge \mathrm{i}_v \tilde{H}_1. \tag{5.67}$$

By the antiderivation property of exterior derivative the last two terms may be written as

$$-\int_{o_2} \mathrm{d}M_2 \wedge \mathrm{i}_v\tilde{H}_1 - \int_{\partial o_2} [\mathrm{t}_2 M_2]_2 \wedge \mathrm{i}_v\tilde{H}_1 = \int_{o_2} M_2 \wedge \mathrm{d}\mathrm{i}_v\tilde{H}_1 - \int_{o_2} \mathrm{d}(M_2 \wedge \mathrm{i}_v\tilde{H}_1) - \int_{\partial o_2} [\mathrm{t}_2 M_2]_2 \wedge \mathrm{i}_v\tilde{H}_1. \tag{5.68}$$

Finally, by using the familiar integration argument to the second term on the right hand side, and taking into account that $\mathrm{i}_v\tilde{H}_1$ is smooth on ∂o_2, we get

$$F_{12}(v) = \int_{o_2} \tilde{J}_2 \wedge \mathrm{i}_v B_1 + \int_{\partial o_2} \tilde{j}_2 \wedge \mathrm{i}_v B_1 + \int_{o_2} M_2 \wedge \mathrm{d}\mathrm{i}_v\tilde{H}_1. \tag{5.69}$$

We observe that the contribution of magnetic charges to total force may be considered as the result of partial integration applied to the rightmost term of (5.69).

To see that (5.69) coincides with (5.66) we first write the terms $\tilde{M}_2 \wedge \mathrm{d}\mathrm{i}_v B_1$ and $M_2 \wedge \mathrm{d}\mathrm{i}_v\tilde{H}_1$ by using the Lie derivative as

$$\tilde{M}_2 \wedge \mathrm{d}\mathrm{i}_v B_1 = \tilde{M}_2 \wedge (\mathcal{L}_v B_1 - \mathrm{i}_v \mathrm{d}B_1), \tag{5.70}$$

$$M_2 \wedge \mathrm{d}\mathrm{i}_v\tilde{H}_1 = M_2 \wedge (\mathcal{L}_v\tilde{H}_1 - \mathrm{i}_v \mathrm{d}\tilde{H}_1). \tag{5.71}$$

Lie derivative (continued). The Lie derivative of a differential form may be expressed by using exterior derivative and interior product. For a smooth p-form ω and a smooth vector field u we have

$$\mathcal{L}_u\omega = \mathrm{i}_u \mathrm{d}\omega + \mathrm{d}\mathrm{i}_u\omega,$$

as proved in [1, 3].

Then, by taking into account that $\mathrm{d}B_1$ and $\mathrm{d}\tilde{H}_1$ vanish outside of o_1, we have

$$\tilde{M}_2 \wedge \mathrm{d}\mathrm{i}_v B_1 = \tilde{M}_2 \wedge \mathcal{L}_v B_1, \tag{5.72}$$

$$M_2 \wedge \mathrm{d}\mathrm{i}_v\tilde{H}_1 = M_2 \wedge \mathcal{L}_v\tilde{H}_1. \tag{5.73}$$

By using (5.59) in (5.73), we get

$$\begin{aligned} M_2 \wedge \mathcal{L}_v \tilde{H}_1 &= \mu_0 \star \tilde{M}_2 \wedge \mathcal{L}_v \tilde{H}_1 \\ &= \tilde{M}_2 \wedge (\mu_0 \star \mathcal{L}_v \tilde{H}_1), \end{aligned}$$

where the second equality follows by a property of the Hodge operator because $\tilde{M}_2$ is a 2-form and $\mathcal{L}_v \tilde{H}_1$ is a 1-form. Now, because v is a Killing vector field, the Hodge operator commutes with the Lie derivative $\mathcal{L}_v$, that is

$$\tilde{M}_2 \wedge (\mu_0 \star \mathcal{L}_v \tilde{H}_1) = \tilde{M}_2 \wedge \mathcal{L}_v(\mu_0 \star \tilde{H}_1).$$

Finally, the equivalence of the electric current model and magnetic charge model with respect to total force is obtained by noticing that (5.58)-(5.63) imply that the fields $(B_1, \tilde{H}_1)$ given by the two models coincide outside of o_1 and satisfy $B_1 = \mu_0 \star \tilde{H}_1$.

Killing vector fields (continued). On a Riemannian manifold (M, g) the Lie derivative with respect to a Killing vector field commutes with the Hodge operator induced by the metric g. To verify this, we recall that the condition of Killing vector field may be given as

$$\mathcal{L}_u(g(v, w)) = g(\mathcal{L}_u v, w) + g(v, \mathcal{L}_u w)$$

for all smooth vector fields v and w. Multiplying both sides by a unit n-vector field σ, and using a property of the Lie derivative, we get

$$\mathcal{L}_u(g(v, w)\sigma) - g(v, w)\mathcal{L}_u \sigma = g(\mathcal{L}_u v, w)\sigma + g(v, \mathcal{L}_u w)\sigma$$

for all smooth vector fields v and w. Next, we take into account that $\mathcal{L}_u \sigma$ vanishes. Because much of this thesis will depend on the result that follows, let us verify this claim in detail. Let us denote as $< \cdot, \cdot >_x$ the inner product of $T_x^n M$. That σ is a unit n-vector field means we have $< \sigma, \sigma >= 1$ on the neighbourhood where σ is defined. We need to prove the second equality in

$$0 = \mathcal{L}_u < \sigma, \sigma >=< \mathcal{L}_u \sigma, \sigma > + < \sigma, \mathcal{L}_u \sigma > .$$

Because $T_x^n M$ is 1-dimensional, we have $\mathcal{L}_u \sigma = f\sigma$ for some function f, which must then be zero by the equality above. To prove the equality we express σ in terms of orthonormal basis vector fields as $\sigma = e_1 \wedge \cdots \wedge e_n$. The inner product $< \sigma, \sigma >$ is by definition the determinant $\det\big(g(e_i, e_j)\big)$, which reduces to

$$< \sigma, \sigma >= g(e_1, e_1)g(e_2, e_2) \cdots g(e_n, e_n)$$

because $e_1, \ldots, e_n$ are orthogonal. Accordingly, we have

$$\begin{aligned}\mathcal{L}_u < \sigma, \sigma >=& g(\mathcal{L}_u e_1, e_1) + g(e_1, \mathcal{L}_u e_1) + g(\mathcal{L}_u e_2, e_2) + g(e_2, \mathcal{L}_u e_2) \\ &+ \cdots + g(\mathcal{L}_u e_n, e_n) + g(e_n, \mathcal{L}_u e_n),\end{aligned}$$

where we have used the product rule of derivatives, the definition of Killing vector field, and taken into account that $e_1, \ldots, e_n$ are of unit length. But note then that $< \mathcal{L}_u \sigma, \sigma >$ may be given as

$$\begin{aligned}< \mathcal{L}_u \sigma, \sigma >=& < \mathcal{L}_u e_1 \wedge e_2 \wedge \cdots \wedge e_n, \sigma > \\ &+ < e_1 \wedge \mathcal{L}_u e_2 \wedge \cdots \wedge e_n, \sigma > \\ &\vdots \\ &+ < e_1 \wedge e_2 \wedge \cdots \wedge \mathcal{L}_u e_n, \sigma > .\end{aligned}$$

This makes use of a product property of the Lie derivative and the bilinearity of the inner product $< \cdot, \cdot >$. By further expressing σ as $e_1 \wedge \cdots \wedge e_n$ this yields

$$< \mathcal{L}_u \sigma, \sigma >= g(\mathcal{L}_u e_1, e_1) + g(\mathcal{L}_u e_2, e_2) + \cdots + g(\mathcal{L}_u e_n, e_n),$$

and the claim follows by the symmetry of $< \cdot, \cdot >$ and g. Thus we find that the defining condition for Killing vector field is equivalent to

$$\mathcal{L}_u(g(v, w)\sigma) = g(\mathcal{L}_u v, w)\sigma + g(v, \mathcal{L}_u w)\sigma$$

for all smooth vector fields v and w. Applying the definition of the Hodge operator yields

$$\mathcal{L}_u(\star v \wedge w) = (\star \mathcal{L}_u v) \wedge w + \star v \wedge \mathcal{L}_u w$$

for all smooth vector fields v and w. Finally, by a property of the Lie derivative, this is equivalent to

$$(\mathcal{L}_u \star v) \wedge w = (\star \mathcal{L}_u v) \wedge w$$

for all smooth vector fields v and w. This means that when operating on vector fields the Lie derivative with respect to a

Killing vector field commutes with the Hodge operator. This commutation result holds also for general multivector fields and differential forms. To derive it for 1-forms we use the definition of Hodge operator and a property of the Lie derivative to write

$$(\star\mathcal{L}_u\omega)(\sigma) = (\mathcal{L}_u \star \omega)(\sigma) + \omega(\star\mathcal{L}_u\sigma - \mathcal{L}_u \star \sigma)$$

for all smooth 1-forms ω and (n-1)-vector fields σ. Thus, the result for 1-forms follows from that for vector fields. To deal with arbitrary p-forms one can use mathematical induction, see [5].

5.3 Conclusion

Forces and torques may be determined by using equivalent charges or equivalent currents. In particular, the electric current and magnetic charge models of magnetic materials give the same predictions for net forces and torques. The result is not obvious because locally the force densities of the two models are different. From a strictly mathematical point of view the situation with dielectric objects is similar, as instead of polarization charges one may evoke magnetic currents to describe the objects' behavior.

Chapter 6

Forces and torques in terms of polarization and magnetization

So far we have used equivalent currents and equivalent charges to accommodate materials. For magnetic materials we have equivalent currents and equivalent magnetic charges as alternative models. Because the two stand as models for the same phenomenon (the interaction of rigid material body with magnetic field) we wanted them to yield the same observable quantities, that is, the same forces and torques. This was achieved essentially by using a metric by which the objects are seen as rigid to relate $\tilde{M}_2$ and M_2 by $M_2 = \mu_0 \star \tilde{M}_2$ (and similarly for $\tilde{M}_1$ and M_1). When proving the equivalence of the two models with respect to total force, we observed that in the two models the virtual work done on o_2 by o_1 may be given (in the absence of free currents) by integrating either $\tilde{M}_2 \wedge \mathcal{L}_v B_1$ or $M_2 \wedge \mathcal{L}_v \tilde{H}_1$ over o_2, and that these 3-forms coincide at each point of o_2. In the present chapter I will introduce starting points that regard these 3-forms as densities of virtual work. Before laying down the starting points in full I will motivate the expressions $\tilde{M}_2 \wedge \mathcal{L}_v B_1$ and $M_2 \wedge \mathcal{L}_v \tilde{H}_1$ by considering basic microscopic models for magnetic materials. In the case of electrostatics the density of virtual work is $\tilde{P}_2 \wedge \mathcal{L}_v E_1$, which is analogous to $M_2 \wedge \mathcal{L}_v \tilde{H}_1$.

6.1 Heuristic derivation of the density of virtual work from microscopic material models

In classical electromagnetism the basic microscopic models for magnetic materials are virtual current loops (Ampèrian dipoles) and virtually displaced pairs of oppositely charged magnetic monopoles (Coulombian dipoles). Here, we consider these two models separately. The quantities used in this section are independent of those used elsewhere in this thesis. If a quantity of this section is to be identified with a quantity used elsewhere it will be notified explicitly.

6.1.1 Ampèrian dipoles

To determine the virtual work done on a virtual current loop in magnetic field we begin with a rigid loop ∂S carrying current I and bounding a surface S (the specification of the surface turns out to be irrelevant). In the following we will use the Levi-Civita connection of the metric by which the loop is seen as rigid. We first consider the virtual work done on the loop under a virtual displacement given by a constant vector field v defined on a parallellizable neighbourhood containing S. By denoting as B_{ext} the magnetic induction field caused by external sources (sources other than the current loop itself) the virtual work done on the loop is given as $I \int_{\partial S} \mathrm{i}_v B_{ext}$. The virtual work $F(v)$ done on a virtual loop is obtained by taking the limit of a sequence of loops shrinking towards a point, that is

$$F(v) = \lim_{\substack{I\to\infty \\ S\to 0}} I \int_{\partial S} \mathrm{i}_v B_{ext}, \quad IS = \text{constant}.$$

By using Stokes' theorem the integration above may be transformed to an integration over S. We have

$$F(v) = \lim_{\substack{I\to\infty \\ S\to 0}} I \int_{S} \mathrm{d}\mathrm{i}_v B_{ext}, \quad IS = \text{constant}.$$

The integrand may be expressed by using the Lie derivative as $\mathrm{d}\mathrm{i}_v B_{ext} = \mathcal{L}_v B_{ext} - \mathrm{i}_v \mathrm{d}B_{ext}$. Then, since $\mathrm{d}B_{ext}$ vanishes, we get

$$F(v) = \lim_{\substack{I\to\infty \\ S\to 0}} I \int_{S} \mathcal{L}_v B_{ext}, \quad IS = \text{constant}.$$

The integration above may be performed by using triangulations as

$$\begin{aligned}\int_S \mathcal{L}_v B_{ext} &= \lim_{k\to\infty} \sum_{i=1}^{k} (\mathcal{L}_v B_{ext})_i(\{s_i\}) \\ &= (\mathcal{L}_v B_{ext})_x(\lim_{k\to\infty} \sum_{i=1}^{k} \{s_i\}),\end{aligned}$$

where the 2-vectors $\{s_i\}$ to be added are first transported to the point $x \in S$ from their original base points, while keeping them constant by using the Levi-Civita connection. The existence of a point $x \in S$ such that the last equality above holds follows from the mean value theorem for multiple integrals. We will denote the sum of the 2-vectors $\{s_i\}$ as $\{S\}$. It can be shown that $\{S\}$ depends only on the boundary ∂S, see [6] (pp. 83-84). In the above limiting process we in fact let $\{S\}$ approach zero such that $I\{S\}$ is constant. We thus have

$$F(v) = (\mathcal{L}_v B_{ext})_x(\mathrm{m}),$$

where $\mathrm{m} = I\{S\}$ is the *magnetic dipole moment 2-vector*. By using this formula for virtual work we may define a covector F_x, whose value on vector v_x is given by

$$F_x(v_x) = (\mathcal{L}_v B_{ext})_x(\mathrm{m}) \tag{6.1}$$

where v on the right-hand side is the constant vector field that coincides with v_x at x. This is the *force* on Ampèrian dipole at x (see [16]).

Ampèrian dipole cannot be taken as a point particle (although we have just attempted to do so by describing it as 2-vector at a point). This is because to evaluate the virtual work (6.1) we need a vector field defined in a neighbourhood of the dipole and not just a vector at the point of the dipole. Another reason is that besides the translations described above also the rotation of the dipole contributes to virtual work. To include the contribution that comes from rotation we begin with the current loop ∂S as above, but now take the virtual displacement vector field v to be rotational vector field with respect to the center of mass of the loop defined on a Euclidean neighbourhood containing the loop. So we have $v = \star(\tilde{\theta} \wedge r)$, where r is a position vector field with respect to the center of mass of the loop. Also now the virtual work done on the loop is given as $I \int_{\partial S} \mathrm{i}_v B_{ext}$. Note that for a tangent vector u_x of ∂S we have

$$\begin{aligned}
(\mathrm{i}_v B_{ext})_x(u_x) &= -\mathrm{i}_{u_x}(B_{ext})_x(v_x) \\
&= \mathrm{i}_{u_x}(B_{ext})_x\big(\star(r_x \wedge \tilde{\theta}_x)\big) \\
&= \star \mathrm{i}_{u_x}(B_{ext})_x(r_x \wedge \tilde{\theta}_x) \\
&= \mathrm{i}_{r_x} \star \mathrm{i}_{u_x}(B_{ext})_x(\tilde{\theta}_x)
\end{aligned}$$

which gives the familiar expression $\mathrm{i}_{r_x} \star \mathrm{i}_{Iu_x}(B_{ext})_x$ for the torque on current element Iu_x. By considering a sequence of loops shrinking towards a point we have for the virtual work done on a virtual current loop

$$F(v) = (\mathcal{L}_v B_{ext})_x(\mathrm{m}),$$

by just following the steps above. At this point, the peculiarity of the Lie derivative becomes evident: although the vector field $v = \star(\tilde{\theta} \wedge r)$ vanishes at the point x of the dipole where $r = 0$, the Lie derivative $\mathcal{L}_v B_{ext}$ does not vanish at x. Thus, there will be a nonzero contribution to the virtual work that comes from rotation. To find out this contribution we extract from $\mathcal{L}_v B_{ext}$ the part that has tensorial dependence on the derivatives of v (the *derivatives* of v do *not* vanish at the point of the dipole). First note that we may give an arbitrary 2-vector field w as $\star u$ where u is a vector field. We have

$$\begin{aligned}
(\mathcal{L}_v B_{ext})(w) &= (\mathcal{L}_v B_{ext})(\star u) \\
&= (\star \mathcal{L}_v B_{ext})(u) \\
&= (\mathcal{L}_v \star B_{ext})(u) \\
&= \mathcal{L}_v(\star B_{ext}(u)) - (\star B_{ext})(\mathcal{L}_v u) \\
&= \mathcal{L}_v(\star B_{ext}(u)) - (\star B_{ext})([v, u]) \\
&= \nabla_v(\star B_{ext}(u)) - (\star B_{ext})(\nabla_v u - \nabla_u v) \\
&= (\nabla_v \star B_{ext})(u) + (\star B_{ext})(\nabla_u v) \\
&= (\star \nabla_v B_{ext})(u) + (\star B_{ext})(\nabla_u v),
\end{aligned}$$

where the last equality follows because ∇ is metric compatible.

Levi-Civita connection (continued). On a Riemannian manifold (M, g) the covariant derivative induced by the Levi-Civita connection commutes with the Hodge operator induced by the metric g. The situation may be contrasted with the Lie derivative with respect to Killing vector fields. The difference is that this time the metric information is included in the connection – not in the vector field appearing in the direction argument. Consequently, the vector field in the direction argument is arbitrary.

The commutation result follows from the metric compatibility of the Levi-Civita connection by exactly the same arguments as in the case of the Lie derivative with respect to Killing vector fields. This is because the metric compatibility means that

$$\nabla_u(g(v, w)) = g(\nabla_u v, w) + g(v, \nabla_u w)$$

for all smooth vector fields u, v and w. Contrast this to the Lie derivative of $g(v, w)$ with respect to Killing vector field u. Since the covariant derivative and the Lie derivative have similar behavior with respect to interior and exterior products we may follow familiar steps to arrive at the commutation result. (The result is independent from the symmetry property of the Levi-Civita connection, and thus it holds for all connections that are metric compatible.)

The term $(\star B_{ext})(\nabla_u v)$ on the right hand side contains the dependence of $(\mathcal{L}_v B_{ext})(w)$ on the derivatives of v. Now, in the case of a rotational vector field $v = \star(\tilde{\theta} \wedge r)$, we may express this dependence as

$$\begin{aligned}(\star B_{ext})(\nabla_u v) &= (\star B_{ext})\big(\nabla_u \star (\tilde{\theta} \wedge r)\big)\\ &= (\star B_{ext})\big(\star \nabla_u(\tilde{\theta} \wedge r)\big)\\ &= (B_{ext})\big(\nabla_u(\tilde{\theta} \wedge r)\big)\\ &= B_{ext}(\nabla_u \tilde{\theta} \wedge r + \tilde{\theta} \wedge \nabla_u r)\\ &= B_{ext}(\tilde{\theta} \wedge \nabla_u r),\end{aligned}$$

where in the third equality we have taken into account that $\star\star$ is the identity map on 2-vector fields. The last equality uses the fact that $\tilde{\theta}$ is constant. Note now that ∇r is the identity on smooth vector fields, that is, in local Euclidean coordinates x^1, x^2, x^3 with the origin at the point of the dipole, we have

$$\nabla_u r = \nabla_{u^i \frac{\partial}{\partial x^i}} \Big(x^j \frac{\partial}{\partial x^j}\Big) = u^i \nabla_{\frac{\partial}{\partial x^i}} \Big(x^j \frac{\partial}{\partial x^j}\Big) = u^i \mathrm{d}x^j (\frac{\partial}{\partial x^i}) \frac{\partial}{\partial x^j} = u^j \frac{\partial}{\partial x^j} = u.$$

We get

$$\begin{aligned}B_{ext}(\tilde{\theta} \wedge \nabla_u r) &= B_{ext}(\tilde{\theta} \wedge u)\\ &= \mathrm{i}_{\tilde{\theta}} B_{ext}(u)\\ &= \star \mathrm{i}_{\tilde{\theta}} B_{ext}(w).\end{aligned}$$

Summing up, we have the decomposition

$$\mathcal{L}_{\star(\tilde{\theta}\wedge r)} B_{ext} = \nabla_{\star(\tilde{\theta}\wedge r)} B_{ext} + \star \mathrm{i}_{\tilde{\theta}} B_{ext}. \tag{6.2}$$

At the point x of the dipole we have $r = 0$ so that

$$(\mathcal{L}_{\star(\tilde{\theta}\wedge r)} B_{ext})_x = (\star \mathrm{i}_{\tilde{\theta}} B_{ext})_x.$$

We thus have for the virtual work

$$F(v) = (\star \mathrm{i}_{\tilde{\theta}} B_{ext})_x(\mathrm{m}).$$

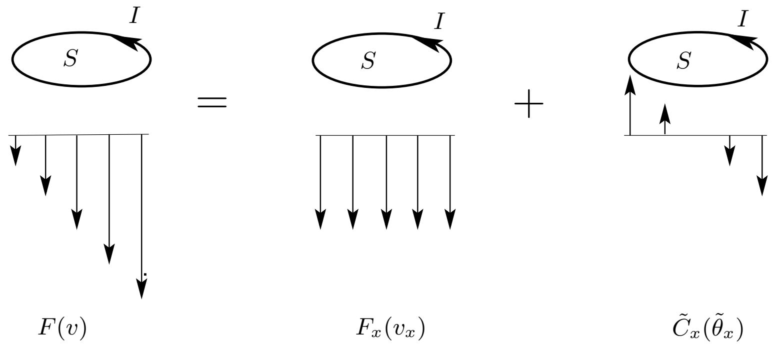


Figure 6.1: Decomposition of virtual work done on an Ampèrian dipole by external magnetic field under rigid virtual displacement.

We may define a twisted covector $\tilde{C}_x$ whose value on vector $\tilde{\theta}_x$ is defined by

$$\tilde{C}_x(\tilde{\theta}_x) = \star \mathrm{i}_{\tilde{\theta}_x}(B_{ext})_x(\mathrm{m}). \tag{6.3}$$

This is called the *couple* (torque) on an Ampèrian dipole at x.

Let us return at this point to the force expression (6.1). Because of the Lie derivative this expression requires a vector field v on the neighbourhood of x. But since v is constant we may remedy this excess requirement by using the covariant derivative, that is, we have

$$F_x(v_x) = (\mathcal{L}_v B_{ext})_x(\mathrm{m}) = (\nabla_{v_x} B_{ext})(\mathrm{m}). \tag{6.4}$$

From this expression it is clearly visible that we only need the vector v_x to evaluate the virtual work. The metric information on the neighbourhood of x that was before in the constant vector field v is now included in ∇.

Summing up, the virtual work done on an Ampèrian dipole in an external magnetic field under a rigid virtual displacement is composed of force term related to virtual translation and a couple term related to virtual rotation. This is elucidated in Figure 6.1.

The above results may be used as an aid to suggest expressions for force and couple densities in magnetic materials. We will follow the standard approach, see [15]. To describe the density of magnetic dipole moments, that is, the density of 2-vectors, we introduce *magnetization* $\tilde{m}$ as a 2-vector valued twisted 3-form. We try to construct force and couple densities by giving the 2-vector values of $\tilde{m}$ to the 2-forms $\nabla_v B_{ext}$ and $\star \mathrm{i}_{\tilde{\theta}} B_{ext}$ in accordance with the above dipole formulas. For this, we note that the 2-forms $\nabla_v B_{ext}$ and $\star \mathrm{i}_{\tilde{\theta}} B_{ext}$ may also be taken as a 2-covector-valued 0-forms. By using this point of view, we introduce *force density* $\tilde{F}$ and *couple density* $\tilde{C}$ by using the *generalized* *exterior product* $\dot{\wedge}$ such that

$$\mathcal{G}(\tilde{F}, v) = \tilde{m}\dot{\wedge}\nabla_v B_{ext} \tag{6.5}$$

$$\mathcal{G}(\tilde{C}, \tilde{\theta}) = \tilde{m}\dot{\wedge} \star \mathrm{i}_{\tilde{\theta}} B_{ext} \tag{6.6}$$

for all vector fields v and $\tilde{\theta}$. Both force and couple densities are thus taken as covector-valued twisted 3-forms (with the difference that the covector values of the couple density are twisted).

Generalized exterior product. The exterior product of vector-valued p-form and covector-valued q-form defined earlier may be generalized for r-vector valued p-form and r-covector-valued q-form in a straightforward way. That is, we just use the pairing of r-vector and r-covector values in the exterior product of real valued objects to get a real-valued (p+q)-form. For instance, in the case of a r-vector-valued 1-form ν and a r-covector-valued 2-form η their exterior product is the (real valued) 3-form defined by

$$\begin{aligned}\nu\dot{\wedge}\eta(u_1 \wedge u_2 \wedge u_3) =& \eta(u_2 \wedge u_3)\big(\nu(u_1)\big) + \eta(u_3 \wedge u_1)\big(\nu(u_2)\big) \\ &+ \eta(u_1 \wedge u_2)\big(\nu(u_3)\big)\end{aligned}$$

for all vector fields u_1, u_2, u_3. In the case of r-vector-valued 3-form ν and r-covector-valued 0-form η we have for an arbitrary 3-vector field σ

$$\begin{aligned}\nu\dot{\wedge}\eta(\sigma) &= \nu\dot{\wedge}\eta\big(\sigma \wedge 1\big) \\ &= \eta(1)\big(\nu(\sigma)\big) \\ &= \eta\big(\nu(\sigma)\big),\end{aligned}$$

where 1 is taken as a 0-vector field whose value at each point is $1 \in \mathbb{R}$, and in the last row η is identified with the r-form $\eta(1)$.

To clarify the above, let us select local basis 1-forms $\omega_1, \omega_2, \omega_3$ and denote as $\tilde{V}$ the (locally defined) twisted 3-form represented by $\pm\omega_1 \wedge \omega_2 \wedge \omega_3$. If $\omega_1, \omega_2, \omega_3$ were dual to orthonormal basis vector fields, $\tilde{V}$ would measure volumes according to the used metric. By using $\tilde{V}$ we may express the magnetic dipole moments in an arbitrary virtual volume at point x as

$$\tilde{m}_x(\tilde{\sigma_x}) = m_x \tilde{V}_x(\tilde{\sigma}_x),$$

where $\tilde{\sigma}_x$ is the virtual volume 3-vector, and m_x is a 2-vector to be identified with the magnetic dipole moment in expressions (6.3) and (6.4). The virtual work done on the virtual volume $\tilde{\sigma}_x$ by forces is thus

$$\begin{aligned}\mathcal{G}(\tilde{F}, v)_x(\tilde{\sigma}_x) &= (\tilde{m}\dot{\wedge}\nabla_v B_{ext})_x(\tilde{\sigma_x}) \\ &= (\nabla_v B_{ext})_x\big(\tilde{m}_x(\tilde{\sigma_x})\big) \\ &= (\nabla_v B_{ext})_x(m_x)\tilde{V}_x(\tilde{\sigma}_x).\end{aligned}$$

Similarly, the virtual work done on $\tilde{\sigma}_x$ by couples is

$$\mathcal{G}(\tilde{C}, \tilde{\theta})_x(\tilde{\sigma}_x) = (\star \mathrm{i}_{\tilde{\theta}} B_{ext})_x(m_x)\tilde{V}_x(\tilde{\sigma}_x).$$

We have thus constructed force and couple densities $\tilde{F}$ and $\tilde{C}$ from the 2-vector valued twisted 3-form $\tilde{m}$ and the 2-forms $\nabla_v B_{ext}$ and $\star \mathrm{i}_{\tilde{\theta}} B_{ext}$. Now, it should be possible to introduce a (twisted) 1-form that contains the same information as $\tilde{m}$, and whose exterior product with $\nabla_v B_{ext}$ and $\star \mathrm{i}_{\tilde{\theta}} B_{ext}$ yield $\mathcal{G}(\tilde{F}, v)$ and $\mathcal{G}(\tilde{C}, \tilde{\theta})$, respectively. For this, we introduce a twisted 1-form $\tilde{M}$ by contracting $\tilde{m}$, that is, we set pointwise

$$\tilde{M}_x(\tilde{u}) = \tilde{V}_x(m_x \wedge \tilde{u}) \tag{6.7}$$

for all (twisted) vectors $\tilde{u}$. We will identify this 1-form with the magnetization of previous chapters, so we use the same symbol here. To see that $\tilde{M}$ has the desired performance we first look for the operation of $(\tilde{M} \wedge \nabla_v B_{ext})_x$ on an arbitrary (nonzero) 3-vector $u_1 \wedge u_2 \wedge u_3$. Note first that we may give m_x as $m^1 u_2 \wedge u_3 + m^2 u_3 \wedge u_1 + m^3 u_1 \wedge u_2$ because u_1, u_2, u_3 form a set of basis vectors at x. We have

$$\begin{aligned}(\tilde{M} \wedge \nabla_v B_{ext})_x(u_1 \wedge u_2 \wedge u_3) &= \tilde{M}_x(u_1)(\nabla_v B_{ext})_x(u_2 \wedge u_3) \\ &\quad + \tilde{M}_x(u_2)(\nabla_v B_{ext})_x(u_3 \wedge u_1) \\ &\quad + \tilde{M}_x(u_3)(\nabla_v B_{ext})_x(u_1 \wedge u_2) \\ &= (\nabla_v B_{ext})_x(\tilde{M}_x(u_1)u_2 \wedge u_3) \\ &\quad + (\nabla_v B_{ext})_x(\tilde{M}_x(u_2)u_3 \wedge u_1) \\ &\quad + (\nabla_v B_{ext})_x(\tilde{M}_x(u_3)u_1 \wedge u_2) \\ &= (\nabla_v B_{ext})_x(m^1\tilde{V}_x(u_1 \wedge u_2 \wedge u_3)u_2 \wedge u_3) \\ &\quad + (\nabla_v B_{ext})_x(m^2\tilde{V}_x(u_1 \wedge u_2 \wedge u_3)u_3 \wedge u_1) \\ &\quad + (\nabla_v B_{ext})_x(m^3\tilde{V}_x(u_1 \wedge u_2 \wedge u_3)u_1 \wedge u_2) \\ &= \big((\nabla_v B_{ext})_x(m^1 u_2 \wedge u_3) + (\nabla_v B_{ext})_x(m^2 u_3 \wedge u_1) \\ &\quad + (\nabla_v B_{ext})_x(m^3 u_1 \wedge u_2)\big)\tilde{V}_x(u_1 \wedge u_2 \wedge u_3) \\ &= (\nabla_v B_{ext})_x(m_x)\tilde{V}_x(u_1 \wedge u_2 \wedge u_3) \\ &= (\tilde{m}\dot{\wedge}\nabla_v B_{ext})_x(u_1 \wedge u_2 \wedge u_3).\end{aligned}$$

In the same way, we have

$$(\tilde{M} \wedge \star \mathrm{i}_{\tilde{\theta}} B_{ext})_x(u_1 \wedge u_2 \wedge u_3) = (\tilde{m} \dot{\wedge} \star \mathrm{i}_{\tilde{\theta}} B_{ext})_x(u_1 \wedge u_2 \wedge u_3).$$

Thus, when $\tilde{M}$ is obtained by contracting $\tilde{m}$, we have

$$\mathcal{G}(\tilde{F}, v) = \tilde{m} \dot{\wedge} \nabla_v B_{ext} = \tilde{M} \wedge \nabla_v B_{ext}, \tag{6.8}$$

$$\mathcal{G}(\tilde{C}, \tilde{\theta}) = \tilde{m} \dot{\wedge} \star \mathrm{i}_{\tilde{\theta}} B_{ext} = \tilde{M} \wedge \star \mathrm{i}_{\tilde{\theta}} B_{ext}. \tag{6.9}$$

Finally, when v is a linear combination of a constant vector field v_{const} and a rotational vector field $\star(\tilde{\theta} \wedge r)$, we have

$$\tilde{M} \wedge \nabla_{v_{const}} B_{ext} + \tilde{M} \wedge \nabla_{\star(\tilde{\theta} \wedge r)} B_{ext} + \tilde{M} \wedge \star \mathrm{i}_{\tilde{\theta}} B_{ext} = \tilde{M} \wedge \mathcal{L}_v B_{ext}$$

as may be verified by using similar calculation as that preceding (6.2).

Before concluding this subsection we note that the expressions (6.8) and (6.9) do not yet have meaning since we have not specified what quantity in our mesoscopic model should be used in place of B_{ext}. Recently Bobbio approached this question by modeling the magnetic material by finite number of dipoles, and using average values to arrive at a formula for force on a small (but not infinitesimal) volume, see [15]. (Actually Bobbio deals mainly with the analogous situation of forces in dielectrics.) To arrive at his result, we first assume that in a small (parallelizable) volume U containing point x there are n dipoles with dipole moment 2-vectors $\{\mathrm{m}_i\}_{i=1}^n$. (The base points of these 2-vectors will also be indexed by $i = 1, \ldots, n$.) We describe the virtual displacements of the dipoles in U by the point values v_i of a constant vector field v. To determine the force on the assembly of dipoles in U we note that the assembly causes no net force to itself. (This follows from the superposition principle and Newton's law of action and reaction.) It follows that the virtual work done on the assembly may be given as

$$\sum_{i=1}^{n} (\nabla_{v_i} B_{ext})(\mathrm{m}_i),$$

where B_{ext} is the magnetic induction field caused by sources outside of U. To get mesoscopic quantities we define 2-vector field m and 2-form $\hat{B}_{ext}$ pointwise by

$$m_x = \lim_{U \to 0} \Big(\int_U \tilde{V} \Big)^{-1} \sum_{i=1}^{n} \mathrm{m}_i$$

and

$$(\hat{B}_{ext})_x = \frac{1}{n}\sum_{i=1}^{n}(B_{ext})_i,$$

where $\tilde{V}$ now measures volumes according to the used metric. Here the addition of 2-vectors and 2-covectors at different points of U is performed by using the Levi-Civita connection ∇. According to Bobbio's arguments this leads to the result that the virtual work done on U is given as

$$(\nabla_{v_x}\hat{B}_{ext})(m_x)\int_U \tilde{V},$$

and, again according to Bobbio's arguments, the limit of this when $U \to 0$ is not uniquely defined but depends on the shape of the limiting volume U. More precisely, the limit value of $\hat{B}_{ext}$ depends on the shape of the limiting volume U. The shape dependence arises when one subtracts from the total magnetic field the contribution caused by sources inside U. From this shape dependence it follows that the modeling of the distribution of magnetic forces inside materials by a density function is in contradiction with the basic dipole formula $(\nabla_{v_x}B_{ext})(\mathrm{m})$, and thus with the basic model of magnetic materials. See the discussion of Smith-White and Cade [17, 18, 19, 20]. Note that this situation only becomes a problem when attempting to define force densities that fit for determining deformations – in the case of a rigid body we may take B_{ext} to be the field caused by sources outside of the rigid body resulting in a concept of force density suitable for total force and torque calculation.

6.1.2 Coulombian dipoles

When the model of magnetic materials is based on virtually displaced pairs of oppositely charged magnetic monopoles, we will find very similar results to those above . Let us consider a pair of magnetic monopoles with magnetic charges $\pm\tilde{q}_m$ sitting at the boundary points of a path C. We take this system to be rigid, and describe its displacement by a constant vector field v. By letting $\tilde{H}_{ext}$ be the external magnetic field intensity the virtual work done on this system is $\tilde{q}_m \int_{\partial C} \mathrm{i}_v \tilde{H}_{ext}$. The virtual work done on a virtually displaced pair of magnetic charges is obtained by taking the limit of a sequence of paths shrinking towards a point, that is

$$F(v) = \lim_{\substack{\tilde{q}_m\to\infty \\ C\to 0}} \tilde{q}_m \int_{\partial C} \mathrm{i}_v \tilde{H}_{ext}, \quad \tilde{q}_m C = \text{constant}.$$

Next, we use Stokes' theorem to transfer the above integration to an integration over C, and express $\mathrm{di}_v\tilde{H}_{ext}$ by using the Lie derivative as $\mathrm{di}_v\tilde{H}_{ext} = \mathcal{L}_v\tilde{H}_{ext} - \mathrm{i}_v\mathrm{d}\tilde{H}_{ext}$. Because $\mathrm{d}\tilde{H}_{ext}$ vanishes on C, we get

$$F(v) = \lim_{\substack{\tilde{q}_m\to\infty \\ C\to 0}} \tilde{q}_m \int_C \mathcal{L}_v\tilde{H}_{ext}, \quad \tilde{q}_m C = \text{constant}.$$

By taking the limit we get

$$F(v) = (\mathcal{L}_v\tilde{H}_{ext})_x(\tilde{\mathrm{m}}),$$

where $\tilde{\mathrm{m}} = \tilde{q}_m\{C\}$ is the *magnetic dipole moment (twisted) vector* sitting at point x. We define a covector F_x such that its operation on a vector v_x is given as

$$F_x(v_x) = (\mathcal{L}_v\tilde{H}_{ext})_x(\tilde{\mathrm{m}}), \tag{6.10}$$

where v on the right hand side is the constant vector field, whose value at x is v_x. This is the force on a Coulombian dipole at x. By using the Levi-Civita connection we have

$$F_x(v_x) = (\nabla_{v_x}\tilde{H}_{ext})(\tilde{\mathrm{m}}). \tag{6.11}$$

Note that by setting $\tilde{\mathrm{m}} = \mu_0 \star \mathrm{m}$, where m is the magnetic dipole moment 2-vector of the previous subsection, we have

$$\begin{aligned}(\nabla_{v_x}\tilde{H}_{ext})(\tilde{\mathrm{m}}) &= (\nabla_{v_x}\tilde{H}_{ext})(\mu_0 \star \mathrm{m}) \\ &= (\mu_0 \star \nabla_{v_x}\tilde{H}_{ext})(\mathrm{m}) \\ &= \big(\nabla_{v_x}(\mu_0 \star \tilde{H}_{ext})\big)(\mathrm{m}) \\ &= (\nabla_{v_x}B_{ext})(\mathrm{m}).\end{aligned}$$

Thus, because the field $(B_{ext}, \tilde{H}_{ext})$ of this subsection coincides at x with the field $(B_{ext}, \tilde{H}_{ext})$ of the previous subsection, the two types of dipoles experience the same force F_x.

To include the contribution to virtual work that comes from rotation we take v to be a rotational vector field with respect the center of mass of the charge pair, so we have $v = \star(\tilde{\theta} \wedge r)$. The virtual work done on the system of magnetic charges is given as $\tilde{q}_m \int_{\partial C} \mathrm{i}_v\tilde{H}_{ext}$. Note that now we have for a point $x \in \partial C$

$$\begin{aligned}(\mathrm{i}_v\tilde{H}_{ext})_x &= \mathrm{i}_{v_x}(\tilde{H}_{ext})_x \\ &= (\tilde{H}_{ext})_x\big(\star(\tilde{\theta}_x \wedge r_x)\big) \\ &= -\mathrm{i}_{r_x} \star (\tilde{H}_{ext})_x(\tilde{\theta}_x),\end{aligned}$$

which gives the familiar expression $-\mathrm{i}_{r_x}\star\tilde{q}_m(\tilde{H}_{ext})_x$ for torque on the magnetic charge $\tilde{q}_m$ at x. Next, by considering a shrinking sequence of paths as above, we get

$$F(v) = (\mathcal{L}_v\tilde{H}_{ext})_x(\tilde{\mathrm{m}}).$$

Extracting from $\mathcal{L}_v\tilde{H}_{ext}$ the part that depends on the derivatives of v, we get

$$\mathcal{L}_{\star(\tilde{\theta}\wedge r)}\tilde{H}_{ext} = \nabla_{\star(\tilde{\theta}\wedge r)}\tilde{H}_{ext} + \mathrm{i}_{\tilde{\theta}} \star \tilde{H}_{ext}, \tag{6.12}$$

which may be compared to (6.2). Since $r = 0$ at the point of the dipole, we get

$$F(v) = (\mathrm{i}_{\tilde{\theta}} \star \tilde{H}_{ext})_x(\tilde{\mathrm{m}}).$$

We define a twisted covector $\tilde{C}_x$ such that its operation on vector $\tilde{\theta}_x$ is given as

$$\tilde{C}_x(\tilde{\theta}_x) = \mathrm{i}_{\tilde{\theta}_x} \star (\tilde{H}_{ext})_x(\tilde{\mathrm{m}}). \tag{6.13}$$

This is the couple on Coulombian dipole at x. Note that by setting $\tilde{\mathrm{m}} = \mu_0\star\mathrm{m}$ as above, we have

$$\begin{aligned}\mathrm{i}_{\tilde{\theta}_x} \star (\tilde{H}_{ext})_x(\tilde{\mathrm{m}}) &= \mathrm{i}_{\tilde{\theta}_x} \star (\tilde{H}_{ext})_x(\mu_0 \star \mathrm{m}) \\ &= \mu_0 \star \mathrm{i}_{\tilde{\theta}_x} \star (\tilde{H}_{ext})_x(\mathrm{m}) \\ &= \star\mathrm{i}_{\tilde{\theta}_x}(\mu_0 \star \tilde{H}_{ext})_x(\mathrm{m}) \\ &= \star\mathrm{i}_{\tilde{\theta}_x}(B_{ext})_x(\mathrm{m}),\end{aligned}$$

so we may conclude that Ampèrian and Coulombian dipoles also experience the same couple $\tilde{C}_x$.

Expressions for force and couple densities in magnetic materials are deduced in the same way as in the previous subsection. This involves the description of the density of magnetic dipole moments by using a vector-valued twisted 3-form (whose vector values are twisted). By contracting this object to *magnetic polarization* 2-form M one ends up in defining the force density $\tilde{F}$ and couple density $\tilde{C}$ by

$$\mathcal{G}(\tilde{F}, v) = M \wedge \nabla_v\tilde{H}_{ext} \tag{6.14}$$

$$\mathcal{G}(\tilde{C}, \tilde{\theta}) = M \wedge \mathrm{i}_{\tilde{\theta}} \star \tilde{H}_{ext} \tag{6.15}$$

for all vector fields v and $\tilde{\theta}$. The 2-form M is to be identified with the magnetic polarization of previous chapters. When the virtual displacement vector field v is a linear combination of a constant vector field v_{const} and a rotational vector field $\star(\tilde{\theta} \wedge r)$ one has

$$M \wedge \nabla_{v_{const}} \tilde{H}_{ext} + M \wedge \nabla_{\star(\tilde{\theta}\wedge r)} \tilde{H}_{ext} + M \wedge \mathrm{i}_{\tilde{\theta}} \star \tilde{H}_{ext} = M \wedge \mathcal{L}_v \tilde{H}_{ext}. \quad (6.16)$$

The problem with the quantity $\tilde{H}_{ext}$ to be used in (6.14) and (6.15) is the same as before, and it concerns the modeling of deformable bodies only.

6.2 Forces and torques in electrostatics

Now we are ready to introduce a model for the behavior of rigid dielectric objects that is based on the idea of distributed polarity in dielectric materials. First, we take the free charge densities and polarizations of the objects o_1 and o_2 as primary quantities. Then, the defining properties of $(E_1, \tilde{D}'_1)$ are given as

$$\mathrm{d}E_1 = 0, \quad (6.17)$$
$$\mathrm{d}\tilde{D}'_1 = \tilde{\rho}^f_1, \quad (6.18)$$
$$\tilde{D}'_1 = \epsilon_0 \star E_1 + \tilde{P}_1, \quad (6.19)$$

and

$$[\mathrm{t}_1 E_1]_1 = 0, \quad (6.20)$$
$$[\mathrm{t}_1 \tilde{D}'_1]_1 = \tilde{\sigma}^f_1. \quad (6.21)$$

In our previous treatment of dielectric objects these relations were taken as implications of (5.1)-(5.5) when (5.6)-(5.15) are understood. The implication also goes to the opposite direction so that (6.17)-(6.21) are equivalent to (5.1)-(5.5). Similar defining properties concern $(E_2, \tilde{D}'_2)$. In the present analysis we call $\tilde{D}'_1$ (resp. $\tilde{D}'_2$) the electric displacement of object o_1 (resp. object o_2).

Based on the discussion of the previous section, by using the analogy between the electric charge and magnetic charge models, we know that besides force density one also needs couple density to model the behavior of dielectric objects. Like force density also couple density is a covector valued twisted 3-form. However, because couple density is related to torques on material elements, its covector values are twisted (as notified in the previous section). For the force density $\tilde{F}_{12}$ we set

$$\mathcal{G}(\tilde{F}_{12}, v) = \tilde{\rho}^f_2 \wedge \mathrm{i}_v E_1 + \tilde{P}_2 \wedge \nabla_v E_1 \quad (6.22)$$

for all smooth vector fields v. For the *couple density* $\tilde{C}_{12}$ we set

$$\mathcal{G}(\tilde{C}_{12}, \tilde{\theta}) = \tilde{P}_2 \wedge \mathrm{i}_{\tilde{\theta}} \star E_1 \tag{6.23}$$

for all vector fields $\tilde{\theta}$. For the surface force density there is only a contribution from free surface charges, so we set

$$\mathcal{G}(\tilde{f}_{12}, v) = \tilde{\sigma}_2^f \wedge \mathrm{t}_2 \mathrm{i}_v E_1 \tag{6.24}$$

for all smooth vector fields v. Similar expressions are used for the force densities $\tilde{F}_{21}$ and $\tilde{f}_{21}$, and for the couple density $\tilde{C}_{21}$. For the basis representations of force densities $\tilde{F}_{12} + \tilde{F}_{21}$ and $\tilde{f}_{12} + \tilde{f}_{21}$, and the stress $\tilde{T}_{12} + \tilde{T}_{21}$, see section A.1.2. Basis representation of the couple density $\tilde{C}_{12}$ is given in section B.1.2.

The generalized force F_{12} is realized in terms of force and couple densities as

$$F_{12}(v) = \int_{o_2} \big(\mathcal{G}(\tilde{F}_{12}, v) + \mathcal{G}(\tilde{C}_{12}, \tilde{\theta})\big) + \int_{\partial o_2} \mathcal{G}(\tilde{f}_{12}, v), \tag{6.25}$$

where v is a linear combination of a constant vector field and a rotational vector field $\star(\tilde{\theta} \wedge r)$ about the center of mass of o_2. In the case of a constant vector field v we have $\tilde{\theta} = 0$, resulting in the total force

$$F_{12} = \int_{o_2} \tilde{F}_{12} + \int_{\partial o_2} \tilde{f}_{12},$$

and when $v = \star(\tilde{\theta} \wedge r)$ we get the torque

$$\tilde{\tau}_{12} = \int_{o_2} (-\mathrm{i}_r \star \tilde{F}_{12} + \tilde{C}_{12}) + \int_{\partial o_2} (-\mathrm{i}_r \star \tilde{f}_{12}).$$

Here the quantity $-\mathrm{i}_r \star \tilde{F}_{12}$ is determined according to (6.22), such that

$$\begin{aligned} \mathcal{G}(-\mathrm{i}_r \star \tilde{F}_{12}, \tilde{\theta}) &= \mathcal{G}\big(\tilde{F}_{12}, \star(\tilde{\theta} \wedge r)\big) \\ &= \tilde{\rho}_2^f \wedge \mathrm{i}_r \mathrm{i}_{\tilde{\theta}} \star E_1 + \tilde{P}_2 \wedge \nabla_{\star(\tilde{\theta} \wedge r)} E_1 \end{aligned} \tag{6.26}$$

for all vector fields $\tilde{\theta}$. The surface term $-\mathrm{i}_r \star \tilde{f}_{12}$ is determined according to (6.24), such that

$$\begin{aligned} \mathcal{G}(-\mathrm{i}_r \star \tilde{f}_{12}, \tilde{\theta}) &= \mathcal{G}\big(\tilde{f}_{12}, \star(\tilde{\theta} \wedge r)\big) \\ &= \tilde{\sigma}_2^f \wedge \mathrm{t}_2 \mathrm{i}_r \mathrm{i}_{\tilde{\theta}} \star E_1 \end{aligned} \tag{6.27}$$

for all vector fields $\tilde{\theta}$. For basis representations of $-\mathrm{i}_r \star \tilde{F}_{12}$ and $-\mathrm{i}_r \star \tilde{f}_{12}$, see section B.1.2.

Let us next verify that the present model of dielectric objects yields the same forces and torques as the model based on equivalent charges. For this, we first note that the volume integrand in (6.25) may be given by using the Lie derivative as

$$\begin{aligned}\mathcal{G}(\tilde{F}_{12}, v) + \mathcal{G}(\tilde{C}_{12}, \tilde{\theta}) &= \tilde{\rho}_2^f \wedge \mathrm{i}_v E_1 + \tilde{P}_2 \wedge \nabla_v E_1 + \tilde{P}_2 \wedge \mathrm{i}_{\tilde{\theta}} \star E_1 \\ &= \tilde{\rho}_2^f \wedge \mathrm{i}_v E_1 + \tilde{P}_2 \wedge \mathcal{L}_v E_1\end{aligned}$$

as shown in section 6.1. We may thus apply partial integration as in subsection 5.2.3 to get

$$\begin{aligned}F_{12}(v) &= \int_{o_2} (\tilde{\rho}_2^f \wedge \mathrm{i}_v E_1 + \tilde{P}_2 \wedge \mathcal{L}_v E_1) + \int_{\partial o_2} \tilde{\sigma}_2^f \wedge \mathrm{i}_v E_1 \\ &= \int_{o_2} (\tilde{\rho}_2^f \wedge \mathrm{i}_v E_1 - \mathrm{d}\tilde{P}_2 \wedge \mathrm{i}_v E_1) + \int_{\partial o_2} (\tilde{\sigma}_2^f \wedge \mathrm{i}_v E_1 - [\mathrm{t}_2 \tilde{P}_2]_2 \wedge \mathrm{i}_v E_1) \\ &= \int_{o_2} \tilde{\rho}_2 \wedge \mathrm{i}_v E_1 + \int_{\partial o_2} \tilde{\sigma}_2 \wedge \mathrm{i}_v E_1,\end{aligned}$$

which is the virtual work according to the electric charge model of dielectric objects.

As in our previous treatment of dielectric objects we will employ the total field approach to determine the unknown source quantities. To obtain the defining properties for the total field $(E, \tilde{D}')$ we first observe that the above starting points imply properties (5.21)-(5.25), and then we add to the starting points the constitutive law (5.26). In our example case, where free charge densities vanish, we are thus led to the familiar solution strategy. That is, by first determining the pair $(E, \tilde{D}')$ from the boundary value problem defined by (5.21)-(5.26) we obtain $\tilde{P}$ from (5.23) so that we know $\tilde{P}_1$ and $\tilde{P}_2$. Then, by using $\tilde{P}_1$ as the source in the boundary value problem defined by (6.17)-(6.21), we obtain the pair $(E_1, \tilde{D}'_1)$. The pair $(E_2, \tilde{D}'_2)$ is obtained similarly. At this stage we have all that is needed to determine the force and couple densities which yield forces and torques by integration. However, our earlier problem remains as this strategy cannot be used directly to compute the volume force densities from the finite element approximation of fields. This is because the force density expression (6.22) includes the derivative of E_1 whose approximation is elementwise constant. A method to circumvent this problem is given in appendix D. The volume force density $\tilde{F}_{12}$ of our example case is visualized in Figure 6.2.

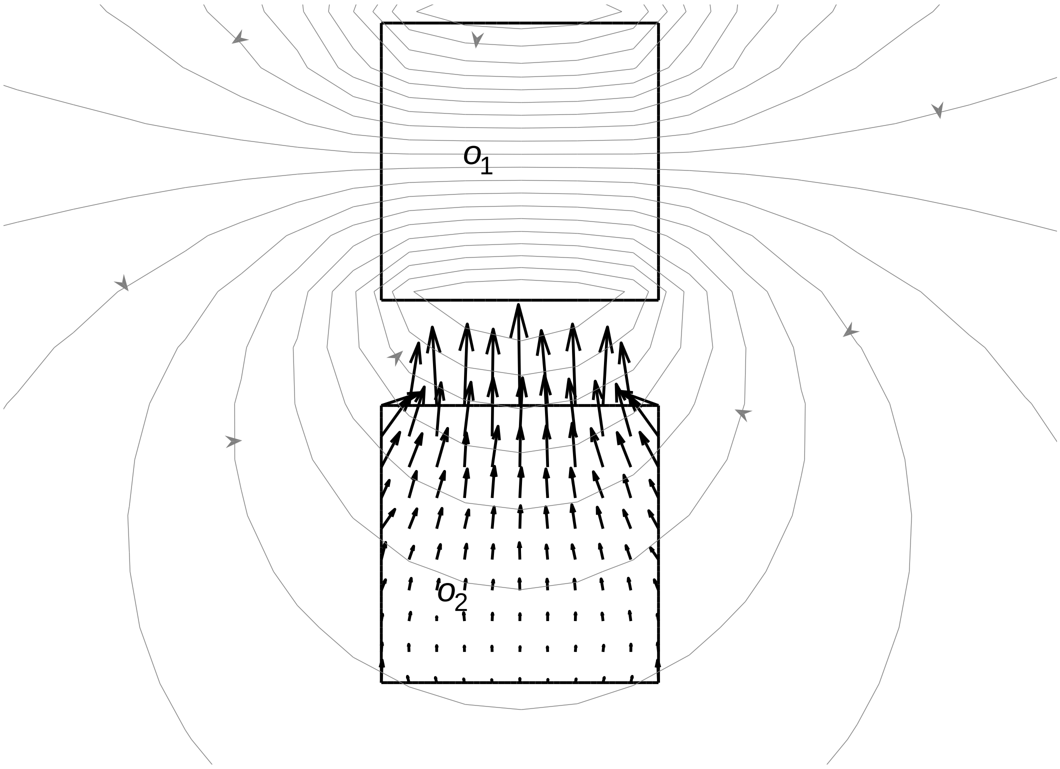


Figure 6.2: The example system of dielectric objects. The arrows show the vector representation of the volume force density $\tilde{F}_{12}$ on object o_2, and the gray lines with arrowheads visualize the electric field intensity E_1 of object o_1 (equivalue lines of the potential of E_1, the arrowheads indicating the direction of the field).

6.2.1 Forces and torques from the total fields

The issue of determining forces and torques directly from the total fields is somewhat different from our previous treatment of dielectric objects. One reason for this is that now we need to include the couple density term. But there is also another reason, and this is what easily causes confusion. The other reason is that now there will be a surface contribution to the self-force and self-torque of the dielectric object o_2 even when the surface force density $\tilde{f}_{12}$ vanishes. This means (6.24) does not yield the correct decomposition of $\tilde{f}_{12}$ as $\tilde{f}_2 - \tilde{f}_{22}$. The situation may be clarified by looking for integrands involving total fields that yield the correct generalized force F_{12}. By the electric charge model of dielectric materials we have

$$F_{12}(v) = \int\limits_{o_2} \tilde{\rho}_2^f \wedge \mathrm{i}_v E + \int\limits_{\partial o_2} \tilde{\sigma}_2^f \wedge (\mathrm{t}_2 \mathrm{i}_v E)^{av} - \int\limits_{o_2} \mathrm{d}\tilde{P}_2 \wedge \mathrm{i}_v E$$
$$- \int\limits_{\partial o_2} [\mathrm{t}_2 \tilde{P}_2]_2 \wedge (\mathrm{t}_2 \mathrm{i}_v E)^{av}$$

because the self-force and self-torque of o_2 vanish according to our previous analysis. Taking under consideration the last two terms of this, and using $\mathrm{d}(\tilde{P}_2 \wedge \mathrm{i}_v E) = \mathrm{d}\tilde{P}_2 \wedge \mathrm{i}_v E + \tilde{P}_2 \wedge \mathrm{d}\mathrm{i}_v E$, we get

$$- \int\limits_{o_2} \mathrm{d}\tilde{P}_2 \wedge \mathrm{i}_v E - \int\limits_{\partial o_2} [\mathrm{t}_2 \tilde{P}_2]_2 \wedge (\mathrm{t}_2 \mathrm{i}_v E)^{av} = \int\limits_{o_2} \tilde{P}_2 \wedge \mathrm{d}\mathrm{i}_v E - \int\limits_{o_2} \mathrm{d}(\tilde{P}_2 \wedge \mathrm{i}_v E)$$
$$- \int\limits_{\partial o_2} [\mathrm{t}_2 \tilde{P}_2]_2 \wedge (\mathrm{t}_2 \mathrm{i}_v E)^{av}.$$

Now, contrary to the situation with the field E_1, the second volume integral on the right hand side does not cancel the surface term because E is not continuous at the points of ∂o_2. Once the left-over surface term is figured out, we get

$$F_{12}(v) = \int\limits_{o_2} \tilde{\rho}_2^f \wedge \mathrm{i}_v E + \int\limits_{\partial o_2} \tilde{\sigma}_2^f \wedge (\mathrm{t}_2 \mathrm{i}_v E)^{av} + \int\limits_{o_2} \tilde{P}_2 \wedge \mathrm{d}\mathrm{i}_v E$$
$$+ \int\limits_{\partial o_2} (\mathrm{t}_2 \tilde{P}_2)^{av} \wedge [\mathrm{t}_2 \mathrm{i}_v E]_2.$$

By further using $\mathrm{d}\mathrm{i}_v E = \mathcal{L}_v E - \mathrm{i}_v \mathrm{d}E$, and taking into account that $\mathcal{L}_v E = \nabla_v E + \mathrm{i}_{\tilde{\theta}} \star E$, we get

$$F_{12}(v) = \int\limits_{o_2} (\tilde{\rho}_2^f \wedge \mathrm{i}_v E + \tilde{P}_2 \wedge \nabla_v E - \tilde{P}_2 \wedge \mathrm{i}_v \mathrm{d}E + \tilde{P}_2 \wedge \mathrm{i}_{\tilde{\theta}} \star E)$$
$$+ \int\limits_{\partial o_2} \big(\tilde{\sigma}_2^f \wedge (\mathrm{t}_2 \mathrm{i}_v E)^{av} + (\mathrm{t}_2 \tilde{P}_2)^{av} \wedge [\mathrm{t}_2 \mathrm{i}_v E]_2\big), \qquad (6.28)$$

where we have kept the zero term $\tilde{P}_2 \wedge \mathrm{i}_v \mathrm{d}E$ to maintain an analogy with the magnetic case. This is the expression for the virtual work done on o_2 in terms of the total field as given by the electric polarization approach. We have a similar expression for the (generalized) self-force of o_2 – the only difference being that in place of E we have E_2. Thus both (6.28) and this self-force will contain a nonzero surface term even when $\tilde{\sigma}_2^f$ vanishes, and these surface terms will cancel each other to yield zero surface force density $\tilde{f}_{12}$ in accordance with (6.24).

For further clarification we repeat some of the analysis of section (3.1). First, we have the decomposition of $\tilde{F}_{12}$ as

$$\tilde{F}_{12} = \tilde{F}_2 - \tilde{F}_{22},$$

where $\tilde{F}_2$ and $\tilde{F}_{22}$ are defined by

$$\mathcal{G}(\tilde{F}_2, v) = \tilde{\rho}_2^f \wedge \mathrm{i}_v E + \tilde{P}_2 \wedge \nabla_v E - \tilde{P}_2 \wedge \mathrm{i}_v \mathrm{d}E, \tag{6.29}$$

$$\mathcal{G}(\tilde{F}_{22}, v) = \tilde{\rho}_2^f \wedge \mathrm{i}_v E_2 + \tilde{P}_2 \wedge \nabla_v E_2 - \tilde{P}_2 \wedge \mathrm{i}_v \mathrm{d}E_2, \tag{6.30}$$

for all vector fields v. Here, the derivatives are restricted to points where E and E_2 are smooth. The decomposition of $\tilde{C}_{12}$ is

$$\tilde{C}_{12} = \tilde{C}_2 - \tilde{C}_{22},$$

where $\tilde{C}_2$ and $\tilde{C}_{22}$ are defined by

$$\mathcal{G}(\tilde{C}_2, \tilde{\theta}) = \tilde{P}_2 \wedge \mathrm{i}_{\tilde{\theta}} \star E, \tag{6.31}$$

$$\mathcal{G}(\tilde{C}_{22}, \tilde{\theta}) = \tilde{P}_2 \wedge \mathrm{i}_{\tilde{\theta}} \star E_2, \tag{6.32}$$

for all vector fields $\tilde{\theta}$. The decomposition of $\tilde{f}_{12}$ needs special care. By the above discussion we have

$$\tilde{f}_{12} = \tilde{f}_2 - \tilde{f}_{22},$$

where $\tilde{f}_2$ and $\tilde{f}_{22}$ are defined by

$$\mathcal{G}(\tilde{f}_2, v) = \tilde{\sigma}_2^f \wedge (\mathrm{t}_2 \mathrm{i}_v E)^{av} + (\mathrm{t}_2 \tilde{P}_2)^{av} \wedge [\mathrm{t}_2 \mathrm{i}_v E]_2, \tag{6.33}$$

$$\mathcal{G}(\tilde{f}_{22}, v) = \tilde{\sigma}_2^f \wedge (\mathrm{t}_2 \mathrm{i}_v E_2)^{av} + (\mathrm{t}_2 \tilde{P}_2)^{av} \wedge [\mathrm{t}_2 \mathrm{i}_v E_2]_2, \tag{6.34}$$

for all vector fields v. The terms $[\mathrm{t}_2 \mathrm{i}_v E]_2$ and $[\mathrm{t}_2 \mathrm{i}_v E_2]_2$ should be understood in the same way as $(\mathrm{t}_2 \mathrm{i}_v E)^{av}$ and $(\mathrm{t}_2 \mathrm{i}_v E_2)^{av}$ so that they require the values of v only at points of ∂o_2 (and not on the two sides wherein E and E_2 are evaluated). Now the force (covector) on o_2 may be written as

$$\begin{aligned} F_{12} &= \int_{o_2} \tilde{F}_{12} + \int_{\partial o_2} \tilde{f}_{12} \\ &= \int_{o_2} (\tilde{F}_2 - \tilde{F}_{22}) + \int_{\partial o_2} (\tilde{f}_2 - \tilde{f}_{22}) \\ &= \int_{o_2} \tilde{F}_2 + \int_{\partial o_2} \tilde{f}_2 - \Big(\int_{o_2} \tilde{F}_{22} + \int_{\partial o_2} \tilde{f}_{22} \Big). \end{aligned}$$

By defining $\tilde{T}_{22}$ in the familiar way, such that

$$\mathrm{d}_\nabla \tilde{T}_{22} = \tilde{F}_{22},$$
$$[\mathrm{t}_2 \tilde{T}_{22}]_2 = \tilde{f}_{22},$$

and using the familiar observation surface $\partial o_2'$, we get for the term in parenthesis

$$\int_{o_2} \tilde{F}_{22} + \int_{\partial o_2} \tilde{f}_{22} = \int_{o_2} \mathrm{d}_\nabla \tilde{T}_{22} + \int_{o_2' - o_2} \mathrm{d}_\nabla \tilde{T}_{22} + \int_{\partial o_2} [\mathrm{t}_2 \tilde{T}_{22}]_2 = \int_{\partial o_2'} \tilde{T}_{22} = 0,$$

where the final equality follows in the same way as before because $\tilde{F}_{22}$ vanishes outside of o_2'. Note now the necessity of the surface term $\tilde{f}_{22}$ as defined in (6.34). Without this term the discontinuity of $\tilde{T}_{22}$ at points of ∂o_2 would not cancel as it does in the second equality above. Thus, when the correct surface term is included, we have

$$F_{12} = \int_{o_2} \tilde{F}_2 + \int_{\partial o_2} \tilde{f}_2,$$

so the force on o_2 may be evaluated directly from $(E, \tilde{D}')$. As before, we may also determine F_{12} by integrating $\tilde{T}$ over the observation surface $\partial o_2'$. Similar analysis concerns torques, resulting in

$$\tilde{\tau}_{12} = \int_{o_2} (-\mathrm{i}_r \star \tilde{F}_2 + \tilde{C}_2) + \int_{\partial o_2} (-\mathrm{i}_r \star \tilde{f}_2).$$

6.3 Forces and torques in magnetostatics

Let us next use the idea of distributed polarity to model the behavior of rigid magnetic objects. In the following we will use the magnetizations of the objects as primary quantities. The other possible model (that takes magnetic polarizations of the objects as primary) is analogous to the model of dielectric objects given in the previous section. Now the defining properties for $(B_1, \tilde{H}_1')$ are given as

$$\mathrm{d}B_1 = 0, \tag{6.35}$$
$$\mathrm{d}\tilde{H}_1' = \tilde{J}_1^f, \tag{6.36}$$
$$B_1 = \mu_0 \star (\tilde{H}_1' + \tilde{M}_1), \tag{6.37}$$

and

$$[\mathrm{t}_1 B_1]_1 = 0, \tag{6.38}$$

$$[\mathrm{t}_1 \tilde{H}'_1]_1 = \tilde{j}_1^f. \tag{6.39}$$

Similar properties concern $(B_2, \tilde{H}'_2)$. We observe that (6.35)-(6.39) are equivalent to (3.35), (3.36), and (3.40)-(3.42) when (5.27)-(5.34) are understood. In the present theory we call $\tilde{H}'_1$ (resp. $\tilde{H}'_2$) the magnetic field intensity of object o_1 (resp. object o_2).

Expressions for the force and couple densities are obtained based on section 6.1. The force density $\tilde{F}_{12}$ is determined by

$$\mathcal{G}(\tilde{F}_{12}, v) = \tilde{J}_2^f \wedge \mathrm{i}_v B_1 + \tilde{M}_2 \wedge \nabla_v B_1 \tag{6.40}$$

for all smooth vector fields v. For the couple density $\tilde{C}_{12}$ we set

$$\mathcal{G}(\tilde{C}_{12}, \tilde{\theta}) = \tilde{M}_2 \wedge \star \mathrm{i}_{\tilde{\theta}} B_1 \tag{6.41}$$

for all vector fields $\tilde{\theta}$. For the surface force density there is only a contribution from free surface currents. We set

$$\mathcal{G}(\tilde{f}_{12}, v) = \tilde{j}_2^f \wedge \mathrm{t}_2 \mathrm{i}_v B_1 \tag{6.42}$$

for all smooth vector fields v. Similar expressions are used for the force densities $\tilde{F}_{21}$ and $\tilde{f}_{21}$, and for the couple density $\tilde{C}_{21}$. For basis representations, see sections A.2.3 and B.2.2.

To verify that the present model of magnetic materials yields the same forces and torques as the electric current model of magnetic materials we proceed as in the previous section. We take a virtual displacement vector field v that is a linear combination of a constant vector field and a rotational vector field with respect to the center of mass of o_2, and give it to the generalized force F_{12}. We have

$$\begin{aligned} F_{12}(v) &= \int_{o_2} \big(\mathcal{G}(\tilde{F}_{12}, v) + \mathcal{G}(\tilde{C}_{12}, \tilde{\theta})\big) + \int_{\partial o_2} \mathcal{G}(\tilde{f}_{12}, v) \\ &= \int_{o_2} (\tilde{J}_2^f \wedge \mathrm{i}_v B_1 + \tilde{M}_2 \wedge \nabla_v B_1 + \tilde{M}_2 \wedge \star \mathrm{i}_{\tilde{\theta}} B_1) + \int_{\partial o_2} \tilde{j}_2^f \wedge \mathrm{i}_v B_1. \end{aligned}$$

By using our earlier results this may be given as

$$\begin{aligned}
F_{12}(v) &= \int_{o_2} (\tilde{J}_2^f \wedge \mathrm{i}_v B_1 + \tilde{M}_2 \wedge \mathcal{L}_v B_1) + \int_{\partial o_2} \tilde{j}_2^f \wedge \mathrm{i}_v B_1 \\
&= \int_{o_2} (\tilde{J}_2^f \wedge \mathrm{i}_v B_1 + \mathrm{d}\tilde{M}_2 \wedge \mathrm{i}_v B_1) + \int_{\partial o_2} (\tilde{j}_2^f \wedge \mathrm{i}_v B_1 + [\mathrm{t}_2 \tilde{M}_2]_2 \wedge \mathrm{i}_v B_1) \\
&= \int_{o_2} \tilde{J}_2 \wedge \mathrm{i}_v B_1 + \int_{\partial o_2} \tilde{j}_2 \wedge \mathrm{i}_v B_1,
\end{aligned}$$

which is the virtual work according to the electric current model of magnetic objects.

Let us compare at this stage the force and couple densities of the present model of magnetic materials to those of the alternative model in which magnetic polarizations are taken as primary quantities. Based on section 6.1 the alternative model has the force density $M_2 \wedge \nabla_v \tilde{H}_1$ and couple density $M_2 \wedge \mathrm{i}_{\tilde{\theta}} \star \tilde{H}_1$ (in the absence of free currents). Relating magnetization and magnetic polarization by $M_2 = \mu_0 \star \tilde{M}_2$, we have

$$\begin{aligned}
M_2 \wedge \nabla_v \tilde{H}_1 &= (\mu_0 \star \tilde{M}_2) \wedge \nabla_v \tilde{H}_1 \\
&= \tilde{M}_2 \wedge (\mu_0 \star \nabla_v \tilde{H}_1) \\
&= \tilde{M}_2 \wedge \nabla_v (\mu_0 \star \tilde{H}_1)
\end{aligned}$$

for all vector fields v. Also, we have

$$\begin{aligned}
M_2 \wedge \mathrm{i}_{\tilde{\theta}} \star \tilde{H}_1 &= (\mu_0 \star \tilde{M}_2) \wedge \mathrm{i}_{\tilde{\theta}} \star \tilde{H}_1 \\
&= \tilde{M}_2 \wedge \star \mathrm{i}_{\tilde{\theta}} (\mu_0 \star \tilde{H}_1)
\end{aligned}$$

for all vector fields $\tilde{\theta}$. Thus, because the field $(B_1, \tilde{H}_1)$ given by the two models of magnetic materials coincide outside of o_1 and satisfy $B_1 = \mu_0 \star \tilde{H}_1$, we see that the two models are equivalent with respect to force and couple densities. For demonstration we compute the force density $\tilde{F}_{12}$ in the example case considered in section 5.2. The method of computation is described in appendix D, and the result is visualized in Figure 6.3. There is a small difference in the force density pattern compared to that of Figure 6.2, which is a consequence of using magnetic vector potential formulation of the field problem (instead of magnetic scalar potential formulation).

6.3.1 Forces and torques in terms of total fields

The determination of forces directly from the total fields follows the lines of the previous section. So let us look for integrands involving total fields

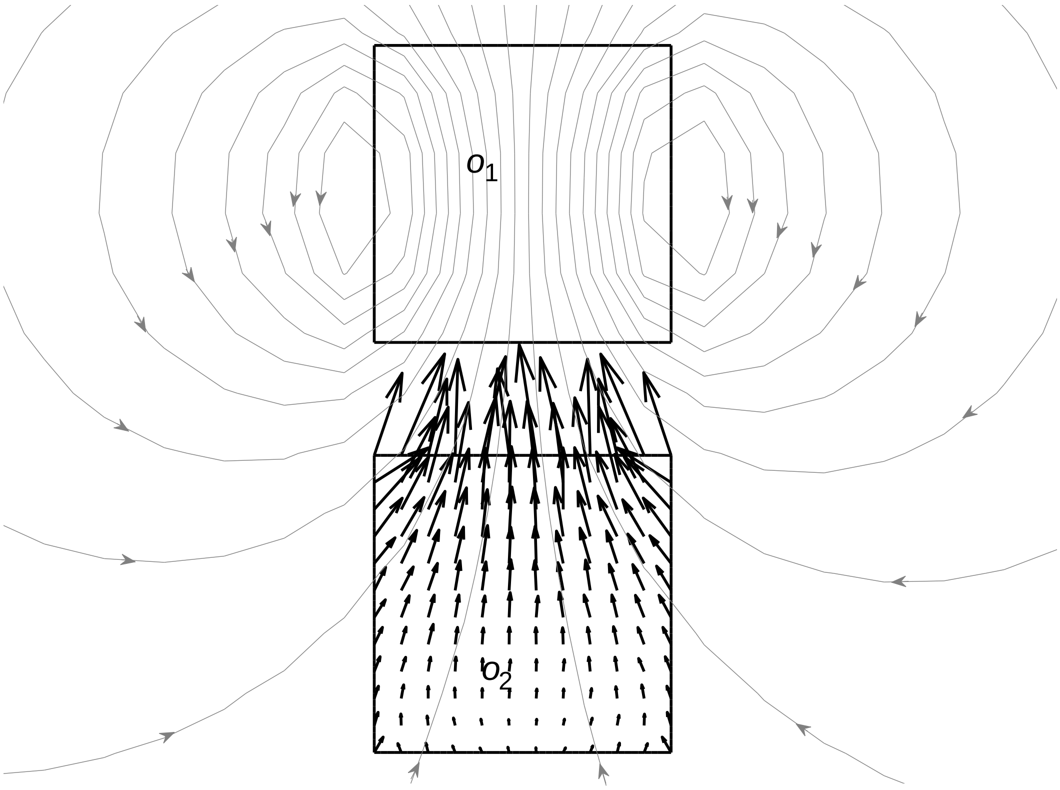


Figure 6.3: The example system of magnetic objects. The arrows show the vector representation of the volume force density $\tilde{F}_{12}$ on object o_2, and the gray lines with arrowheads visualize the magnetic induction B_1 of object o_1 (field lines of B_1, the arrowheads indicating the direction of the field).

that yield the correct generalized force F_{12}. By the electric current model of magnetic materials we have

$$F_{12}(v) = \int_{o_2} \tilde{J}_2^f \wedge \mathrm{i}_v B + \int_{\partial o_2} \tilde{j}_2^f \wedge (\mathrm{t}_2 \mathrm{i}_v B)^{av} + \int_{o_2} \mathrm{d}\tilde{M}_2 \wedge \mathrm{i}_v B$$
$$+ \int_{\partial o_2} [\mathrm{t}_2 \tilde{M}_2]_2 \wedge (\mathrm{t}_2 \mathrm{i}_v B)^{av},$$

since the self-force and self-torque of o_2 vanish. Considering the last two terms, and using $\mathrm{d}(\tilde{M}_2 \wedge \mathrm{i}_v B) = \mathrm{d}\tilde{M}_2 \wedge \mathrm{i}_v B - \tilde{M}_2 \wedge \mathrm{d}\mathrm{i}_v B$, we get

$$\int_{o_2} \mathrm{d}\tilde{M}_2 \wedge \mathrm{i}_v B + \int_{\partial o_2} [\mathrm{t}_2 \tilde{M}_2]_2 \wedge (\mathrm{t}_2 \mathrm{i}_v B)^{av} = \int_{o_2} \tilde{M}_2 \wedge \mathrm{d}\mathrm{i}_v B + \int_{o_2} \mathrm{d}(\tilde{M}_2 \wedge \mathrm{i}_v B) + \int_{\partial o_2} [\mathrm{t}_2 \tilde{M}_2]_2 \wedge (\mathrm{t}_2 \mathrm{i}_v B)^{av}.$$

The second term on the right hand side does not cancel the surface term because B is not continuous at the points of ∂o_2. Taking into account the left-over surface term, we have for the virtual work

$$F_{12}(v) = \int_{o_2} \tilde{J}_2^f \wedge \mathrm{i}_v B + \int_{\partial o_2} \tilde{j}_2^f \wedge (\mathrm{t}_2 \mathrm{i}_v B)^{av} + \int_{o_2} \tilde{M}_2 \wedge \mathrm{d}\mathrm{i}_v B - \int_{\partial o_2} (\mathrm{t}_2 \tilde{M}_2)^{av} \wedge [\mathrm{t}_2 \mathrm{i}_v B]_2.$$

By further using $\mathrm{d}\mathrm{i}_v B = \mathcal{L}_v B - \mathrm{i}_v \mathrm{d}B$, and taking into account that $\mathrm{d}B = 0$, and that $\mathcal{L}_v B = \nabla_v B + \star \mathrm{i}_{\tilde{\theta}} B$, we get

$$F_{12}(v) = \int_{o_2} (\tilde{J}_2^f \wedge \mathrm{i}_v B + \tilde{M}_2 \wedge \nabla_v B + \tilde{M}_2 \wedge \mathrm{i}_{\tilde{\theta}} B) + \int_{\partial o_2} \big(\tilde{j}_2^f \wedge (\mathrm{t}_2 \mathrm{i}_v B)^{av} - (\mathrm{t}_2 \tilde{M}_2)^{av} \wedge [\mathrm{t}_2 \mathrm{i}_v B]_2\big). \tag{6.43}$$

This is the total field expression for virtual work in the present magnetization approach. As in the electric case, we have a nonzero surface term in the total field expression even when the surface force density $\tilde{f}_{12}$ vanishes. This surface term is canceled by the surface term in the (generalized) self-force expression, which is obtained by using B_2 in place of B in the above calculation. Accordingly, $\tilde{F}_{12}$ is decomposed as

$$\tilde{F}_{12} = \tilde{F}_2 - \tilde{F}_{22},$$

where $\tilde{F}_2$ and $\tilde{F}_{22}$ are defined by

$$\mathcal{G}(\tilde{F}_2, v) = \tilde{J}_2^f \wedge \mathrm{i}_v B + \tilde{M}_2 \wedge \nabla_v B, \tag{6.44}$$

$$\mathcal{G}(\tilde{F}_{22}, v) = \tilde{J}_2^f \wedge \mathrm{i}_v B_2 + \tilde{M}_2 \wedge \nabla_v B_2, \tag{6.45}$$

for all vector fields v. Again, the derivatives are restricted to points the fields are smooth. The decomposition of $\tilde{C}_{12}$ is

$$\tilde{C}_{12} = \tilde{C}_2 - \tilde{C}_{22},$$

where $\tilde{C}_2$ and $\tilde{C}_{22}$ are defined by

$$\mathcal{G}(\tilde{C}_2, \tilde{\theta}) = \tilde{M}_2 \wedge \star \mathrm{i}_{\tilde{\theta}} B, \tag{6.46}$$

$$\mathcal{G}(\tilde{C}_{22}, \tilde{\theta}) = \tilde{M}_2 \wedge \star \mathrm{i}_{\tilde{\theta}} B_2, \tag{6.47}$$

for all vector fields $\tilde{\theta}$. Finally, the decomposition of $\tilde{f}_{12}$ is

$$\tilde{f}_{12} = \tilde{f}_2 - \tilde{f}_{22},$$

where $\tilde{f}_2$ and $\tilde{f}_{22}$ are defined by

$$\mathcal{G}(\tilde{f}_2, v) = \tilde{j}_2^f \wedge (\mathrm{t}_2 \mathrm{i}_v B)^{av} - (\mathrm{t}_2 \tilde{M}_2)^{av} \wedge [\mathrm{t}_2 \mathrm{i}_v B]_2, \tag{6.48}$$

$$\mathcal{G}(\tilde{f}_{22}, v) = \tilde{j}_2^f \wedge (\mathrm{t}_2 \mathrm{i}_v B_2)^{av} - (\mathrm{t}_2 \tilde{M}_2)^{av} \wedge [\mathrm{t}_2 \mathrm{i}_v B_2]_2, \tag{6.49}$$

for all vector fields v. As before, these expressions require the values of v only at the points of ∂o_2 and not on its two sides. The rest of the analysis of the previous section applies as such to the present magnetic case. Thus, also the present model of magnetic materials allows us to determine forces and torques directly from the total fields.

6.4 Discussion of force densities for determining deformations

At this point it is reasonable to further examine whether the material models of the present chapter allow us to define force densities that fit for the determination of deformations of objects o_1 and o_2. We thus ask for the distribution of forces given by these material models. The classical thinking goes as follows. Because the objects are deformable it is necessary to take into account the interactions between field sources that reside in one and the same object. Based on the discussion of section 6.1 it seems that the basic microscopic models of dielectric and magnetic materials do not allow the definition of electric or magnetic force density that takes into account these interactions. However, if the microscopic model is disregarded we may ask for mesoscopic force density expression that is to be taken as a starting point for modeling. The resulting model should be a generalization of the classical model of rigid objects. In electrostatics the obvious candidate for volume force density is the quantity $\tilde{F}$ determined in terms of the total fields (in the absence of free charges) by

$$\mathcal{G}(\tilde{F}, v) = \tilde{P} \wedge \nabla_v E$$

for all vector fields v. The associated surface force density is the quantity $\tilde{f}$ determined by

$$\mathcal{G}(\tilde{f}, v) = (\mathrm{t}\tilde{P})^{av} \wedge [\mathrm{ti}_v E]$$

for all vector fields v. In the case of rigid objects these quantities integrate to the correct total force as shown above. However, this generalization is questionable because it seems to contradict the basic microscopic model from which these expressions were "derived" in the first place.

In magnetostatics there are two different candidates for volume and surface force densities, corresponding to the two alternative models of magnetic materials. One of the models regards magnetization as a primary quantity, resulting in the volume and surface force densities $\tilde{F}$ and $\tilde{f}$ determined by

$$\begin{aligned}\mathcal{G}(\tilde{F}, v) &= \tilde{M} \wedge \nabla_v B \\ \mathcal{G}(\tilde{f}, v) &= -(\mathrm{t}\tilde{M})^{av} \wedge [\mathrm{ti}_v B]\end{aligned}$$

for all smooth vector fields v. The other model has magnetic polarization as primary, resulting in the volume and surface force densities $\tilde{F}$ and $\tilde{f}$ determined by

$$\begin{aligned}\mathcal{G}(\tilde{F}, v) &= M \wedge \nabla_v \tilde{H} - M \wedge \mathrm{i}_v \mathrm{d}\tilde{H} \\ \mathcal{G}(\tilde{f}, v) &= (\mathrm{t}M)^{av} \wedge [\mathrm{ti}_v \tilde{H}]\end{aligned}$$

for all smooth vector fields v. As shown above, both of these force density candidates integrate to the correct total force on rigid objects when M and $\tilde{M}$ are related by $M = \mu_0 \star \tilde{M}$. Note that for the equivalence of the two candidates with respect to total force we first need to subtract from $\tilde{F}$ and $\tilde{f}$ the integrands of the corresponding self-force expressions. The resulting local quantities describing the left-over external interactions then coincide at each point implying the total force equivalence. The difference in the above force density candidates is thus a difference in the integrands of the corresponding self-force expressions. The difference means that the two models could at best be regarded as models for different phenomena, that is, as models for the behavior of different deformable objects in a magnetic field.

It seems that the distribution of electric and magnetic forces in materials is not yet fully understood. A common opinion seems to be that of DiCarlo (see [22]), who argues that "...to obtain a decent general theory of electromagnetic forces in material media it is necessary to treat the material medium as fundamentally as the electromagnetic field." I understand this sentence such that electromagnetic force density for the determination of deformations cannot be uniquely defined, and different candidates for it need to be accompanied by additional terms that come from continuum mechanics. Recently, Bossavit has shown how to formulate a coupled magnetoelastic problem for determining deformations of magnetic objects, see [23].

Chapter 7

Contact forces

This chapter deals with the limitation in the above construction, whereby objects o_1 and o_2 are assumed to be distinct objects whose boundaries have no common points. When considering objects in contact there will be an immediate problem with the surface force density expressions. To elucidate this, let us consider our example system of magnetic objects when the objects are in contact. When considering the force on o_2, for instance, it is clear that the magnetic field intensity $\tilde{H}_1$ of object o_1 is not continuous at all points of ∂o_2, see Figure 7.1. At points of contact the tangential trace of the 2-form $\star\tilde{H}_1$ has discontinuity determined by magnetic surface charge density σ_1^m according to (5.43) and (5.44) as

$$[\mathrm{t}_2 \star \tilde{H}_1]_2 = -[\mathrm{t}_1 \star \tilde{H}_1]_1 = -\frac{1}{\mu_0}\sigma_1^m.$$

Because of the discontinuity it is not possible to determine the surface force density $\tilde{f}_{12}$ from the expression $\sigma_2^m \wedge \mathrm{t}_2 \mathrm{i}_v \tilde{H}_1$ as was done earlier in the magnetic charge starting point. The situation is similar with the electric current model of magnetic materials. In Figure 7.2 the magnetic induction B_1 of object o_1 is not continuous at all points of ∂o_2. At points of contact the tangential trace of the 1-form $\star B_1$ has discontinuity determined by magnetization surface current density $\tilde{j}_1^m$ as

$$[\mathrm{t}_2 \star B_1]_2 = -[\mathrm{t}_1 \star B_1]_1 = -\mu_0 \tilde{j}_1^m.$$

We thus observe that the expression $\tilde{j}_2^m \wedge \mathrm{t}_2 \mathrm{i}_v B_1$ for the surface force density is not applicable. In the following I will approach the above problem by considering a system of separate objects in the limit when the separation becomes infinitely small. Such an approach has been empoyed recently in [24, 25, 26, 27]. I will focus on magnetostatics and leave the results of electrostatics to be obtained by analogy.

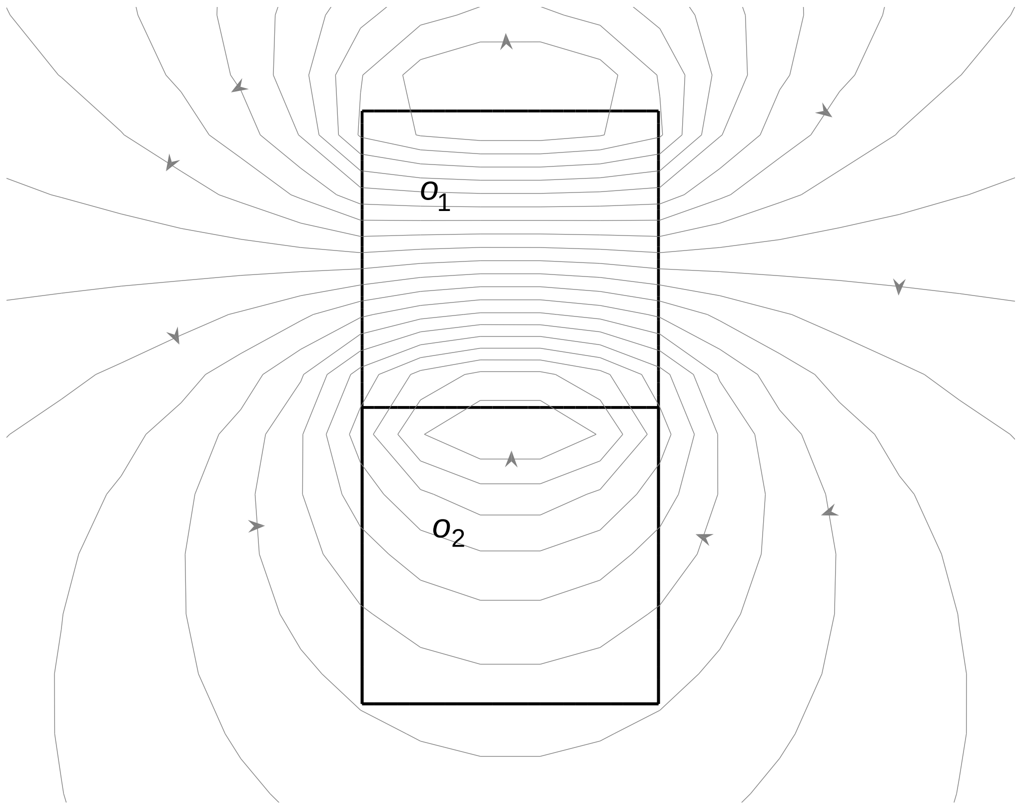


Figure 7.1: Magnetic objects in contact. Object o_1 has upward directed remanent magnetic polarization. The magnetic field intensity H_1 of object o_1 is not continuous at all points of ∂o_2 as there are magnetic surface charges of object o_1 on the contact interface.

7.1 Contact forces in magnetostatics

To consider objects o_1 and o_2 in the limit of infinitely small separation we use a real number parameter d to describe the smoothly varying separation, such that for $d = 0$ the objects are in contact, and for $d > 0$ they are separated by free space. This can be made more precise if needed, see [27]. Let us consider the virtual work $F_{12}(v)$ in the various starting points introduced earlier as a function of d. By the above discussion, $F_{12}(v)$ is defined in the electric current and magnetic charge starting points only for positive values of d. We define the virtual work at contact to be the limit value $\lim_{d \to 0+} F_{12}(v)$, given

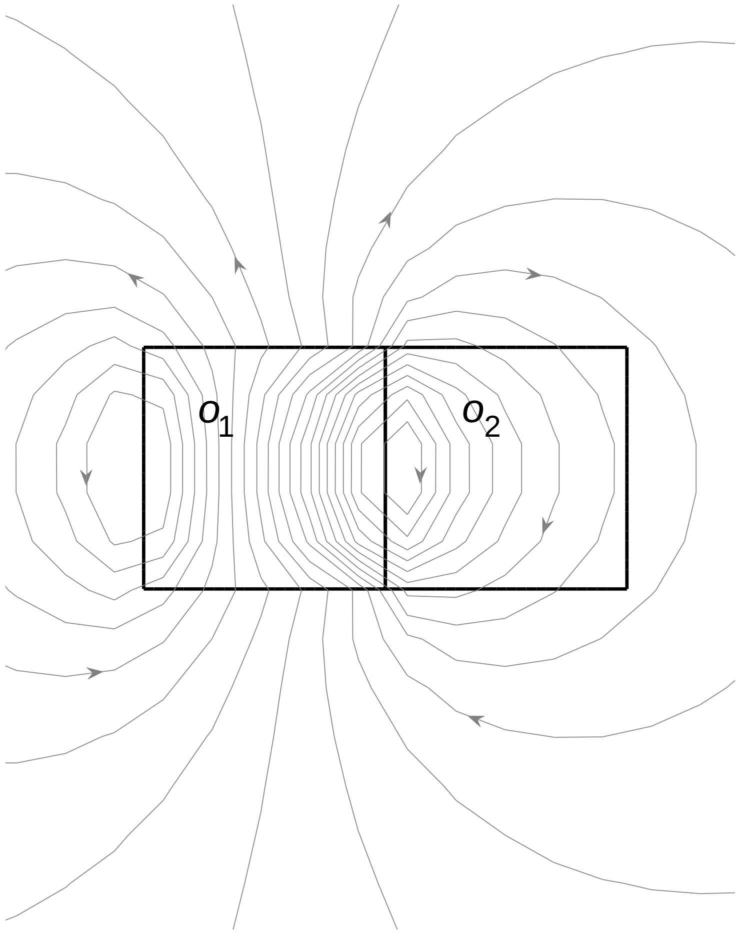


Figure 7.2: Magnetic objects in contact. Object o_1 has upward directed remanent magnetization. The magnetic induction B_1 of object o_1 is not continuous at all points of ∂o_2 as there are magnetization surface currents of object o_1 on the contact interface.

from (5.64) as

$$\lim_{d\to 0+} F_{12}(v) = \lim_{d\to 0+} \Big(\int_{o_2} \tilde{J}_2^f \wedge \mathrm{i}_v B_1 + \int_{\partial o_2} \tilde{j}_2^f \wedge \mathrm{i}_v B_1 + \int_{o_2} \mathrm{d}\tilde{M}_2 \wedge \mathrm{i}_v B_1 + \int_{\partial o_2} [\mathrm{t}_2 \tilde{M}_2]_2 \wedge \mathrm{i}_v B_1 \Big),$$

or from (5.67) as

$$\lim_{d\to 0+} F_{12}(v) = \lim_{d\to 0+} \Big(\int\limits_{o_2} \tilde{J}_2 \wedge \mathrm{i}_v B_1 + \int\limits_{\partial o_2} \tilde{j}_2 \wedge \mathrm{i}_v B_1 - \int\limits_{o_2} \mathrm{d}M_2 \wedge \mathrm{i}_v \tilde{H}_1$$
$$- \int\limits_{\partial o_2} [\mathrm{t}_2 M_2]_2 \wedge \mathrm{i}_v \tilde{H}_1 \Big).$$

To further examine the limit, we restrict the analysis to situations in which free surface currents vanish, that is, we require

$$\tilde{j}_2^f = \tilde{j}_2 = 0$$

in the above expressions for $F_{12}(v)$. In this case the only terms in these expressions that are not defined at $d = 0$ are the rightmost terms. Applying the partial integration process of section 5.2.3 we get rid of these surface terms, and have

$$\begin{aligned} \lim_{d\to 0+} F_{12}(v) &= \lim_{d\to 0+} \int\limits_{o_2} (\tilde{J}_2^f \wedge \mathrm{i}_v B_1 + \tilde{M}_2 \wedge \mathcal{L}_v B_1) \\ &= \lim_{d\to 0+} \int\limits_{o_2} (\tilde{J}_2 \wedge \mathrm{i}_v B_1 + M_2 \wedge \mathcal{L}_v \tilde{H}_1), \end{aligned} \tag{7.1}$$

which are the expressions for the virtual work at contact in the magnetization and magnetic polarization starting points. Provided that $F_{12}(v)$ above, considered as a function of d, is continuous at $d = 0$, the limit may be evaluated by evaluating $F_{12}(v)$ at the contact $d = 0$. Intuitively this seems to be the situation because the above expression for $F_{12}(v)$ involves the values of fields only inside of o_2 (and not the values in the free space separation of the objects). These values should depend continuously on d also in the limit $d = 0$. I make the following conjecture.

Conjecture 1. The function $F_{12}(v)$ in (7.1) is continuous at $d = 0$, so that

$$\lim_{d\to 0+} F_{12}(v)(d) = F_{12}(v)(0).$$

Even if this conjecture is correct, a computational problem remains as the method of appendix D cannot be used to compute the values of volume force density (involving ∇B_1 or $\nabla \tilde{H}_1$) near the contact interface.

7.2 Contact forces from the total fields

To determine contact forces directly from the total fields we need to reconsider the above limiting process with the exception that now we use the expressions for the virtual work $F_{12}(v)$ involving total fields. I will consider separately the magnetization and magnetic polarization starting points.

7.2.1 Magnetization approach

In the magnetization starting point we have $F_{12}(v)$ given from (6.43) for positive values of d. The virtual work at contact is thus

$$\lim_{d\to 0+} F_{12}(v) = \lim_{d\to 0+} \Big(\int_{o_2} (\tilde{J}_2^f \wedge \mathrm{i}_v B + \tilde{M}_2 \wedge \mathcal{L}_v B) - \int_{\partial o_2} (\mathrm{t}_2 \tilde{M}_2)^{av} \wedge [\mathrm{t}_2 \mathrm{i}_v B]_2 \Big). \tag{7.2}$$

It seems clear that the integral expression on the right hand side of (7.2), considered as a function of d, is not in general continuous at $d = 0$. This is suggested by the example case shown in Figure 7.3 where two magnetic bodies of the same magnetic susceptibility are in an initially homogeneous magnetic field. In this case, the object o_1 of our model is formed by one of the magnetic bodies and the body generating the initially homogeneous magnetic field (not shown in the figure). Since at the contact situation the magnetic induction B is continuous at the contact interface there is no contribution to the surface integral of (7.2) from the contact interface. On the other hand, for positive values of d the magnetic induction field B is discontinuous at all points of ∂o_2 (including the part that approaches the contact interface). The abrupt change in the surface contribution to $F_{12}(v)$ at $d = 0$ suggests that $F_{12}(v)$ is discontinuous at $d = 0$.

To find an expression for $F_{12}(v)$ in terms of total fields that is continuous at $d = 0$ we examine the surface term in (7.2) for positive values of d. We expect some part of the 1-form $[\mathrm{t}_2 \mathrm{i}_v B]_2$ to vanish because the tangential trace of B is continuous. To see what is left of this term, we first decompose v at the points of ∂o_2 into tangential and normal components as

$$v = v_\parallel + v_\perp, \tag{7.3}$$

where the normal component $v_\perp$ is given by using the metric tensor g and outward unit normal vector field n of ∂o_2 as $v_\perp = g(n, v)n$. Since we have assumed for the 1-form $[\mathrm{t}_2 \mathrm{i}_v B]_2$ to require the values of v only at points of

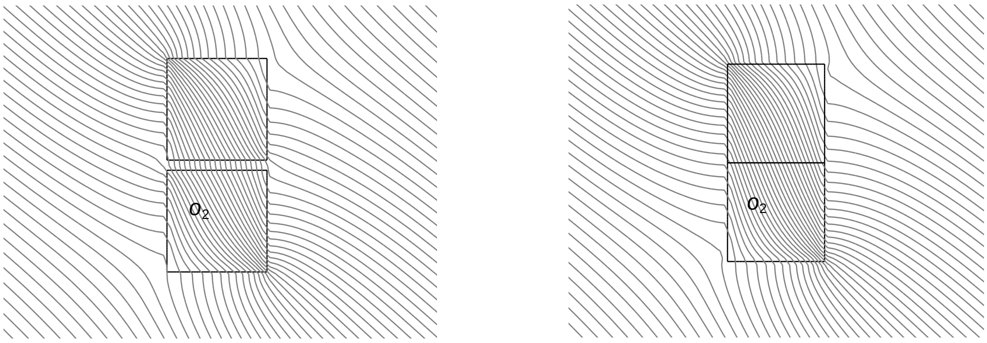


Figure 7.3: Two bodies with equal magnetic susceptibilities in an initially homogeneous magnetic field. The gray lines are field lines of B.

∂o_2, we may use the above decomposition of v to decompose this 1-form as

$$\begin{aligned}[\mathrm{t}_2 \mathrm{i}_v B]_2 &= [\mathrm{t}_2(\mathrm{i}_{v_\parallel} B + \mathrm{i}_{v_\perp} B)]_2 \\ &= [\mathrm{t}_2 \mathrm{i}_{v_\parallel} B]_2 + [\mathrm{t}_2 \mathrm{i}_{v_\perp} B]_2,\end{aligned}$$

where we have further used the linearity of interior product and tangential trace. Because $v_\parallel$ is tangent to ∂o_2, it follows that $[\mathrm{t}_2 \mathrm{i}_{v_\parallel} B]_2 = \mathrm{i}_{v_\parallel}[\mathrm{t}_2 B]_2$ (where on the right hand side $v_\parallel$ is taken as a tangent vector field on ∂o_2). Then, since $[\mathrm{t}_2 B]_2$ vanishes, we have

$$[\mathrm{t}_2 \mathrm{i}_v B]_2 = [\mathrm{t}_2 \mathrm{i}_{v_\perp} B]_2.$$

We see that only the normal component of v contributes to the surface term of (7.2). By expressing this normal component as $g(n, v)n$, and using the function-linearity of the interior product in the vector argument, we get

$$\begin{aligned}[\mathrm{t}_2 \mathrm{i}_v B]_2 &= [\mathrm{t}_2\big(g(n, v)\mathrm{i}_n B\big)]_2 \\ &= g(n, v)[\mathrm{t}_2 \mathrm{i}_n B]_2.\end{aligned}$$

This may further be written by using the *normal trace* n_2, resulting in

$$[\mathrm{t}_2 \mathrm{i}_v B]_2 = g(n, v)[\mathrm{n}_2 B]_2. \tag{7.4}$$

Normal trace. The normal trace of a p-form on a surface with selected outward pointing crossing direction is obtained by putting the outward pointing unit normal vector in the first argument place of the form and letting the resulting (p-1)-form operate only on vectors tangent to the surface. Formally, given a p-form ω on n-dimensional Riemannian manifold (M, g), its *normal trace* on a transverse oriented (n-1)-dimensional submanifold N with continuous outward pointing normal vector field n is the (p-1)-form $\mathrm{n}\omega$ on N defined at each point $x \in N$ by

$$(\mathrm{n}\omega)_x(u_2, \ldots, u_p) = \omega_x(n_x, u_2, \ldots, u_p)$$

for all $u_2, \ldots, u_p \in T^1_x N$. Note that we may express $\mathrm{n}\omega$ by using the interior product and tangential trace as $\mathrm{ti}_n\omega$.

The normal trace of B is certainly not continuous in general. To see that here its discontinuity is determined by the discontinuity of magnetization $\tilde{M}$, we first use $B = \mu_0 \star (\tilde{H}' + \tilde{M})$ in (7.4) to get

$$\begin{aligned}[\mathrm{t}_2\mathrm{i}_v B]_2 &= \mu_0 g(n, v)[\mathrm{n}_2 \star \tilde{H}' + \mathrm{n}_2 \star \tilde{M}]_2 \\ &= \mu_0 g(n, v)[\mathrm{n}_2 \star \tilde{H}']_2 + \mu_0 g(n, v)[\mathrm{n}_2 \star \tilde{M}]_2.\end{aligned}$$

Then we note that the discontinuities $[\mathrm{n}_2 \star \tilde{H}']_2$ and $[\mathrm{n}_2 \star \tilde{M}]_2$ above may be given by using the tangential trace and *surfacic Hodge operator* $\star_s$ to get

$$\begin{aligned}[\mathrm{t}_2\mathrm{i}_v B]_2 &= \mu_0 g(n, v) \star_s [\mathrm{t}_2\tilde{H}']_2 + \mu_0 g(n, v) \star_s [\mathrm{t}_2\tilde{M}]_2 \\ &= \mu_0 g(n, v) \star_s [\mathrm{t}_2\tilde{M}]_2, \qquad (7.5)\end{aligned}$$

where the last equality follows because we have assumed free surface currents to vanish so that $[\mathrm{t}_2\tilde{H}']_2 = 0$.

Surfacic Hodge operator. Given an n-dimensional Riemannian manifold (M, g) and its (n-1)-dimensional submanifold N we can define a surfacic Hodge operator on N by first making N a Riemannian manifold in its own right by restricting g to N. The restriction is performed by using the natural inclusion $i : N \to M$ in exactly the same way as when restricting a differential form. Thus, to evaluate the inner product of two tangent vectors of $T_x N$ we just consider these vectors as tangent vectors of $T_{i(x)} M$ and give them to $g_{i(x)}$. When N is equipped with this *pull-back metric*, we may define a Hodge operator on N in just the same way as it was defined on M. This is called the *surfacic Hodge operator*. Because we have defined twisted forms only on orientable manifolds we need to require here that the submanifold N is orientable.

To verify that $[\mathrm{n}_2 \star \tilde{M}]_2 = \star_s[\mathrm{t}_2\tilde{M}]_2$ as claimed above, we first write this claim in detail as

$$
\begin{aligned}
[\mathrm{n}_2 \star \tilde{M}]_2 &= \mathrm{n}_2^+ \star \tilde{M} - \mathrm{n}_2^- \star \tilde{M} \\
&= \star_s \mathrm{t}_2^+ \tilde{M} - \star_s \mathrm{t}_2^- \tilde{M} \\
&= \star_s [\mathrm{t}_2 \tilde{M}]_2.
\end{aligned}
$$

Thus, it is left to be shown that $\mathrm{n}_2^\pm \star \tilde{M} = \star_s \mathrm{t}_2^\pm \tilde{M}$. For this, it is sufficient to take two orthonormal vectors e_1 and e_2 at a point of ∂o_2 that are tangent to ∂o_2, and show that the covector $\mathrm{n}_2^\pm \star \tilde{M}$ yields the same number for both of these vectors as the covector $\star_s \mathrm{t}_2^\pm \tilde{M}$. (Here I abuse the notation by using the same symbol for a 1-form and its value at a point.) By denoting as $\tilde{M}^\pm$ the limit value of $\tilde{M}$ from outside or from inside of ∂o_2, we get

$$
\begin{aligned}
(\mathrm{n}_2^\pm \star \tilde{M})(e_1) &= \star \tilde{M}^\pm (n \wedge e_1), \\
&= \tilde{M}^\pm \big(\star (n \wedge e_1) \big),
\end{aligned}
$$

where on the right hand side e_1 is taken as a vector on the space manifold Ω. By using the unit 3-vector $n \wedge e_1 \wedge e_2$ to specify local orientation of Ω, the twisted vector $\star(n \wedge e_1)$ will be represented by e_2 (considered as a vector on Ω). We have

$$
\begin{aligned}
\tilde{M}^\pm \big(\star (n \wedge e_1) \big) &= \tilde{M}^\pm (e_2) \\
&= \mathrm{t}_2^\pm \tilde{M}(e_2),
\end{aligned}
$$

where on the last row e_2 is again taken as a vector on ∂o_2. The transverse oriented surface ∂o_2 now has an induced orientation determined by $e_1 \wedge e_2$, see Figure 7.4. Thus the vector that the surfacic Hodge maps to the twisted vector represented by e_2 is e_1. This gives

$$
\begin{aligned}
\mathrm{t}_2^\pm \tilde{M}(e_2) &= \mathrm{t}_2^\pm \tilde{M}(\star_s e_1) \\
&= (\star_s \mathrm{t}_2^\pm \tilde{M})(e_1).
\end{aligned}
$$

The equality $(\mathrm{n}_2^\pm \star \tilde{M})(e_2) = (\star_s \mathrm{t}_2^\pm \tilde{M})(e_2)$ can be shown similarly. We thus have $\mathrm{n}_2^\pm \star \tilde{M} = \star_s \mathrm{t}_2^\pm \tilde{M}$ as promised, and this proves the claim that $[\mathrm{n}_2 \star \tilde{M}]_2 = \star_s [\mathrm{t}_2 \tilde{M}]_2$. Finally, by using (7.5), we may express the surface term in (7.2) as

$$
\begin{aligned}
-(\mathrm{t}_2 \tilde{M}_2)^{av} \wedge [\mathrm{t}_2 \mathrm{i}_v B]_2 &= -(\mathrm{t}_2 \tilde{M}_2)^{av} \wedge \big(\mu_0 g(n, v) \star_s [\mathrm{t}_2 \tilde{M}]_2 \big) \\
&= \frac{\mathrm{t}_2^- \tilde{M}_2}{2} \wedge \big(\mu_0 g(n, v) \star_s \mathrm{t}_2^- \tilde{M} \big) \qquad (7.6)
\end{aligned}
$$

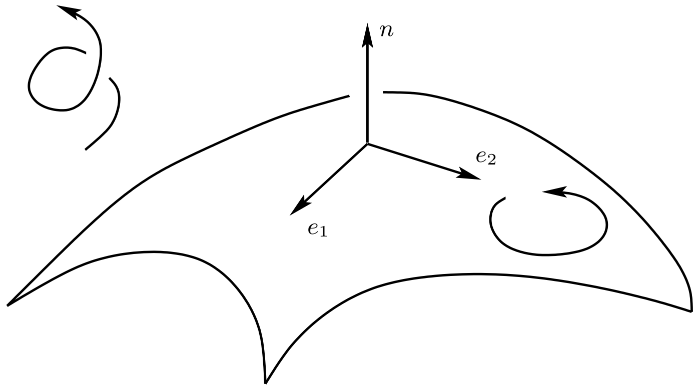


Figure 7.4: A piece of boundary ∂o_2 with its orientation induced from the right handed orientation of Ω. The surfacic Hodge operator turns tangent vectors 90 degrees to the positive direction.

where we have taken into account that $\mathrm{t}_2^+ \tilde{M}$ vanishes because o_2 is surrounded by free space.

By using (7.6) in (7.2), and taking into account that $\tilde{M}$ coincides with $\tilde{M}_2$ on o_2, we have

$$\lim_{d\to 0+} F_{12}(v) = \lim_{d\to 0+} \Big(\int_{o_2} (\tilde{J}_2^f \wedge \mathrm{i}_v B + \tilde{M}_2 \wedge \mathcal{L}_v B) + \int_{\partial o_2} \frac{\mu_0}{2} g(n,v) \mathrm{t}_2^- \tilde{M}_2 \wedge \star_s \mathrm{t}_2^- \tilde{M}_2 \Big) \tag{7.7}$$

for the virtual work at contact. This expression involves the values of B, $\tilde{M}_2$ and $\tilde{J}_2^f$ only inside of o_2 (and not in the free space separation of the objects). These values should depend continuously on d even in the limit $d = 0$, and thus we should be able to evaluate the limit by just evaluating $F_{12}(v)$ corresponding to $d = 0$. I will make a similar conjecture as in the previous section.

Conjecture 2. The function $F_{12}(v)$ in (7.7) is continuous at $d = 0$, so that

$$\lim_{d\to 0+} F_{12}(v)(d) = F_{12}(v)(0).$$

If the conjecture is correct, the virtual work done on o_2 by o_1 when the objects are in contact is thus given as

$$F_{12}(v)(0) = \int_{o_2} (\tilde{J}_2^f \wedge \mathrm{i}_v B + \tilde{M}_2 \wedge \nabla_v B + \tilde{M}_2 \wedge \star \mathrm{i}_{\tilde{\theta}} B) + \int_{\partial o_2} \frac{\mu_0}{2} g(n,v) \mathrm{t}_2^- \tilde{M}_2 \wedge \star_s \mathrm{t}_2^- \tilde{M}_2, \tag{7.8}$$

where the quantities on the right hand side are evaluated at the contact situation. This gives the force and torque formulae that appear in the work of Brown [14, 28]. For basis representation of the surface integrand in (7.8), see Appendix C.

7.2.2 Magnetic polarization approach

To determine contact forces from the total fields in the magnetic polarization approach we follow the steps of the previous subsection. Now the virtual work at contact is given as the limit

$$\lim_{d\to 0+} F_{12}(v) = \lim_{d\to 0+} \Big(\int_{o_2} (\tilde{J}_2 \wedge \mathrm{i}_v B + M_2 \wedge \mathcal{L}_v \tilde{H} - M_2 \wedge \mathrm{i}_v \mathrm{d}\tilde{H}) + \int_{\partial o_2} (\mathrm{t}_2 M_2)^{av} \wedge [\mathrm{t}_2 \mathrm{i}_v \tilde{H}]_2 \Big). \tag{7.9}$$

To obtain an expression that is continuous at $d = 0$ we consider the surface term $(\mathrm{t}_2 M_2)^{av} \wedge [\mathrm{t}_2 \mathrm{i}_v \tilde{H}]_2$ corresponding to positive values of d. Let us first examine what is left of the term $[\mathrm{t}_2 \mathrm{i}_v \tilde{H}]_2$ after we have taken into account that the tangential traces of B' and $\tilde{H}$ are continuous over ∂o_2. By using the decomposition of v into tangential and normal components, and following the steps of the previous subsection, we get

$$\begin{aligned}
[\mathrm{t}_2 \mathrm{i}_v \tilde{H}]_2 &= [\mathrm{t}_2 \mathrm{i}_{v_{||}} \tilde{H}]_2 + [\mathrm{t}_2 \mathrm{i}_{v_\perp} \tilde{H}]_2 \\
&= \mathrm{i}_{v_{||}} [\mathrm{t}_2 \tilde{H}]_2 + [\mathrm{t}_2 \mathrm{i}_{v_\perp} \tilde{H}]_2 \\
&= [\mathrm{t}_2 \mathrm{i}_{v_\perp} \tilde{H}]_2 \\
&= g(n_2, v) [\mathrm{t}_2 \mathrm{i}_{n_2} \tilde{H}]_2 \\
&= g(n_2, v) [\mathrm{n}_2 \tilde{H}]_2,
\end{aligned}$$

where in the third equality we have taken into account that the surface currents vanish so that $[\mathrm{t}_2 \tilde{H}]_2 = 0$. By further using $\tilde{H} = \mu_0^{-1} \star (B' - M)$ this yields

$$\begin{aligned}[\mathrm{t}_2\mathrm{i}_v\tilde{H}]_2 &= \mu_0^{-1}g(n_2,v)[\mathrm{n}_2\star B' - \mathrm{n}_2\star M]_2\\ &= \mu_0^{-1}g(n_2,v)\star_s[\mathrm{t}_2B']_2 - \mu_0^{-1}g(n_2,v)\star_s[\mathrm{t}_2M]_2\\ &= -\mu_0^{-1}g(n_2,v)\star_s[\mathrm{t}_2M]_2, \end{aligned} \tag{7.10}$$

where we have taken into account that $[\mathrm{t}_2B']_2 = 0$.

In (7.10) we have made use of the fact that $\mathrm{n}_2^{\pm}\star M = \star_s\mathrm{t}_2^{\pm}M$ so that

$$\begin{aligned}[\mathrm{n}_2\star M]_2 &= \mathrm{n}_2^{+}\star M - \mathrm{n}_2^{-}\star M\\ &= \star_s\mathrm{t}_2^{+}M - \star_s\mathrm{t}_2^{-}M\\ &= \star_s[\mathrm{t}_2M]_2.\end{aligned}$$

To verify the claim $\mathrm{n}_2^{\pm}\star M = \star_s\mathrm{t}_2^{\pm}M$ we take a point on ∂o_2, and look at the operation of the twisted 0-covector $\mathrm{n}_2^{\pm}\star M$ at the point on the unit 0-vector at the point. (I abuse the notation by using the same symbol for differential form and its value at a point.) We have

$$\begin{aligned}\mathrm{n}_2^{\pm}\star M(1) &= \star M^{\pm}(n_2)\\ &= M^{\pm}(\star n_2).\end{aligned}$$

Let us next take two orthonormal tangent vectors e_1 and e_2 of ∂o_2 at the point in question. As before we may consider these vectors as vectors on Ω, and this way we can specify the local orientation of Ω by using the unit 3-vector $n\wedge e_1\wedge e_2$. With this orientation the twisted 2-vector $\star n_2$ is represented by $e_1\wedge e_2$, so we have

$$\begin{aligned}\mathrm{n}_2^{\pm}\star M(1) &= M^{\pm}(e_1\wedge e_2)\\ &= \mathrm{t}_2^{\pm}M(e_1\wedge e_2),\end{aligned}$$

where on the last row e_1 and e_2 are again taken as vectors on ∂o_2. Now ∂o_2 has an induced local orientation defined by the 2-vector $e_1\wedge e_2$. It follows that the 0-vector which is mapped by $\star_s$ to the twisted 2-vector represented by $e_1\wedge e_2$ is 1. We thus have

$$\begin{aligned}\mathrm{n}_2^{\pm}\star M(1) &= \mathrm{t}_2^{\pm}M(\star_s 1)\\ &= \star_s\mathrm{t}_2^{\pm}M(1).\end{aligned}$$

This shows that $\mathrm{n}_2^{\pm}\star M = \star_s\mathrm{t}_2^{\pm}M$ as claimed above.

By using (7.10) we may write the surface integrand in (7.9) as

$$\begin{aligned}(\mathrm{t}_2M_2)^{av}\wedge[\mathrm{t}_2\mathrm{i}_v\tilde{H}]_2 &= -(\mathrm{t}_2M_2)^{av}\wedge\big(\mu_0^{-1}g(n_2,v)\star_s[\mathrm{t}_2M]_2\big)\\ &= \frac{\mathrm{t}_2^{-}M_2}{2}\wedge\big(\mu_0^{-1}g(n_2,v)\star_s\mathrm{t}_2^{-}M\big),\end{aligned}$$

where we have taken into account that $t_2^+ M = 0$ because o_2 is surrounded by free space. Finally, the virtual work at contact may be given as

$$\lim_{d\to 0+} F_{12}(v) = \lim_{d\to 0+} \Big(\int_{o_2} (\tilde{J}_2 \wedge \mathrm{i}_v B + M_2 \wedge \mathcal{L}_v \tilde{H} - M_2 \wedge \mathrm{i}_v \mathrm{d}\tilde{H}) + \int_{\partial o_2} \frac{1}{2\mu_0} g(n_2, v) \mathrm{t}_2^- M_2 \wedge \star_s \mathrm{t}_2^- M_2 \Big). \tag{7.11}$$

We make the following conjecture.

Conjecture 3. The function $F_{12}(v)$ in (7.11) is continuous at $d = 0$, so that

$$\lim_{d\to 0+} F_{12}(v)(d) = F_{12}(v)(0).$$

If the conjecture holds true, we have for the virtual work at contact

$$F_{12}(v)(0) = \int_{o_2} (\tilde{J}_2 \wedge \mathrm{i}_v B + M_2 \wedge \nabla_v \tilde{H} - M_2 \wedge \mathrm{i}_v \mathrm{d}\tilde{H} + M_2 \wedge \mathrm{i}_{\tilde{\theta}} \star \tilde{H}) + \int_{\partial o_2} \frac{1}{2\mu_0} g(n_2, v) \mathrm{t}_2^- M_2 \wedge \star_s \mathrm{t}_2^- M_2, \tag{7.12}$$

where all quantities on the right hand side correspond to the contact situation $d = 0$. This is in accordance with the force and torque formulae appearing in the work of Brown [14, 28]. For basis representation of the surface integrand in (7.12), see Appendix C.

Chapter 8

Geometric view of magnetic forces

In this chapter I will use the methods of previous chapters to evaluate magnetic forces in example situations and try to find a usable engineering rule by which the forces may be estimated from the field lines of the total fields. If this can be done one does not need to know the field sources to estimate forces. At the end of this chapter a new geometric view of magnetic forces will be suggested. It is precisely this kind of geometric thinking that the methods introduced in the previous chapters help to develop. In the same spirit, these methods constitute the tool for formalizing the developed geometric intuition.

Let us examine magnetic forces in an example system where a current carrying wire resides in the vicinity of a permanent magnet, see Figure 8.1. The magnetic force on the wire tends to pull the wire towards the magnet. How could this be seen from the magnetic field lines shown in the figure? The field lines are not only bent (with respect to Cartesian straight lines) but they also have density variations. In free space these two properties seem to compensate each other such that the bending of field lines and the decrease in density are always in opposite directions with the same magnitude. Inside of the wire this is clearly not the case since both the bending and the decrease in density is towards the magnet. It seems that forces arise from the combined effect of the bending of field lines and decrease in their density.

The above reasoning is in accordance with the classical force density formula $\boldsymbol{J} \times \boldsymbol{B}$. By expressing $\boldsymbol{J}$ as $\mu_0^{-1}\nabla \times \boldsymbol{B}$ we may decompose this as

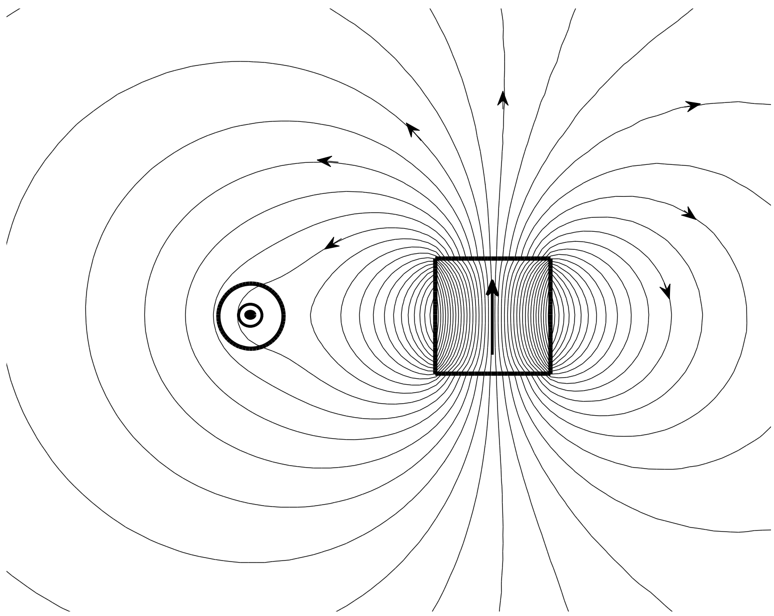


Figure 8.1: Magnetic field lines in the example system consisting of a permanent magnet and a current carrying wire. Current in the wire is directed out of the page and the permanent magnet has upward directed remanent magnetization.

$$\begin{aligned}\boldsymbol{J} \times \boldsymbol{B} &= \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} \\ &= \frac{1}{\mu_0}\boldsymbol{B} \cdot \nabla \boldsymbol{B} - \nabla(\frac{1}{2\mu_0}\boldsymbol{B} \cdot \boldsymbol{B}),\end{aligned} \tag{8.1}$$

where for the last equality we have used the vector analysis formula $\boldsymbol{a} \cdot \nabla \boldsymbol{b} + \boldsymbol{b} \cdot \nabla \boldsymbol{a} + \boldsymbol{b} \times (\nabla \times \boldsymbol{a}) = -\boldsymbol{a} \times (\nabla \times \boldsymbol{b}) + \nabla(\boldsymbol{a} \cdot \boldsymbol{b})$. (This formula follows from the Lie derivative formula $\mathcal{L}_a b = \mathrm{i}_a \mathrm{d}b + \mathrm{d}\mathrm{i}_a b$, where a is a vector field with components $\boldsymbol{a}$, and b is a 1-form whose proxy-vector has components $\boldsymbol{b}$.) The first term in (8.1) describes the contribution that comes from the bending of field lines, whereas the second term describes the contribution that comes from the density variations of field lines. If the field lines are straight lines of Cartesian geometry, the first term vanishes and only the second term contributes to forces. In [30] these two contributions are referred to tension in the field lines and magnetic pressure gradient.

When using the above reasoning to estimate the force on the permanent magnet, we only expect the net effect on the magnet to be described correctly. Equation 8.1 is valid also inside of the permanent magnet provided that equivalent magnetization currents are included in $\boldsymbol{J}$. At the leftmost and rightmost parts of the permanent magnet the field lines bend to opposite directions, suggesting forces that stretch the magnet horizontally. The net effect cannot be easily seen from this in the example of Figure 8.1 because the fields of the magnet are much stronger than the fields of the wire, so the wire has only a very small effect on the field lines inside of the magnet. For this reason we consider another example where the fields of the magnet are weaker than above with respect to the fields of the wire, see Figure 8.2. In this case we see that the field lines inside of the magnet are bent slightly more to the left than to the right, suggesting a net force to the left. However, the density of field lines decreases slightly more to the right than to the left, so the net effect is still not clear. On the surface of the magnet the bending of field lines and their density variation do not occur smoothly. In this case the heuristic application of the above reasoning suggests that the force at the leftmost surface is to the left and the force at the rightmost surface is to the right. However, the net effect is still somewhat blurred as inside of the magnet. The present method is clearly not optimal for figuring out the force on the permanent magnet. Yet there is something in the field lines above, giving the impression that the magnet is pulled to the left.

The properties of field lines inside of the magnet are reflected to their properties in the neighbourhood of the magnet. Thus, it should be possible to see the net effect on the magnet by looking at field lines in the neighbourhood of the magnet. This is realized formally by integrating the Maxwell's stress tensor over an "observation surface" surrounding the magnet (but not the wire). We may express $\boldsymbol{J} \times \boldsymbol{B}$ as

$$\boldsymbol{J} \times \boldsymbol{B} = \nabla \cdot \big(\frac{1}{\mu_0}\boldsymbol{B}\boldsymbol{B} - \frac{1}{2\mu_0}(\boldsymbol{B} \cdot \boldsymbol{B})\bar{I}\big) \tag{8.2}$$

as shown in Appendix A. Stoke's theorem then implies that the net force on the magnet is given by integrating the quantity in brackets over the observation surface. It is not easy to see the Maxwell's tensor and its net effect from the field lines, so we seek here a simple heuristic engineering rule that yields the outcome of the above formal procedure. More to the point, we seek to interpret in example situations Faraday's suggestion rephrased by Cross ([30]) as follows: "...lines of magnetic force exist in a state of tension and have a tendency to shorten themselves."

For the interpretation of Faraday's rule in the example of Figure 8.2 we look at field lines that bypass the magnet from the left and from the right.

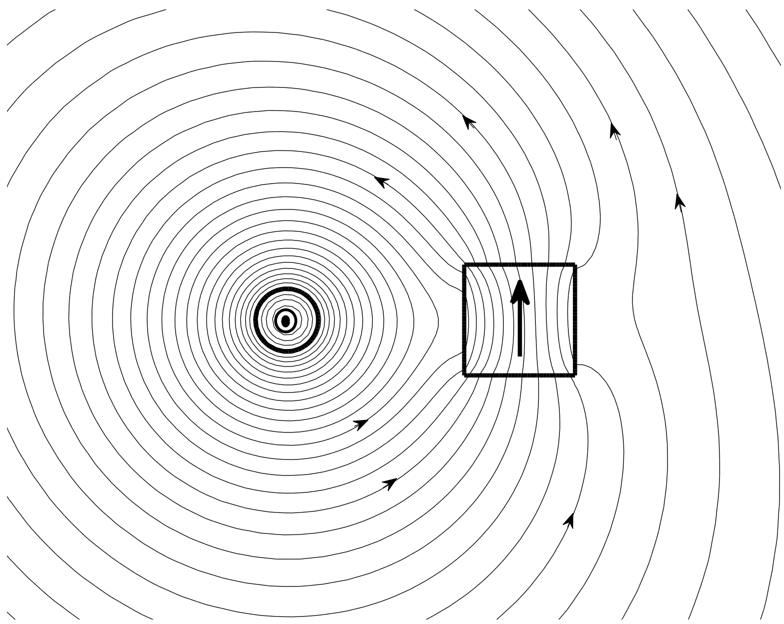


Figure 8.2: Magnetic field lines in a system where the fields of the permanent magnet are of the same order of magnitude as the fields of the wire.

Field lines passing by from the left bend to the left and field lines passing by from the right bend to the right. Since the field lines bending to the left have higher density than those bending to the right, the net force on the magnet is to the left. Let us consider this same example when the direction of current is reversed, see Figure 8.3. In this case the permanent magnet is pushed to the right. To see this by using the above rule we look at (as above) those field lines passing by the magnet that do not go through the magnet. On the left hand side of the magnet these field lines bend to the right, whereas on the right hand side the field lines bend to the left, and since the field lines bending to the right are of higher density, the net force is to the right. Let us consider yet another example where two permanent magnets of the same magnitude are aligned such that their magnetizations point to opposite directions, see Figure 8.4. Here, the two magnets tend to repel each other. How should one interpret Faraday's rule in this example? Which of the field lines tend to shorten themselves to yield a repulsive force, for instance, to the rightmost magnet? My suggestion is that, as above, those field lines

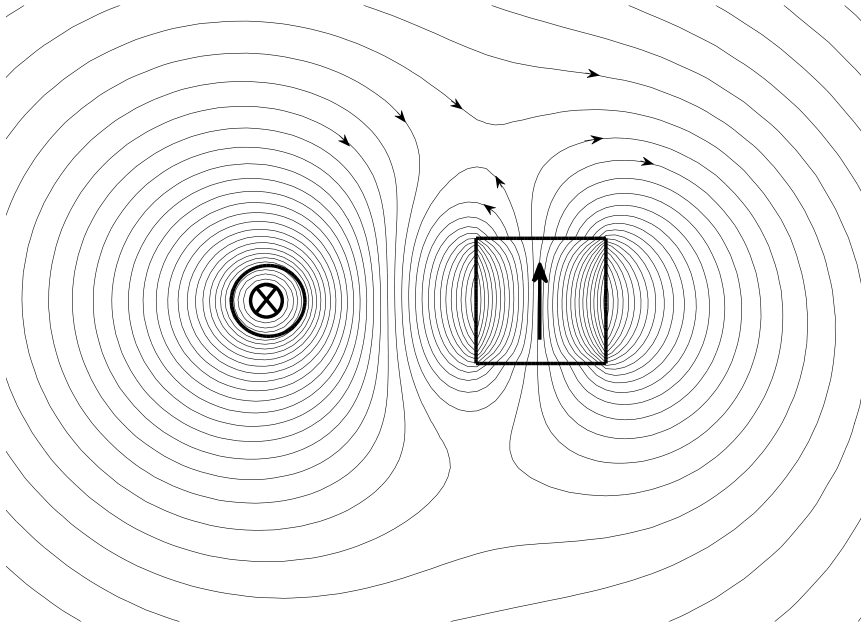


Figure 8.3: Magnetic field lines in a system where the direction of current is reversed with respect to the example of Figure 8.2.

passing by the rightmost magnet that do not go through the magnet tend to shorten themselves to yield a net force to the right. (These are the field lines going through the leftmost magnet.) However, the field lines do not tend to shorten themselves in the usual Cartesian metric, but in another metric defined by the fields of the leftmost magnet. In this other metric (defined outside of the leftmost magnet) the field lines of the leftmost magnet give the shortest paths between pairs of points and thus replace the straight lines of Cartesian geometry, see Figure 8.5. The intuition is thus that the field lines going through the leftmost magnet in Figure 8.4 tend towards the field lines of the leftmost magnet shown in Figure 8.5, pushing the other magnet to the right. This rule works in all the example cases above – in the examples preceding the present one the two concepts of shortening field lines yield the same conclusion about the net force.

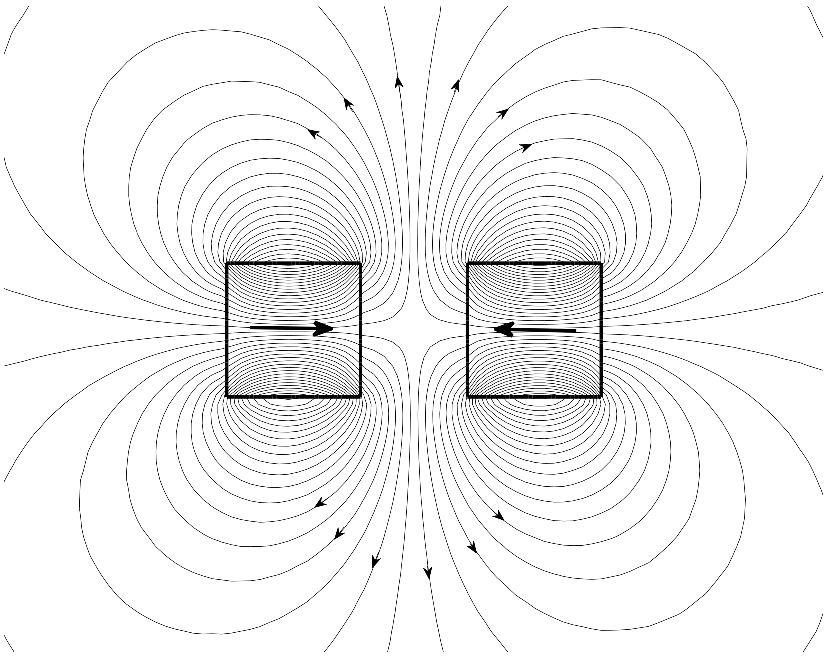


Figure 8.4: Magnetic field lines in a system of two permanent magnets.

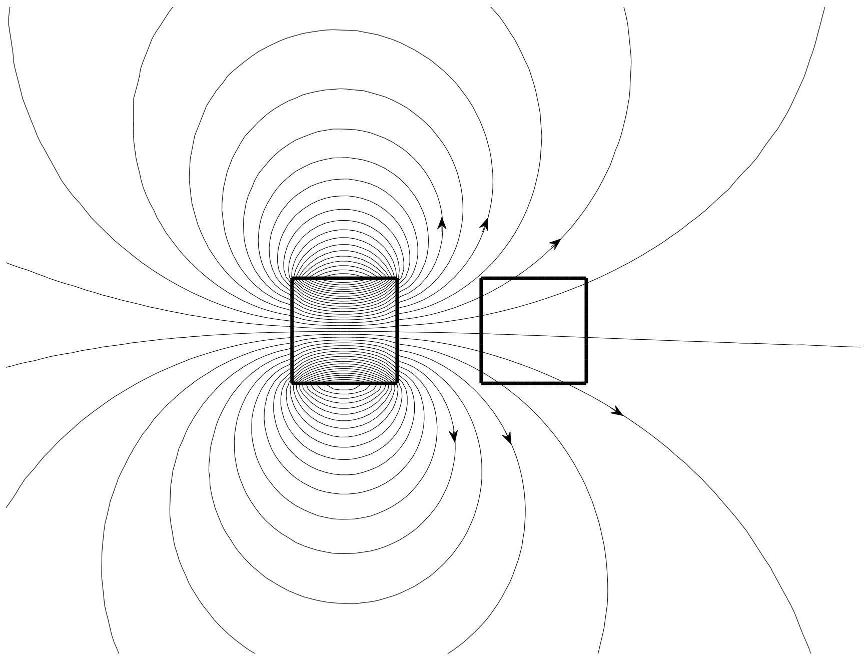


Figure 8.5: Field lines of the leftmost magnet in the system of Figure 8.4. The effect of the rightmost magnet on the magnetization of the leftmost magnet is included.

Chapter 9

Conclusions

In this thesis we have given definitions of electrostatic and magnetostatic forces that take into account the effect of finite test object on the charge and current distributions of the source object. We have constructed these definitions inside of an axiomatized geometry, or more precisely, inside of Riemannian geometry. This is done with an aim to generalize the classical methods for rigid objects to cover also situations where the interacting objects are deformable in the usual sense.

By considering force on finite interacting object as a covector, one accepts that the displacements of all the points of the object may be represented by a single vector. To express this formally one uses a constant virtual displacement vector field. Constancy of vector fields is taken to be a relative notion, and thus it is included in the theory as a separate structure called connection. To ensure that a constant vector field exists we assume a neighbourhood containing the object to be parallelizable. This is an additional requirement for the underlying Riemannian manifold, and it is sufficient for expressing the mathematical predicates defining electrostatic and magnetostatic forces. Intuitively, by using a parallelism we may add up covector-valued contributions to total force from different points of the object. This requirement is less stringent than the global parallelizability required by the Newton's law of action and reaction.

We have restricted the analysis to rigid objects, that is, objects whose virtual displacements do not distort distances and angles. To ensure that constant vector fields fullfill this property we have used a specific connection that is symmetric, and that is compatible with metric. This is the Levi-Civita connection. The virtual displacements of rigid objects are, in general, described by Killing vector fields. These vector fields are infinitesimal descriptions of continuous metric symmetries, which are translations and rotations in the case of Euclidean manifolds. Thus, to consider rigid rotations of an object, we assume the existence of Euclidean coordinates on a neighbourhood containing the object, and redefine force as a map taking Killing vector fields to real numbers.

Forces on material objects may be defined by using equivalent charges and equivalent currents. These terms describe the interacting objects as possessing certain kind of polarity as a whole. The corresponding terms describing distributed polarity inside of the objects are polarization and magnetization. The situation with magnetic forces is interesting because we have equivalent magnetic charges and equivalent electric currents as alternative primary terms. Corresponding to equivalent magnetic charges, there is magnetic polarization describing the distributed polarity. Magnetization and magnetic polarization have a different geometric character, and their relation requires the Hodge operator.

Microscopic counterparts of magnetization and magnetic polarization are virtual current loops and pairs of virtually displaced oppositely charged magnetic monopoles, respectively. That these are not pointlike particles is reflected to our analysis as the appearance of the Lie derivative – an operator that has a peculiar non-tensorial character. We have seen that magnetization and magnetic polarization yield equivalent definitions of magnetic force. In an abstract level this equivalence corresponds to the commutation of the Hodge operator and the Lie derivative with respect to a Killing vector field. Formally, we have

$$\mathcal{L}_v \circ \star = \star \circ \mathcal{L}_v,$$

for all Killing vector fields v. The Hodge operator is needed for the relation between the two kinds of primary terms, and Killing vector fields contain the requirement of rigidity enforced for the equivalence. From this it follows that also equivalent magnetic charges and equivalent electric currents yield equivalent definitions of magnetic forces. The equivalence of the alternative force definitions is thus structural. Although electric and magnetic forces on rigid objects may be defined in terms of polarization and magnetization (and magnetic polarization), the use of these primary terms does not solve the remaining problem concerning the definition of electric and magnetic force densities for the calculation of deformations.

We have attempted to extend the definitions of electric and magnetic forces to situations where the interacting objects are in contact with each other. We have approached such a contact situation by considering a sequence of situations where the separation of two distinct objects becomes infinitely small. The force at contact is defined to be the limit of forces as the separation parameter approaches zero. We have formulated conjectures that state how contact forces may be determined from quantities corresponding to the contact situation of zero separation parameter.

Finally, this study has guided us towards methods that have not yet been fully exploited in electromagnetic engineering. Namely, we aim to see electrostatic and magnetostatic interactions as instances of geometry. Indeed, we naturally have strong intuition about geometry, so thinking in geometric terms should be profitable. In practice, this would mean taking advantage of mathematical methods of Riemannian geometry in the computation of forces. In chapter 8 we have suggested a basis for such a geometrization in the case of magnetostatics.

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Appendix A

Basis representations of force densities and stresses

In this appendix we derive local basis representations of the force densities and the stress corresponding to electrostatic and magnetostatic cases. The results are given by using the classical vector analysis notation. We assume local Cartesian coordinates (holding both of the objects o_1 and o_2) with basis vector fields e_1, e_2, e_3 that are constant and orthonormal. The metric tensor is given in this basis by using the dual basis 1-forms $\omega^1, \omega^2, \omega^3$ as $g = \sum_{i=1}^{3} \omega^i \otimes \omega^i$. Its matrix is thus the identity. For vector fields u and v we thus have

$$g(u, v) = \sum_{i=1}^{3} \omega^i(u)\omega^i(v) = \sum_{i=1}^{3} u^i v^i = \boldsymbol{u} \cdot \boldsymbol{v},$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ stand for the triplets of components of u and v. A 1-form such as E_1 is written in this basis as $(E_1)_i \omega^i$, and its proxy-vector field has the components $g^{ij}(E_1)_j = (E_1)_i$. Because the components of the two coincide in the Cartesian coordinate basis we will not keep up here with the index placement.

We also specify orientation by the array (e_1, e_2, e_3) such that all twisted vectors and twisted forms will be represented by ordinary ones. In particular, the 3-form $\omega^1 \wedge \omega^2 \wedge \omega^3$ represents the volume density that takes in vector arrays such as (u, v, w) and produces their scalar triple product, that is

$$\omega^1 \wedge \omega^2 \wedge \omega^3(u, v, w) = \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}).$$

When (u, v, w) is positively oriented (such as when integrating the 3-form over a positively oriented submanifold) this is the volume of the parallelopiped formed by u, v, w.

We may represent a 2-form such as B_1 by using the 3-form $\omega^1 \wedge \omega^2 \wedge \omega^3$ and a vector field b such that

$$\begin{aligned} B_1 &= \mathrm{i}_b(\omega^1 \wedge \omega^2 \wedge \omega^3) \\ &= \omega^1(b)\omega^2 \wedge \omega^3 + \omega^2(b)\omega^3 \wedge \omega^1 + \omega^3(b)\omega^1 \wedge \omega^2 \\ &= b_1\omega^2 \wedge \omega^3 + b_2\omega^3 \wedge \omega^1 + b_3\omega^1 \wedge \omega^2 \end{aligned}$$

where we have made repeated use of the antiderivation property of interior product. We thus prefer to write B_1 in the basis as $(B_1)_1\omega^2 \wedge \omega^3 + (B_1)_2\omega^3 \wedge \omega^1 + (B_1)_3\omega^1 \wedge \omega^2$ instead of the expression $(B_1)_{12}\omega^1 \wedge \omega^2 + (B_1)_{13}\omega^1 \wedge \omega^3 + (B_1)_{23}\omega^2 \wedge \omega^3$ (that stems from the tensor notation). Note then that the 1-form $\star B_1$ is $(B_1)_i\omega^i$ so the proxy-vector field of this 1-form has the components $g^{ij}(B_1)_j = (B_1)_i$ in the Cartesian coordinate basis. Thus b is the proxy-vector field of $\star B_1$.

We will also need a coordinate expression for the exterior derivative. For this we use the antiderivation property of d. That is, for p-form ω and q-form η we have $\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^p \omega \wedge \mathrm{d}\eta$, see [1, 3]. Now, because $\mathrm{d}\omega^i = 0$ for $i = 1, 2, 3$, we have for a 1-form such as E_1

$$\begin{aligned} \mathrm{d}E_1 &= \mathrm{d}\big((E_1)_i\omega^i\big) \\ &= \mathrm{d}(E_1)_i \wedge \omega^i + (E_1)_i \mathrm{d}\omega^i \\ &= \mathrm{d}(E_1)_i(e_j)\omega^j \wedge \omega^i \\ &= \frac{\partial (E_1)i}{\partial x^j}\omega^j \wedge \omega^i \\ &= \big(\frac{\partial (E_1)_3}{\partial x^2} - \frac{\partial (E_1)_2}{\partial x^3}\big)\omega^2 \wedge \omega^3 + \big(\frac{\partial (E_1)_1}{\partial x^3} - \frac{\partial (E_1)_3}{\partial x^1}\big)\omega^3 \wedge \omega^1 \\ &\quad + \big(\frac{\partial (E_1)_2}{\partial x^1} - \frac{\partial (E_1)_1}{\partial x^2}\big)\omega^1 \wedge \omega^2, \end{aligned}$$

where x^1, x^2, x^3 stand for the Cartesian coordinates. Thus, denoting as $\boldsymbol{E}_1$ the triplet $((E_1)_1, (E_1)_2, (E_1)_3)$, the proxy-vector field of the 1-form $\star\mathrm{d}E_1$ has components given by the triplet $\nabla \times \boldsymbol{E}_1$. In a similar fashion we have for a 2-form such as B_1 the coordinate expression

$$\mathrm{d}B_1 = \big(\frac{\partial (B_1)_1}{\partial x^1} + \frac{\partial (B_1)_2}{\partial x^2} + \frac{\partial (B_1)_3}{\partial x^3}\big)\omega^1 \wedge \omega^2 \wedge \omega^3.$$

Thus, denoting as $\boldsymbol{B}_1$ the triplet $((B_1)_1, (B_1)_2, (B_1)_3)$, the component of the 3-form $\mathrm{d}B_1$ is $\nabla \cdot \boldsymbol{B}_1$.

A.1 Electrostatics

Here, we consider the two different expressions in electrostatics for the force densities $\tilde{F}_{12} + \tilde{F}_{21}$ and $\tilde{f}_{12} + \tilde{f}_{21}$, and the stress $\tilde{T}_{12} + \tilde{T}_{21}$.

A.1.1 Electric charge approach

To begin with, let us denote as $\boldsymbol{E}_1$ the triplet of components of E_1. Also, we use the notation $\mathrm{d}V$ for the volume 3-form $\omega^1 \wedge \omega^2 \wedge \omega^3$ and represent the charge density $\tilde{\rho}_2$ by the function ρ_2 as $\rho_2 \mathrm{d}V$. Note that $\mathrm{d}V$ is only a notation for the volume form – with no exterior derivative there. Then, by defining $\boldsymbol{E}_2$ and ρ_1 in similar way, we have

$$\begin{aligned}\mathcal{G}(\tilde{F}_{12} + \tilde{F}_{21}, v) &= \mathcal{G}(\tilde{F}_{12}, v) + \mathcal{G}(\tilde{F}_{21}, v)\\ &= \tilde{\rho}_2 \wedge \mathrm{i}_v E_1 + \tilde{\rho}_1 \wedge \mathrm{i}_v E_2\\ &= (E_1)_i v^i \rho_2 \mathrm{d}V + (E_2)_i v^i \rho_1 \mathrm{d}V\\ &= (\rho_2 \boldsymbol{E}_1 \cdot \boldsymbol{v})\mathrm{d}V + (\rho_1 \boldsymbol{E}_2 \cdot \boldsymbol{v})\mathrm{d}V\\ &= (\rho_2 \boldsymbol{E}_1 + \rho_1 \boldsymbol{E}_2) \cdot \boldsymbol{v}\mathrm{d}V,\end{aligned} \tag{A.1}$$

Thus the proxy-vector field of the volume force density has the components $\rho_2 \boldsymbol{E}_1 + \rho_1 \boldsymbol{E}_2$.

For the surface force density we first denote as n_1 and n_2 the outward unit normal vector fields of the transverse oriented surfaces ∂o_1 and ∂o_2, respectively. We further extend the domain of definition of n_1 to ∂o_2 by setting it zero there, and similarly for n_2. Then, we denote as n the sum $n_1 + n_2$, and use the symbol $\mathrm{d}A$ for the area 2-form $\mathrm{ti}_n \mathrm{d}V$. By representing $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ as $\sigma_1 \mathrm{d}A$ and $\sigma_2 \mathrm{d}A$, respectively, we get

$$\begin{aligned}\mathcal{G}(\tilde{f}_{12} + \tilde{f}_{21}, v) &= \mathcal{G}(\tilde{f}_{12}, v) + \mathcal{G}(\tilde{f}_{21}, v)\\ &= \tilde{\sigma}_2 \wedge \mathrm{t}_2 \mathrm{i}_v E_1 + \tilde{\sigma}_1 \wedge \mathrm{t}_1 \mathrm{i}_v E_2\\ &= (\sigma_2 \boldsymbol{E}_1 \cdot \boldsymbol{v})\mathrm{d}A + (\sigma_1 \boldsymbol{E}_2 \cdot \boldsymbol{v})\mathrm{d}A\\ &= (\sigma_2 \boldsymbol{E}_1 + \sigma_1 \boldsymbol{E}_2) \cdot \boldsymbol{v}\mathrm{d}A,\end{aligned} \tag{A.2}$$

where in the second row we have extended $\mathrm{t}_2 \mathrm{i}_v E_1$ to ∂o_1 by setting it zero there, and similarly for $\mathrm{t}_1 \mathrm{i}_v E_2$. From now on I will make such an extension whenever needed without separate notification. We thus find that the proxy-vector field of the surface force density has the components $\sigma_2 \boldsymbol{E}_1 + \sigma_1 \boldsymbol{E}_2$.

The representation of $\tilde{T}_{12} + \tilde{T}_{21}$ is not that straightforward. For this, we first express $\tilde{F}_{12} + \tilde{F}_{21}$ as $\tilde{F} - \tilde{F}_{11} - \tilde{F}_{22}$ to get

$$\begin{aligned}\mathcal{G}(\tilde{F}_{12} + \tilde{F}_{21}, v) &= \mathcal{G}(\tilde{F} - \tilde{F}_{11} - \tilde{F}_{22}, v) \\ &= \mathcal{G}(\tilde{F}, v) - \mathcal{G}(\tilde{F}_{11}, v) - \mathcal{G}(\tilde{F}_{22}, v) \\ &= \mathrm{d}\tilde{D} \wedge \mathrm{i}_v E - \mathrm{d}\tilde{D}_1 \wedge \mathrm{i}_v E_1 - \mathrm{d}\tilde{D}_2 \wedge \mathrm{i}_v E_2. \end{aligned} \tag{A.3}$$

Let us concentrate on the first term

$$\mathcal{G}(\tilde{F}, v) = \mathrm{d}\tilde{D} \wedge \mathrm{i}_v E = \mathrm{d}(\epsilon_0 \star E) \wedge \mathrm{i}_v E. \tag{A.4}$$

It turns out that this may be written in the standard basis as

$$\mathcal{G}(\tilde{F}, v) = \mathcal{G}(\mathrm{d}_\nabla \tilde{T}, v) = \omega^i(v)\mathrm{d}\tilde{T}_i = (\nabla \cdot \boldsymbol{T}) \cdot \boldsymbol{v}\mathrm{d}V, \tag{A.5}$$

where the matrix $\boldsymbol{T}$ is given in terms of the identity matrix $\bar{I}$ as

$$\boldsymbol{T} = \epsilon_0 \boldsymbol{E}\boldsymbol{E} - \frac{\epsilon_0}{2}(\boldsymbol{E} \cdot \boldsymbol{E})\bar{I}, \tag{A.6}$$

where $\boldsymbol{E}$ denotes the triplet of components of E. The calculation behind (A.5) and (A.6) is troublesome. Let us go through it in detail. Beginning from (A.4), and using the antiderivation property of d, we have

$$\begin{aligned}\mathcal{G}(\tilde{F}, v) &= \mathrm{d}(\epsilon_0 \star E) \wedge \mathrm{i}_v E \\ &= \omega^i(v)\mathrm{d}(\epsilon_0 \star E) \wedge E_i \\ &= \omega^i(v)\big(\mathrm{d}(\epsilon_0 \star E E_i) - \epsilon_0 \star E \wedge \mathrm{d}E_i\big). \end{aligned} \tag{A.7}$$

Note that here E_i is the ith component of the 1-form E, and not the electric field intensity of object i. Let us take under consideration the term $\epsilon_0 \star E \wedge \mathrm{d}E_1$. We try to give this as the exterior derivative of a 2-form. By expressing the 2-form $\epsilon_0 \star E$ and the 1-form $\mathrm{d}E_1$ in components as

$$\begin{aligned}\epsilon_0 \star E &= \epsilon_0 \star (E_i \omega^i) \\ &= \epsilon_0(E_1\omega^2 \wedge \omega^3 + E_2\omega^3 \wedge \omega^1 + E_3\omega^1 \wedge \omega^2) \end{aligned} \tag{A.8}$$

and

$$\mathrm{d}E_1 = \mathrm{d}E_1(e_i)\omega^i,$$

we may write

$$\begin{aligned}\epsilon_0 \star E \wedge \mathrm{d}E_1 &= \epsilon_0(E_1\omega^2 \wedge \omega^3 \wedge \mathrm{d}E_1 + E_2\omega^3 \wedge \omega^1 \wedge \mathrm{d}E_1 + E_3\omega^1 \wedge \omega^2 \wedge \mathrm{d}E_1) \\ &= \epsilon_0(E_1\mathrm{d}E_1(e_1)\omega^2 \wedge \omega^3 \wedge \omega^1 + E_2\mathrm{d}E_1(e_2)\omega^3 \wedge \omega^1 \wedge \omega^2 \\ &\quad + E_3\mathrm{d}E_1(e_3)\omega^1 \wedge \omega^2 \wedge \omega^3) \\ &= \epsilon_0\big(E_1\mathrm{d}E_1(e_1) + E_2\mathrm{d}E_1(e_2) + E_3\mathrm{d}E_1(e_3)\big)\mathrm{d}V,\end{aligned}$$

where the second and third equalities make use of the antisymmetry property of the exterior product. Let us next add and subtract the terms $\epsilon_0 E_2 \mathrm{d}E_2(e_1)\mathrm{d}V$ and $\epsilon_0 E_3 \mathrm{d}E_3(e_1)\mathrm{d}V$. We get

$$\begin{aligned}\epsilon_0 \star E \wedge \mathrm{d}E_1 = \epsilon_0\big(&E_1\mathrm{d}E_1(e_1) + E_2\mathrm{d}E_2(e_1) + E_3\mathrm{d}E_3(e_1))\mathrm{d}V \\ &+ \epsilon_0\big(E_3\mathrm{d}E_1(e_3) - E_3\mathrm{d}E_3(e_1) + E_2\mathrm{d}E_1(e_2) - E_2\mathrm{d}E_2(e_1)\big)\mathrm{d}V.\end{aligned} \tag{A.9}$$

In the first term we get the E_i's inside the derivative by noticing that $E_i\mathrm{d}E_i = \frac{1}{2}\mathrm{d}(E_i^2)$. Thus, the first term is

$$\epsilon_0\big(E_1\mathrm{d}E_1(e_1) + E_2\mathrm{d}E_2(e_1) + E_3\mathrm{d}E_3(e_1))\mathrm{d}V = \frac{\epsilon_0}{2}\mathrm{d}(E_1^2 + E_2^2 + E_3^2)(e_1)\mathrm{d}V.$$

The second term $\epsilon_0\big(E_3\mathrm{d}E_1(e_3) - E_3\mathrm{d}E_3(e_1) + E_2\mathrm{d}E_1(e_2) - E_2\mathrm{d}E_2(e_1)\big)\mathrm{d}V$ may be written more compactly as $\epsilon_0(E_3\omega^2 - E_2\omega^3) \wedge \mathrm{d}E$. To see this, we give $\mathrm{d}E$ in the basis as $\mathrm{d}E = \mathrm{d}E_i \wedge \omega^i$. In this way we get

$$\begin{aligned}\epsilon_0(E_3\omega^2 - E_2\omega^3) \wedge \mathrm{d}E &= \epsilon_0(E_3\mathrm{d}E_1 \wedge \omega^1 \wedge \omega^2 - E_3\mathrm{d}E_3 \wedge \omega^2 \wedge \omega^3 \\ &\quad + E_2\mathrm{d}E_1 \wedge \omega^3 \wedge \omega^1 - E_2\mathrm{d}E_2 \wedge \omega^2 \wedge \omega^3) \\ &= \epsilon_0(E_3\mathrm{d}E_1(e_3)\omega^3 \wedge \omega^1 \wedge \omega^2 - E_3\mathrm{d}E_3(e_1)\omega^1 \wedge \omega^2 \wedge \omega^3 \\ &\quad + E_2\mathrm{d}E_1(e_2)\omega^2 \wedge \omega^3 \wedge \omega^1 - E_2\mathrm{d}E_2(e_1)\omega^1 \wedge \omega^2 \wedge \omega^3) \\ &= \epsilon_0\big(E_3\mathrm{d}E_1(e_3) - E_3\mathrm{d}E_3(e_1) + E_2\mathrm{d}E_1(e_2) \\ &\quad - E_2\mathrm{d}E_2(e_1)\big)\mathrm{d}V\end{aligned}$$

as promised. Thus, because $\mathrm{d}E = 0$, the second term in (A.9) vanishes, and we have

$$\epsilon_0 \star E \wedge \mathrm{d}E_1 = \frac{\epsilon_0}{2}\mathrm{d}(E_1^2 + E_2^2 + E_3^2)(e_1)\mathrm{d}V.$$

Finally, this may be written as

$$\begin{aligned}\epsilon_0 \star E \wedge \mathrm{d}E_1 &= \frac{\epsilon_0}{2}\mathrm{d}(E_1^2 + E_2^2 + E_3^2)(e_1)\mathrm{d}V \\ &= \frac{\epsilon_0}{2}\mathrm{d}(E_1^2 + E_2^2 + E_3^2)(e_1)\omega^1 \wedge \omega^2 \wedge \omega^3 \\ &= \frac{\epsilon_0}{2}\mathrm{d}(E_1^2 + E_2^2 + E_3^2)(e_i)\omega^i \wedge \omega^2 \wedge \omega^3 \\ &= \frac{\epsilon_0}{2}\mathrm{d}(E_1^2 + E_2^2 + E_3^2) \wedge \omega^2 \wedge \omega^3 \\ &= \mathrm{d}\Big(\frac{\epsilon_0}{2}(E_1^2 + E_2^2 + E_3^2)\omega^2 \wedge \omega^3\Big).\end{aligned}$$

In the same way we have for the terms $\epsilon_0 \star E \wedge \mathrm{d}E_2$ and $\epsilon_0 \star E \wedge \mathrm{d}E_3$ in (A.7)

$$\begin{aligned}\epsilon_0 \star E \wedge \mathrm{d}E_2 &= \mathrm{d}\Big(\frac{\epsilon_0}{2}(E_1^2 + E_2^2 + E_3^2)\omega^3 \wedge \omega^1\Big),\\ \epsilon_0 \star E \wedge \mathrm{d}E_3 &= \mathrm{d}\Big(\frac{\epsilon_0}{2}(E_1^2 + E_2^2 + E_3^2)\omega^1 \wedge \omega^2\Big).\end{aligned}$$

Using these and (A.8) in (A.7), we get

$$\mathcal{G}(\tilde{F}, v) = \omega^i(v)\mathrm{d}\tilde{T}_i,$$

where

$$\begin{aligned}\tilde{T}_1 &= \Big(\epsilon_0 E_1^2 - \frac{\epsilon_0}{2}(E_1^2 + E_2^2 + E_3^2)\Big)\omega^2 \wedge \omega^3 + \epsilon_0 E_2 E_1 \omega^3 \wedge \omega^1 + \epsilon_0 E_3 E_1 \omega^1 \wedge \omega^2,\\ \tilde{T}_2 &= \epsilon_0 E_1 E_2 \omega^2 \wedge \omega^3 + \Big(\epsilon_0 E_2^2 - \frac{\epsilon_0}{2}(E_1^2 + E_2^2 + E_3^2)\Big)\omega^3 \wedge \omega^1 + \epsilon_0 E_3 E_2 \omega^1 \wedge \omega^2,\\ \tilde{T}_2 &= \epsilon_0 E_1 E_3 \omega^2 \wedge \omega^3 + \epsilon_0 E_2 E_3 \omega^3 \wedge \omega^1 + \Big(\epsilon_0 E_3^2 - \frac{\epsilon_0}{2}(E_1^2 + E_2^2 + E_3^2)\Big)\omega^1 \wedge \omega^2.\end{aligned}$$

By definition $\mathrm{d}_\nabla \tilde{T} = \tilde{F}$, so we have

$$\mathcal{G}(\mathrm{d}_\nabla \tilde{T}, v) = \omega^i(v)\mathrm{d}\tilde{T}_i,$$

and because the basis 1-forms are constant this implies that the $\tilde{T}_i$'s are the component 2-forms of $\tilde{T}$. That is, we have

$$\mathcal{G}(\tilde{T}, v) = \omega^i(v)\tilde{T}_i.$$

Finally, to get to the matrix notation above, we write $\tilde{T}_i = \tilde{T}_i^1 \omega^2 \wedge \omega^3 + \tilde{T}_i^2 \omega^3 \wedge \omega^1 + \tilde{T}_i^3 \omega^1 \wedge \omega^2$, and express the exterior derivative of this as

$$\begin{aligned}\mathrm{d}\tilde{T}_i &= \mathrm{d}\tilde{T}_i^1 \wedge \omega^2 \wedge \omega^3 + \mathrm{d}\tilde{T}_i^2 \wedge \omega^3 \wedge \omega^1 + \mathrm{d}\tilde{T}_i^3 \wedge \omega^1 \wedge \omega^2\\ &= \big(\mathrm{d}\tilde{T}_i^1(e_1) + \mathrm{d}\tilde{T}_i^2(e_2) + \mathrm{d}\tilde{T}_i^3(e_3)\big)\omega^1 \wedge \omega^2 \wedge \omega^3\\ &= \partial_j \tilde{T}_i^j \mathrm{d}V,\end{aligned}$$

where ∂_j denotes the partial derivative with respect to the jth coordinate. By further denoting as $\boldsymbol{T}$ the matrix $\tilde{T}_i^j$ (i the row index and j the column index), and using the notation $(\nabla \cdot \boldsymbol{T})_i$ for $\partial_j \tilde{T}_i^j$, we may write

$$\begin{aligned}\mathcal{G}(\tilde{F}, v) &= \omega^i(v)\mathrm{d}\tilde{T}_i\\ &= \omega^i(v)(\nabla \cdot \boldsymbol{T})_i \mathrm{d}V\\ &= v^i(\nabla \cdot \boldsymbol{T})_i \mathrm{d}V\\ &= (\nabla \cdot \boldsymbol{T}) \cdot \boldsymbol{v} \mathrm{d}V\end{aligned}$$

where $\nabla \cdot \boldsymbol{T}$ is the triplet $\big((\nabla \cdot \boldsymbol{T})_1, (\nabla \cdot \boldsymbol{T})_2, (\nabla \cdot \boldsymbol{T})_3\big)$. Note that the matrix $\boldsymbol{T}$ used here is the transpose of that used, for instance, in [15, 16]. This difference is accompanied by a difference in the definition of $\nabla \cdot \boldsymbol{T}$. Here we have defined $(\nabla \cdot \boldsymbol{T})_i = \partial_j \tilde{T}_i^j$ in contrast to the definition $(\nabla \cdot \boldsymbol{T})_i = \partial_j T_j^i$ used in the above references.

Going back to (A.3) we observe that also the elements of pairs $(E_1, \tilde{D}_1)$ and $(E_2, \tilde{D}_2)$ are related by the Hodge operator, and that $\mathrm{d}E_1$ and $\mathrm{d}E_2$ vanish just like $\mathrm{d}E$. It follows that the above procedure may be carried out also for the second and third terms on the right hand side of (A.3) to obtain matrices

$$\boldsymbol{T}_{11} = \epsilon_0 \boldsymbol{E}_1 \boldsymbol{E}_1 - \frac{\epsilon_0}{2}(\boldsymbol{E}_1 \cdot \boldsymbol{E}_1)\bar{I}, \tag{A.10}$$

$$\boldsymbol{T}_{22} = \epsilon_0 \boldsymbol{E}_2 \boldsymbol{E}_2 - \frac{\epsilon_0}{2}(\boldsymbol{E}_2 \cdot \boldsymbol{E}_2)\bar{I}. \tag{A.11}$$

Finally, by inserting the results back to (A.3), we find

$$\begin{aligned} \mathcal{G}(\tilde{F}_{12} + \tilde{F}_{21}, v) &= \mathcal{G}(\mathrm{d}_\nabla \tilde{T}, v) - \mathcal{G}(\mathrm{d}_\nabla \tilde{T}_{11}, v) - \mathcal{G}(\mathrm{d}_\nabla \tilde{T}_{22}, v) \\ &= \mathcal{G}\big(\mathrm{d}_\nabla (\tilde{T} - \tilde{T}_{11} - \tilde{T}_{22}), v\big) \\ &= \big(\nabla \cdot (\boldsymbol{T} - \boldsymbol{T}_{11} - \boldsymbol{T}_{22})\big) \cdot \boldsymbol{v} \mathrm{d}V, \end{aligned} \tag{A.12}$$

and we observe that the stress $\tilde{T}_{12} + \tilde{T}_{21}$ is represented by the matrix $\boldsymbol{T} - \boldsymbol{T}_{11} - \boldsymbol{T}_{22}$.

It may be instructive to consider the basis representation of the tangential trace of, for instance, $\tilde{T}$. Since the tangential trace operates only on the 2-form part of $\tilde{T}$, we have

$$\mathcal{G}(\mathrm{t}\tilde{T}, v) = \mathrm{t}(\omega^i(v)\tilde{T}_i).$$

This is a 2-form on $\partial o_1 \cup \partial o_2$ so we may express it in the basis provided by the area 2-form $\mathrm{d}A$. For this, we take the outward unit normal n, and consider the 2-vector field $\star n$. This is a 2-vector field defined at points of the space manifold Ω. However, because it is tangent to $\partial o_1 \cup \partial o_2$ we may consider it as a 2-vector field on the surface $\partial o_1 \cup \partial o_2$. To avoid confusion we denote as σ the 2-vector field $\star n$ considered as a 2-vector field on $\partial o_1 \cup \partial o_2$. Let us give σ to the 2-form $\mathrm{d}A$. By making repeated use of the antiderivation property of the interior product we get

$$\begin{aligned}
\mathrm{d}A(\sigma) &= \mathrm{t}\mathrm{i}_n \mathrm{d}V(\sigma) \\
&= \mathrm{i}_n \mathrm{d}V(\star n) \\
&= \mathrm{i}_n(\omega^1 \wedge \omega^2 \wedge \omega^3)(\star n) \\
&= \big(\mathrm{i}_n\omega^1 \wedge \omega^2 \wedge \omega^3 - \omega^1 \wedge \mathrm{i}_n(\omega^2 \wedge \omega^3)\big)(\star n) \\
&= (\mathrm{i}_n\omega^1 \wedge \omega^2 \wedge \omega^3 - \omega^1 \wedge \mathrm{i}_n\omega^2 \wedge \omega^3 \\
&\quad + \omega^1 \wedge \omega^2 \wedge \mathrm{i}_n\omega^3)(\star n) \\
&= n^1\omega^2 \wedge \omega^3(\star n) + n^2\omega^3 \wedge \omega^1(\star n) + n^3\omega^1 \wedge \omega^2(\star n) \\
&= n^1\omega^1(n) + n^2\omega^2(n) + n^3\omega^3(n) \\
&= \boldsymbol{n} \cdot \boldsymbol{n} = 1.
\end{aligned}$$

We may thus express $\mathrm{t}(\omega^i(v)\tilde{T}_i)$ as

$$\begin{aligned}
\mathrm{t}(\omega^i(v)\tilde{T}_i) &= \mathrm{t}(\omega^i(v)\tilde{T}_i)(\sigma)\mathrm{d}A \\
&= \omega^i(v)\tilde{T}_i(\star n)\mathrm{d}A \\
&= \omega^i(v)(\tilde{T}_i^1\omega^2 \wedge \omega^3 + \tilde{T}_i^2\omega^3 \wedge \omega^1 + \tilde{T}_i^3\omega^1 \wedge \omega^2)(\star n)\mathrm{d}A \\
&= \omega^i(v)(\tilde{T}_i^1\omega^1(n) + \tilde{T}_i^2\omega^2(n) + \tilde{T}_i^3\omega^3(n))\mathrm{d}A \\
&= \big(\boldsymbol{v} \cdot (\boldsymbol{T} \cdot \boldsymbol{n})\big)\mathrm{d}A.
\end{aligned}$$

Thus the matrix $\boldsymbol{T}$ takes in $\boldsymbol{n}$ from the right according to our convention. Finally, we note that the Stokes' theorem "$\int_V \mathrm{d}_\nabla \tilde{T} = \int_{\partial V} \tilde{T}$ for all volumes V over which $\tilde{T}$ is smooth" translates to

$$\int_V \nabla \cdot \boldsymbol{T} \mathrm{d}V = \int_{\partial V} \boldsymbol{T} \cdot \boldsymbol{n} \mathrm{d}A \tag{A.13}$$

for all volumes V over which $\boldsymbol{T}$ is smooth.

A.1.2 Electric polarization approach

When dielectric materials are modeled by electric polarization (section 6.2) we have

$$\begin{aligned}
\mathcal{G}(\tilde{F}_{12} + \tilde{F}_{21}, v) &= \mathcal{G}(\tilde{F}_{12}, v) + \mathcal{G}(\tilde{F}_{21}, v) \\
&= \tilde{\rho}_2^f \wedge \mathrm{i}_v E_1 + \tilde{P}_2 \wedge \nabla_v E_1 + \tilde{\rho}_1^f \wedge \mathrm{i}_v E_2 + \tilde{P}_1 \wedge \nabla_v E_2.
\end{aligned} \tag{A.14}$$

For the basis representation, let us first consider the second term on the right hand side. By representing the polarization $\tilde{P}_2$ as $(\tilde{P}_2)_1\omega^2 \wedge \omega^3 + (\tilde{P}_2)_2\omega^3 \wedge \omega^1 +$

$(\tilde{P}_2)_3\omega^1 \wedge \omega^2$, and by using the symbol $\boldsymbol{P}_2$ for the triplet of the components $(\tilde{P}_2)_i$, we have in the standard basis

$$\begin{aligned}
\tilde{P}_2 \wedge \nabla_v E_1 &= \tilde{P}_2 \wedge \Big(\nabla_v\big((E_1)_i\big)\omega^i + (E_1)_i \nabla_v \omega^i\Big) \\
&= \tilde{P}_2 \wedge \mathrm{d}\big((E_1)_i\big)(v)\omega^i \\
&= \tilde{P}_2 \wedge v^j \mathrm{d}\big((E_1)_i\big)(e_j)\omega^i \\
&= \tilde{P}_2 \wedge v^j \partial_j\big((E_1)_i\big)\omega^i \\
&= \Big((\tilde{P}_2)_1 v^j \partial_j\big((E_1)_1\big) + (\tilde{P}_2)_2 v^j \partial_j\big((E_1)_2\big) + (\tilde{P}_2)_3 v^j \partial_j\big((E_1)_3\big)\Big)\mathrm{d}V \\
&= \big(\boldsymbol{P}_2 \cdot (\boldsymbol{v} \cdot \nabla \boldsymbol{E}_1)\big)\mathrm{d}V, \qquad \text{(A.15)}
\end{aligned}$$

where the last equality defines the notation such that $(\boldsymbol{v}\cdot\nabla\boldsymbol{E}_1)_i = v^j\partial_j\big((E_1)_i\big)$. Note that in the second equality we have taken into account that the basis 1-forms are constant. We thus have for (A.14) the basis expression

$$\begin{aligned}
\mathcal{G}(\tilde{F}_{12} + \tilde{F}_{21}, v) =& \big(\rho_2^f \boldsymbol{E}_1 \cdot \boldsymbol{v} + \boldsymbol{P}_2 \cdot (\boldsymbol{v} \cdot \nabla \boldsymbol{E}_1) \\
&+ \rho_1^f \boldsymbol{E}_2 \cdot \boldsymbol{v} + \boldsymbol{P}_1 \cdot (\boldsymbol{v} \cdot \nabla \boldsymbol{E}_2)\big)\mathrm{d}V. \qquad \text{(A.16)}
\end{aligned}$$

To find a convenient formula for the force density proxy-vector field we first take under consideration the term $\boldsymbol{P}_2 \cdot (\boldsymbol{v} \cdot \nabla \boldsymbol{E}_1)$. Since $\mathrm{d}E_1$ vanish (implying that $\nabla \times \boldsymbol{E}_1$ vanish), we have

$$\begin{aligned}
\boldsymbol{P}_2 \cdot (\boldsymbol{v} \cdot \nabla \boldsymbol{E}_1) &= (\boldsymbol{P}_2 \cdot \nabla \boldsymbol{E}_1) \cdot \boldsymbol{v} - \boldsymbol{P}_2 \cdot (\nabla \times \boldsymbol{E}_1) \times \boldsymbol{v} \\
&= (\boldsymbol{P}_2 \cdot \nabla \boldsymbol{E}_1) \cdot \boldsymbol{v}, \qquad \text{(A.17)}
\end{aligned}$$

where the first equality can be verified by direct calculation. Accordingly, we may write (A.16) as

$$\mathcal{G}(\tilde{F}_{12} + \tilde{F}_{21}, v) = \big(\rho_2^f \boldsymbol{E}_1 + \boldsymbol{P}_2 \cdot \nabla \boldsymbol{E}_1 + \rho_1^f \boldsymbol{E}_2 + \boldsymbol{P}_1 \cdot \nabla \boldsymbol{E}_2\big) \cdot \boldsymbol{v}\mathrm{d}V, \qquad \text{(A.18)}$$

so that the proxy-vector field of volume force density has the components $\rho_2^f \boldsymbol{E}_1 + \boldsymbol{P}_2 \cdot \nabla \boldsymbol{E}_1 + \rho_1^f \boldsymbol{E}_2 + \boldsymbol{P}_1 \cdot \nabla \boldsymbol{E}_2$.

For the representation of the stress $\tilde{T}_{12} + \tilde{T}_{21}$ we proceed in the same way as in section A.1.1. That is, we first express $\tilde{F}_{12} + \tilde{F}_{21}$ as $\tilde{F} - \tilde{F}_{11} - \tilde{F}_{22}$ to get

$$\begin{aligned}
\mathcal{G}(\tilde{F}_{12} + \tilde{F}_{21}, v) &= \mathcal{G}(\tilde{F} - \tilde{F}_{11} - \tilde{F}_{22}, v) \\
&= \mathcal{G}(\tilde{F}, v) - \mathcal{G}(\tilde{F}_{11}, v) - \mathcal{G}(\tilde{F}_{22}, v) \\
&= \mathrm{d}\tilde{D}' \wedge \mathrm{i}_v E + \tilde{P} \wedge \nabla_v E - (\mathrm{d}\tilde{D}'_1 \wedge \mathrm{i}_v E_1 + \tilde{P}_1 \wedge \nabla_v E_1) \\
&\quad - (\mathrm{d}\tilde{D}'_2 \wedge \mathrm{i}_v E_2 + \tilde{P}_2 \wedge \nabla_v E_2). \qquad \text{(A.19)}
\end{aligned}$$

Then, by taking into account that $\mathrm{d}E$, $\mathrm{d}E_1$ and $\mathrm{d}E_2$ vanish, we find that the above may be written in the standard basis as

$$\begin{aligned}\mathcal{G}(\tilde{F}_{12}+\tilde{F}_{21},v) &= \mathcal{G}\big(\mathrm{d}_\nabla(\tilde{T}-\tilde{T}_{11}-\tilde{T}_{22}),v\big)\\ &= \big(\nabla\cdot(\boldsymbol{T}-\boldsymbol{T}_{11}-\boldsymbol{T}_{22})\big)\cdot\boldsymbol{v}\mathrm{d}V,\end{aligned} \tag{A.20}$$

where

$$\boldsymbol{T} = \boldsymbol{E}\boldsymbol{D}' - \frac{\epsilon_0}{2}(\boldsymbol{E}\cdot\boldsymbol{E})\bar{I}, \tag{A.21}$$

$$\boldsymbol{T}_{11} = \boldsymbol{E}_1\boldsymbol{D}_1' - \frac{\epsilon_0}{2}(\boldsymbol{E}_1\cdot\boldsymbol{E}_1)\bar{I}, \tag{A.22}$$

$$\boldsymbol{T}_{22} = \boldsymbol{E}_2\boldsymbol{D}_2' - \frac{\epsilon_0}{2}(\boldsymbol{E}_2\cdot\boldsymbol{E}_2)\bar{I}. \tag{A.23}$$

A.2 Magnetostatics

For completeness, here we consider all four expressions for $\tilde{F}_{12}+\tilde{F}_{21}$, $\tilde{f}_{12}+\tilde{f}_{21}$ and $\tilde{T}_{12}+\tilde{T}_{21}$ in magnetostatics.

A.2.1 Electric current approach

According to section 3.2 we have

$$\begin{aligned}\mathcal{G}(\tilde{F}_{12}+\tilde{F}_{21},v) &= \mathcal{G}(\tilde{F}_{12},v)+\mathcal{G}(\tilde{F}_{21},v)\\ &= \tilde{J}_2\wedge \mathrm{i}_v B_1 + \tilde{J}_1\wedge \mathrm{i}_v B_2.\end{aligned} \tag{A.24}$$

Let us consider the term $\tilde{J}_2\wedge \mathrm{i}_v B_1$. For the basis representation we represent the current density $\tilde{J}_2$ as $(\tilde{J}_2)_1\omega^2\wedge\omega^3+(\tilde{J}_2)_2\omega^3\wedge\omega^1+(\tilde{J}_2)_3\omega^1\wedge\omega^2$, and denote the triplet of the components as $\boldsymbol{J}_2$. We also represent B_1 as $(B_1)_1\omega^2\wedge\omega^3+(B_1)_2\omega^3\wedge\omega^1+(B_1)_3\omega^1\wedge\omega^2$, and denote the triplet of the components as $\boldsymbol{B}_1$.

We then have

$$
\begin{aligned}
\tilde{J}_2 \wedge \mathrm{i}_v B_1 &= \tilde{J}_2 \wedge \mathrm{i}_v\big((B_1)_1\omega^2 \wedge \omega^3 + (B_1)_2\omega^3 \wedge \omega^1 + (B_1)_3\omega^1 \wedge \omega^2\big) \\
&= \tilde{J}_2 \wedge \big((B_1)_1 \mathrm{i}_v\omega^2 \wedge \omega^3 - (B_1)_1\omega^2 \wedge \mathrm{i}_v\omega^3 \\
&\quad + (B_1)_2 \mathrm{i}_v\omega^3 \wedge \omega^1 - (B_1)_2\omega^3 \wedge \mathrm{i}_v\omega^1 \\
&\quad + (B_1)_3 \mathrm{i}_v\omega^1 \wedge \omega^2 - (B_1)_3\omega^1 \wedge \mathrm{i}_v\omega^2\big) \\
&= \tilde{J}_2 \wedge \Big(\big((B_1)_2 v^3 - (B_1)_3 v^2\big)\omega^1 + \big((B_1)_3 v^1 - (B_1)_1 v^3\big)\omega^2 \\
&\quad + \big((B_1)_1 v^2 - (B_1)_2 v^1\big)\omega^3\Big) \\
&= \Big((\tilde{J}_2)_1\big((B_1)_2 v^3 - (B_1)_3 v^2\big) + (\tilde{J}_2)_2\big((B_1)_3 v^1 - (B_1)_1 v^3\big) \\
&\quad + (\tilde{J}_2)_3\big((B_1)_1 v^2 - (B_1)_2 v^1\big)\Big)\mathrm{d}V \\
&= \boldsymbol{J}_2 \cdot (\boldsymbol{B}_1 \times \boldsymbol{v})\mathrm{d}V \\
&= \boldsymbol{J}_2 \times \boldsymbol{B}_1 \cdot \boldsymbol{v}\mathrm{d}V. \qquad \text{(A.25)}
\end{aligned}
$$

By defining $\boldsymbol{J}_1$ and $\boldsymbol{B}_2$ in the same way as $\boldsymbol{J}_2$ and $\boldsymbol{B}_1$, we get for (A.24) the basis expression

$$
\mathcal{G}(\tilde{F}_{12} + \tilde{F}_{21}, v) = (\boldsymbol{J}_2 \times \boldsymbol{B}_1 + \boldsymbol{J}_1 \times \boldsymbol{B}_2) \cdot \boldsymbol{v}\mathrm{d}V, \qquad \text{(A.26)}
$$

so that the proxy-vector field of the volume force density has the components $\boldsymbol{J}_2 \times \boldsymbol{B}_1 + \boldsymbol{J}_1 \times \boldsymbol{B}_2$.

For the basis representation of the surface force density $\tilde{j}_2$ we first express it in the form $\mathrm{t}_2\hat{j}_2$, where $\hat{j}_2$ is a 1-form on Ω. This 1-form is defined at the points of ∂o_2 (considered as points of Ω). We require that its normal component vanishes, that is, $\mathrm{i}_{n_2}\hat{j}_2 = 0$. The surface current density $\tilde{j}_1$ is expressed similarly as $\mathrm{t}_1\hat{j}_1$. Then, according to section 3.2, we have

$$
\begin{aligned}
\mathcal{G}(\tilde{f}_{12} + \tilde{f}_{21}, v) &= \mathcal{G}(\tilde{f}_{12}, v) + \mathcal{G}(\tilde{f}_{21}, v) \\
&= \tilde{j}_2 \wedge \mathrm{t}_2 \mathrm{i}_v B_1 + \tilde{j}_1 \wedge \mathrm{t}_1 \mathrm{i}_v B_2 \\
&= \mathrm{t}_2\hat{j}_2 \wedge \mathrm{t}_2 \mathrm{i}_v B_1 + \mathrm{t}_1\hat{j}_1 \wedge \mathrm{t}_1 \mathrm{i}_v B_2 \\
&= \mathrm{t}_2(\hat{j}_2 \wedge \mathrm{i}_v B_1) + \mathrm{t}_1(\hat{j}_1 \wedge \mathrm{i}_v B_2), \qquad \text{(A.27)}
\end{aligned}
$$

where the last equality is a property of tangential trace [1, 3]. Let us focus on the term $\mathrm{t}_2(\hat{j}_2 \wedge \mathrm{i}_v B_1)$. We can express this 2-form by using the area form $\mathrm{d}A$ and the unit 2-vector field σ as in section A.1.1 to get

$$
\begin{aligned}
\mathrm{t}_2(\hat{j}_2 \wedge \mathrm{i}_v B_1) &= \mathrm{t}_2(\hat{j}_2 \wedge \mathrm{i}_v B_1)(\sigma)\mathrm{d}A \\
&= (\hat{j}_2 \wedge \mathrm{i}_v B_1)(\star n)\mathrm{d}A. \qquad \text{(A.28)}
\end{aligned}
$$

Next we use a triplet $\boldsymbol{j}_2$ to express the components of $\hat{j}_2$ by the triplet $-\boldsymbol{n}\times\boldsymbol{j}_2$. Thus

$$\hat{j}_2 = (n^3j^2 - n^2j^3)\omega^1 + (n^1j^3 - n^3j^1)\omega^2 + (n^2j^1 - n^1j^2)\omega^3.$$

We can write the 2-form $\hat{j}_2 \wedge \mathrm{i}_v B_1$ as

$$\begin{aligned}
\hat{j}_2 \wedge \mathrm{i}_v B_1 =& \Big((n^1j^3 - n^3j^1)\big((B_1)_1 v^2 - (B_1)_2 v^1\big) \\
&- (n^2j^1 - n^1j^2)\big((B_1)_3 v^1 - (B_1)_1 v^3\big)\Big)\omega^2 \wedge \omega^3 \\
&+ \Big((n^2j^1 - n^1j^2)\big((B_1)_2 v^3 - (B_1)_3 v^2\big) \\
&- (n^3j^2 - n^2j^3)\big((B_1)_1 v^2 - (B_1)_2 v^1\big)\Big)\omega^3 \wedge \omega^1 \\
&+ \Big((n^3j^2 - n^2j^3)\big((B_1)_3 v^1 - (B_1)_1 v^3\big) \\
&- (n^1j^3 - n^3j^1)\big((B_1)_2 v^3 - (B_1)_3 v^2\big)\Big)\omega^1 \wedge \omega^2.
\end{aligned}$$

Using this in (A.28) we get

$$\begin{aligned}
\mathrm{t}_2(\hat{j}_2 \wedge \mathrm{i}_v B_1) &= (-\boldsymbol{n} \times \boldsymbol{j}_2) \times (\boldsymbol{B}_1 \times \boldsymbol{v}) \cdot \boldsymbol{n}\mathrm{d}A \\
&= (\boldsymbol{B}_1 \times \boldsymbol{v}) \times (\boldsymbol{n} \times \boldsymbol{j}_2) \cdot \boldsymbol{n}\mathrm{d}A.
\end{aligned}$$

Finally, by using the vector analysis formula $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$, we have

$$\begin{aligned}
\mathrm{t}_2(\hat{j}_2 \wedge \mathrm{i}_v B_1) &= \big(\boldsymbol{n}(\boldsymbol{B}_1 \times \boldsymbol{v} \cdot \boldsymbol{j}_2) - (\boldsymbol{n} \times \boldsymbol{j}_2)(\boldsymbol{B}_1 \times \boldsymbol{v} \cdot \boldsymbol{n})\big) \cdot \boldsymbol{n}\mathrm{d}A \\
&= \boldsymbol{B}_1 \times \boldsymbol{v} \cdot \boldsymbol{j}_2 \mathrm{d}A \\
&= \boldsymbol{j}_2 \times \boldsymbol{B}_1 \cdot \boldsymbol{v}\mathrm{d}A, \qquad \text{(A.29)}
\end{aligned}$$

where the second equality follows because $\boldsymbol{n} \times \boldsymbol{j}_2 \cdot \boldsymbol{n} = 0$ (the normal component of $\hat{j}_2$ vanishes). By performing similar calculation with the second term in (A.27), and defining $\boldsymbol{j}_1$ in the same way as $\boldsymbol{j}_2$, we have

$$\mathcal{G}(\tilde{f}_{12} + \tilde{f}_{21}, v) = (\boldsymbol{j}_2 \times \boldsymbol{B}_1 + \boldsymbol{j}_1 \times \boldsymbol{B}_2) \cdot \boldsymbol{v}\mathrm{d}A, \qquad \text{(A.30)}$$

so that the proxy-vector field of the surface force density has the components $\boldsymbol{j}_2 \times \boldsymbol{B}_1 + \boldsymbol{j}_1 \times \boldsymbol{B}_2$.

For the representation of the stress $\tilde{T}_{12} + \tilde{T}_{21}$ we proceed on familiar lines and first express $\tilde{F}_{12} + \tilde{F}_{21}$ as $\tilde{F} - \tilde{F}_{11} - \tilde{F}_{22}$. This yields

$$\begin{aligned}
\mathcal{G}(\tilde{F}_{12} + \tilde{F}_{21}, v) &= \mathcal{G}(\tilde{F} - \tilde{F}_{11} - \tilde{F}_{22}, v) \\
&= \mathcal{G}(\tilde{F}, v) - \mathcal{G}(\tilde{F}_{11}, v) - \mathcal{G}(\tilde{F}_{22}, v) \\
&= \mathrm{d}\tilde{H} \wedge \mathrm{i}_v B - \mathrm{d}\tilde{H}_1 \wedge \mathrm{i}_v B_1 - \mathrm{d}\tilde{H}_2 \wedge \mathrm{i}_v B_2. \qquad \text{(A.31)}
\end{aligned}$$

By taking into account that $\mathrm{d}B$, $\mathrm{d}B_1$ and $\mathrm{d}B_2$ vanish, we find that the above may be written in the standard basis as

$$\begin{aligned}\mathcal{G}(\tilde{F}_{12}+\tilde{F}_{21},v) &= \mathcal{G}\big(\mathrm{d}_\nabla(\tilde{T}-\tilde{T}_{11}-\tilde{T}_{22}),v\big)\\ &= \big(\nabla\cdot(\boldsymbol{T}-\boldsymbol{T}_{11}-\boldsymbol{T}_{22})\big)\cdot\boldsymbol{v}\mathrm{d}V, \end{aligned} \tag{A.32}$$

where

$$\boldsymbol{T} = \frac{1}{\mu_0}\boldsymbol{B}\boldsymbol{B} - \frac{1}{2\mu_0}(\boldsymbol{B}\cdot\boldsymbol{B})\bar{I}, \tag{A.33}$$

$$\boldsymbol{T}_{11} = \frac{1}{\mu_0}\boldsymbol{B}_1\boldsymbol{B}_1 - \frac{1}{2\mu_0}(\boldsymbol{B}_1\cdot\boldsymbol{B}_1)\bar{I}, \tag{A.34}$$

$$\boldsymbol{T}_{22} = \frac{1}{\mu_0}\boldsymbol{B}_2\boldsymbol{B}_2 - \frac{1}{2\mu_0}(\boldsymbol{B}_2\cdot\boldsymbol{B}_2)\bar{I}. \tag{A.35}$$

A.2.2 Magnetic charge approach

In addition to our earlier notation, let us denote as $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ the triplets of the components of $\tilde{H}_1$ and $\tilde{H}_2$, respectively. Further, I will abuse the notation by denoting the functions that represent the 3-forms ρ_2^m and ρ_1^m by the same symbols ρ_2^m and ρ_1^m. Then, we may write

$$\begin{aligned}\mathcal{G}(\tilde{F}_{12}+\tilde{F}_{21},v) &= \mathcal{G}(\tilde{F}_{12},v)+\mathcal{G}(\tilde{F}_{21},v)\\ &= \tilde{J}_2\wedge \mathrm{i}_v B_1 + \rho_2^m\wedge \mathrm{i}_v\tilde{H}_1 + \tilde{J}_1\wedge \mathrm{i}_v B_2 + \rho_1^m\wedge \mathrm{i}_v\tilde{H}_2,\\ &= (\boldsymbol{J}_2\times\boldsymbol{B}_1\cdot\boldsymbol{v})\mathrm{d}V + (\rho_2^m\boldsymbol{H}_1\cdot\boldsymbol{v})\mathrm{d}V + (\boldsymbol{J}_1\times\boldsymbol{B}_2\cdot\boldsymbol{v})\mathrm{d}V\\ &\quad + (\rho_1^m\boldsymbol{H}_2\cdot\boldsymbol{v})\mathrm{d}V\\ &= (\boldsymbol{J}_2\times\boldsymbol{B}_1 + \rho_2^m\boldsymbol{H}_1 + \boldsymbol{J}_1\times\boldsymbol{B}_2 + \rho_1^m\boldsymbol{H}_2)\cdot\boldsymbol{v}\mathrm{d}V \end{aligned} \tag{A.36}$$

so that the proxy-vector field of the volume force density has the components $\boldsymbol{J}_2\times\boldsymbol{B}_1+\rho_2^m\boldsymbol{H}_1+\boldsymbol{J}_1\times\boldsymbol{B}_2+\rho_1^m\boldsymbol{H}_2$.

For the basis representation of the surface force density I will abuse the notation by denoting the functions that represent the 2-forms σ_2^m and σ_1^m by the same symbols σ_2^m and σ_1^m. Then, we have

$$\begin{aligned}\mathcal{G}(\tilde{f}_{12}+\tilde{f}_{21},v) &= \mathcal{G}(\tilde{f}_{12},v)+\mathcal{G}(\tilde{f}_{21},v)\\ &= \tilde{j}_2\wedge \mathrm{t}_2\mathrm{i}_v B_1 + \sigma_2^m\wedge \mathrm{t}_2\mathrm{i}_v\tilde{H}_1 + \tilde{j}_1\wedge \mathrm{t}_2\mathrm{i}_v B_2 + \sigma_1^m\wedge \mathrm{t}_2\mathrm{i}_v\tilde{H}_2\\ &= (\boldsymbol{j}_2\times\boldsymbol{B}_1\cdot\boldsymbol{v})\mathrm{d}A + (\sigma_2^m\boldsymbol{H}_1\cdot\boldsymbol{v})\mathrm{d}A + (\boldsymbol{j}_1\times\boldsymbol{B}_2\cdot\boldsymbol{v})\mathrm{d}A\\ &\quad + (\sigma_1^m\boldsymbol{H}_2\cdot\boldsymbol{v})\mathrm{d}A\\ &= (\boldsymbol{j}_2\times\boldsymbol{B}_1 + \sigma_2^m\boldsymbol{H}_1 + \boldsymbol{j}_1\times\boldsymbol{B}_2 + \sigma_1^m\boldsymbol{H}_2)\cdot\boldsymbol{v}\mathrm{d}A, \end{aligned} \tag{A.37}$$

so that the proxy-vector field of the surface force density has the components $\boldsymbol{j}_2\times\boldsymbol{B}_1+\sigma_2^m\boldsymbol{H}_1+\boldsymbol{j}_1\times\boldsymbol{B}_2+\sigma_1^m\boldsymbol{H}_2$.

For the representation of the stress $\tilde{T}_{12} + \tilde{T}_{21}$ we proceed in the familiar way. We first write

$$\begin{aligned}
\mathcal{G}(\tilde{F}_{12} + \tilde{F}_{21}, v) &= \mathcal{G}(\tilde{F} - \tilde{F}_{11} - \tilde{F}_{22}, v) \\
&= \mathcal{G}(\tilde{F}, v) - \mathcal{G}(\tilde{F}_{11}, v) - \mathcal{G}(\tilde{F}_{22}, v) \\
&= \mathrm{d}\tilde{H} \wedge \mathrm{i}_v B + \mathrm{d}B \wedge \mathrm{i}_v \tilde{H} - (\mathrm{d}\tilde{H}_1 \wedge \mathrm{i}_v B_1 + \mathrm{d}B_1 \wedge \mathrm{i}_v \tilde{H}_1) \\
&\quad - (\mathrm{d}\tilde{H}_2 \wedge \mathrm{i}_v B_2 + \mathrm{d}B_2 \wedge \mathrm{i}_v \tilde{H}_2). \qquad \text{(A.38)}
\end{aligned}$$

By taking into account that $\mathrm{d}B'$, $\mathrm{d}B'_1$ and $\mathrm{d}B'_2$ vanish, we find that the above may be written in the standard basis as

$$\begin{aligned}
\mathcal{G}(\tilde{F}_{12} + \tilde{F}_{21}, v) &= \mathcal{G}\big(\mathrm{d}_\nabla(\tilde{T} - \tilde{T}_{11} - \tilde{T}_{22}), v\big) \\
&= \big(\nabla \cdot (\boldsymbol{T} - \boldsymbol{T}_{11} - \boldsymbol{T}_{22})\big) \cdot \boldsymbol{v}\mathrm{d}V, \qquad \text{(A.39)}
\end{aligned}$$

where

$$\boldsymbol{T} = \mu_0 \boldsymbol{H}\boldsymbol{H} - \frac{\mu_0}{2}(\boldsymbol{H} \cdot \boldsymbol{H})\bar{I}, \qquad \text{(A.40)}$$

$$\boldsymbol{T}_{11} = \mu_0 \boldsymbol{H}_1\boldsymbol{H}_1 - \frac{\mu_0}{2}(\boldsymbol{H}_1 \cdot \boldsymbol{H}_1)\bar{I}, \qquad \text{(A.41)}$$

$$\boldsymbol{T}_{22} = \mu_0 \boldsymbol{H}_2\boldsymbol{H}_2 - \frac{\mu_0}{2}(\boldsymbol{H}_2 \cdot \boldsymbol{H}_2)\bar{I}. \qquad \text{(A.42)}$$

A.2.3 Magnetization approach

When magnetic materials are modeled by magnetization (section 6.3), we have

$$\begin{aligned}
\mathcal{G}(\tilde{F}_{12} + \tilde{F}_{21}, v) &= \mathcal{G}(\tilde{F}_{12}, v) + \mathcal{G}(\tilde{F}_{21}, v) \\
&= \tilde{J}_2^f \wedge \mathrm{i}_v B_1 + \tilde{M}_2 \wedge \nabla_v B_1 + \tilde{J}_1^f \wedge \mathrm{i}_v B_2 + \tilde{M}_1 \wedge \nabla_v B_2. \qquad \text{(A.43)}
\end{aligned}$$

Let us consider the second term on the right hand side. By representing the magnetization $\tilde{M}_2$ as $(\tilde{M}_2)_i \omega^i$, and by using the symbol $\boldsymbol{M}_2$ for the triplet of the components $(\tilde{M}_2)_i$, we have in the standard basis

$$\begin{aligned}
\tilde{M}_2 \wedge \nabla_v B_1 &= \tilde{M}_2 \wedge \nabla_v\big((B_1)_1\omega^2 \wedge \omega^3 + (B_1)_2\omega^3 \wedge \omega^1 + (B_1)_3\omega^1 \wedge \omega^2\big) \\
&= \tilde{M}_2 \wedge \Big(\mathrm{d}\big((B_1)_1\big)(v)\omega^2 \wedge \omega^3 + \mathrm{d}\big((B_1)_2\big)(v)\omega^3 \wedge \omega^1 \\
&\quad + \mathrm{d}\big((B_1)_3\big)(v)\omega^1 \wedge \omega^2\Big) \\
&= \Big((\tilde{M}_2)_1 v^i \partial_i\big((B_1)_1\big) + (\tilde{M}_2)_2 v^i \partial_i\big((B_1)_2\big) \\
&\quad + (\tilde{M}_2)_3 v^i \partial_i\big((B_1)_3\big)\Big)\mathrm{d}V \\
&= \big(\boldsymbol{M}_2 \cdot (\boldsymbol{v} \cdot \nabla \boldsymbol{B}_1)\big)\mathrm{d}V, \qquad \text{(A.44)}
\end{aligned}$$

where the second equality follows because the basis 1-forms are constant. The expression for (A.43) in the basis is

$$\mathcal{G}(\tilde{F}_{12}+\tilde{F}_{21},v) = \big(\boldsymbol{J}_2^f \times \boldsymbol{B}_1 \cdot \boldsymbol{v} + \boldsymbol{M}_2 \cdot (\boldsymbol{v}\cdot\nabla\boldsymbol{B}_1) + \boldsymbol{J}_1^f \times \boldsymbol{B}_2 \cdot \boldsymbol{v} \\ + \boldsymbol{M}_1 \cdot (\boldsymbol{v}\cdot\nabla\boldsymbol{B}_2)\big)\mathrm{d}V. \qquad \text{(A.45)}$$

To find a convenient formula for the force density proxy-vector field we first take under consideration the second term on the right hand side. Since $\mathrm{d}\star B_1$ vanish on o_2 where $\tilde{M}_2$ is supported (implying that $\boldsymbol{M}_2 \times (\nabla \times \boldsymbol{B}_1)$ vanish), we have

$$\begin{aligned}\boldsymbol{M}_2 \cdot (\boldsymbol{v}\cdot\nabla\boldsymbol{B}_1) &= \big(\boldsymbol{M}_2 \cdot \nabla\boldsymbol{B}_1 + \boldsymbol{M}_2 \times (\nabla \times \boldsymbol{B}_1)\big)\cdot \boldsymbol{v} \\ &= (\boldsymbol{M}_2 \cdot \nabla\boldsymbol{B}_1)\cdot\boldsymbol{v}, \end{aligned} \qquad \text{(A.46)}$$

where the first equality may be verified by direct calculation. Accordingly, we may write (A.45) as

$$\mathcal{G}(\tilde{F}_{12}+\tilde{F}_{21},v) = \big(\boldsymbol{J}_2^f \times \boldsymbol{B}_1 + \boldsymbol{M}_2\cdot\nabla\boldsymbol{B}_1 + \boldsymbol{J}_1^f \times \boldsymbol{B}_2 + \boldsymbol{M}_1\cdot\nabla\boldsymbol{B}_2\big)\cdot\boldsymbol{v}\mathrm{d}V, \qquad \text{(A.47)}$$

so that the proxy-vector field of volume force density has the components $\boldsymbol{J}_2^f \times \boldsymbol{B}_1 + \boldsymbol{M}_2\cdot\nabla\boldsymbol{B}_1 + \boldsymbol{J}_1^f \times \boldsymbol{B}_2 + \boldsymbol{M}_1\cdot\nabla\boldsymbol{B}_2$.

For the representation of the stress $\tilde{T}_{12}+\tilde{T}_{21}$ we first write

$$\begin{aligned}\mathcal{G}(\tilde{F}_{12}+\tilde{F}_{21},v) &= \mathcal{G}(\tilde{F}-\tilde{F}_{11}-\tilde{F}_{22},v) \\ &= \mathcal{G}(\tilde{F},v) - \mathcal{G}(\tilde{F}_{11},v) - \mathcal{G}(\tilde{F}_{22},v) \\ &= \mathrm{d}\tilde{H}' \wedge \mathrm{i}_v B + \tilde{M}\wedge\nabla_v B - (\mathrm{d}\tilde{H}_1' \wedge \mathrm{i}_v B_1 + \tilde{M}_1 \wedge \nabla_v B_1) \\ &\quad - (\mathrm{d}\tilde{H}_2' \wedge \mathrm{i}_v B_2 + \tilde{M}_2 \wedge \nabla_v B_2). \end{aligned} \qquad \text{(A.48)}$$

By taking into account that $\mathrm{d}B$, $\mathrm{d}B_1$ and $\mathrm{d}B_2$ vanish, we find that the above may be written in the standard basis as

$$\begin{aligned}\mathcal{G}(\tilde{F}_{12}+\tilde{F}_{21},v) &= \mathcal{G}\big(\mathrm{d}_\nabla(\tilde{T}-\tilde{T}_{11}-\tilde{T}_{22}),v\big) \\ &= \big(\nabla\cdot(\boldsymbol{T}-\boldsymbol{T}_{11}-\boldsymbol{T}_{22})\big)\cdot\boldsymbol{v}\mathrm{d}V, \end{aligned} \qquad \text{(A.49)}$$

where

$$\boldsymbol{T} = \boldsymbol{H}'\boldsymbol{B} - \frac{\mu_0}{2}(\boldsymbol{H}'\cdot\boldsymbol{H}' - \boldsymbol{M}\cdot\boldsymbol{M})\bar{I}, \qquad \text{(A.50)}$$

$$\boldsymbol{T}_{11} = \boldsymbol{H}_1'\boldsymbol{B}_1 - \frac{\mu_0}{2}(\boldsymbol{H}_1'\cdot\boldsymbol{H}_1' - \boldsymbol{M}_1\cdot\boldsymbol{M}_1)\bar{I}, \qquad \text{(A.51)}$$

$$\boldsymbol{T}_{22} = \boldsymbol{H}_2'\boldsymbol{B}_2 - \frac{\mu_0}{2}(\boldsymbol{H}_2'\cdot\boldsymbol{H}_2' - \boldsymbol{M}_2\cdot\boldsymbol{M}_2)\bar{I}. \qquad \text{(A.52)}$$

A.2.4 Magnetic polarization approach

When magnetic materials are modeled by magnetic polarization, we have

$$\begin{aligned}\mathcal{G}(\tilde{F}_{12}+\tilde{F}_{21},v) &= \mathcal{G}(\tilde{F}_{12},v)+\mathcal{G}(\tilde{F}_{21},v)\\ &= \tilde{J}_2\wedge \mathrm{i}_v B_1 + M_2\wedge \nabla_v \tilde{H}_1 + \tilde{J}_1\wedge \mathrm{i}_v B_2 + M_1\wedge \nabla_v\tilde{H}_2.\end{aligned} \tag{A.53}$$

By employing the notation introduced earlier, with the exception of reusing here the symbols $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ for the triplets of the components of the 2-forms M_1 and M_2, respectively, this may be written in the standard basis as

$$\begin{aligned}\mathcal{G}(\tilde{F}_{12}+\tilde{F}_{21},v) =&\big(\boldsymbol{J}_2\times \boldsymbol{B}_1\cdot\boldsymbol{v}+\boldsymbol{M}_2\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{H}_1)\\ &+\boldsymbol{J}_1\times\boldsymbol{B}_2\cdot\boldsymbol{v}+\boldsymbol{M}_1\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{H}_2)\big)\mathrm{d}V.\end{aligned} \tag{A.54}$$

For a convenient formula for the force density proxy-vector field we look at the second term on the right hand side. Since $\mathrm{d}\tilde{H}_1$ vanish on o_2 where M_2 is supported (implying that $\boldsymbol{M}_2\times(\nabla\times\boldsymbol{H}_1)$ vanish), we have

$$\begin{aligned}\boldsymbol{M}_2\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{H}_1) &= \big(\boldsymbol{M}_2\cdot\nabla\boldsymbol{H}_1+\boldsymbol{M}_2\times(\nabla\times\boldsymbol{H}_1)\big)\cdot\boldsymbol{v}\\ &=(\boldsymbol{M}_2\cdot\nabla\boldsymbol{H}_1)\cdot\boldsymbol{v}.\end{aligned} \tag{A.55}$$

Accordingly, we may write (A.54) as

$$\mathcal{G}(\tilde{F}_{12}+\tilde{F}_{21},v) = (\boldsymbol{J}_2\times\boldsymbol{B}_1+\boldsymbol{M}_2\cdot\nabla\boldsymbol{H}_1+\boldsymbol{J}_1\times\boldsymbol{B}_2+\boldsymbol{M}_1\cdot\nabla\boldsymbol{H}_2)\cdot\boldsymbol{v}\mathrm{d}V, \tag{A.56}$$

so that the proxy-vector field of volume force density has the components $\boldsymbol{J}_2\times\boldsymbol{B}_1+\boldsymbol{M}_2\cdot\nabla\boldsymbol{H}_1+\boldsymbol{J}_1\times\boldsymbol{B}_2+\boldsymbol{M}_1\cdot\nabla\boldsymbol{H}_2$.

For the representation of the stress $\tilde{T}_{12}+\tilde{T}_{21}$ we first write

$$\begin{aligned}\mathcal{G}(\tilde{F}_{12}+\tilde{F}_{21},v) &= \mathcal{G}(\tilde{F}-\tilde{F}_{11}-\tilde{F}_{22},v)\\ &=\mathcal{G}(\tilde{F},v)-\mathcal{G}(\tilde{F}_{11},v)-\mathcal{G}(\tilde{F}_{22},v)\\ &=\mathrm{d}\tilde{H}\wedge\mathrm{i}_v B+M\wedge\nabla_v\tilde{H}-M\wedge\mathrm{i}_v\mathrm{d}\tilde{H}\\ &\quad-(\mathrm{d}\tilde{H}_1\wedge\mathrm{i}_vB_1+M_1\wedge\nabla_v\tilde{H}_1-M_1\wedge\mathrm{i}_v\mathrm{d}\tilde{H}_1)\\ &\quad-(\mathrm{d}\tilde{H}_2\wedge\mathrm{i}_vB_2+M_2\wedge\nabla_v\tilde{H}_2-M_2\wedge\mathrm{i}_v\mathrm{d}\tilde{H}_2).\end{aligned} \tag{A.57}$$

Note that the terms $M\wedge\mathrm{i}_v\mathrm{d}\tilde{H}$, $M_1\wedge\mathrm{i}_v\mathrm{d}\tilde{H}_1$ and $M_2\wedge\mathrm{i}_v\mathrm{d}\tilde{H}_2$ need to be included here as shown in chapter 6. By taking into account that $\mathrm{d}B'$, $\mathrm{d}B'_1$ and $\mathrm{d}B'_2$ vanish, we find that the above may be written in the standard basis as

$$\begin{aligned}\mathcal{G}(\tilde{F}_{12}+\tilde{F}_{21},v) &= \mathcal{G}\big(\mathrm{d}_\nabla(\tilde{T}-\tilde{T}_{11}-\tilde{T}_{22}),v\big)\\ &=\big(\nabla\cdot(\boldsymbol{T}-\boldsymbol{T}_{11}-\boldsymbol{T}_{22})\big)\cdot\boldsymbol{v}\mathrm{d}V,\end{aligned} \tag{A.58}$$

where

$$\boldsymbol{T} = \boldsymbol{H}\boldsymbol{B}' - \frac{\mu_0}{2}(\boldsymbol{H} \cdot \boldsymbol{H})\bar{I}, \tag{A.59}$$

$$\boldsymbol{T}_{11} = \boldsymbol{H}_1\boldsymbol{B}_1' - \frac{\mu_0}{2}(\boldsymbol{H}_1 \cdot \boldsymbol{H}_1)\bar{I}, \tag{A.60}$$

$$\boldsymbol{T}_{22} = \boldsymbol{H}_2\boldsymbol{B}_2' - \frac{\mu_0}{2}(\boldsymbol{H}_2 \cdot \boldsymbol{H}_2)\bar{I}. \tag{A.61}$$

Appendix B

Basis representations of torque densities

Here, we derive local basis representations the torque densities in electrostatic and magnetostatic cases. In general, a volume torque density involves two terms. One is expressed by using the force densities while the other is taken into account by the couple densities. We will use the conventions and notations of Appendix A. In particular, we denote the triplets of components of $\tilde{\theta}$ and r in the Cartesian coordinate basis as $\boldsymbol{\theta}$ and $\boldsymbol{r}$.

B.1 Electrostatics

Here we consider the different expressions for $-\mathrm{i}_r \star \tilde{F}_{12}$ and $\tilde{C}_{12}$ in electrostatics.

B.1.1 Electric charge approach

According to (4.8) we have

$$\mathcal{G}(-\mathrm{i}_r \star \tilde{F}_{12}, \tilde{\theta}) = \tilde{\rho}_2 \wedge \mathrm{i}_r \mathrm{i}_{\tilde{\theta}} \star E_1. \tag{B.1}$$

Let us consider the term $\mathrm{i}_r \mathrm{i}_{\tilde\theta} \star E_1$. We have

$$\begin{aligned}
\mathrm{i}_r \mathrm{i}_{\tilde\theta} \star E_1 &= \mathrm{i}_r \mathrm{i}_{\tilde\theta} \star \big((E_1)_a \omega^a\big) \\
&= \mathrm{i}_r \mathrm{i}_{\tilde\theta} \big((E_1)_1 \omega^2 \wedge \omega^3 + (E_1)_2 \omega^3 \wedge \omega^1 + (E_1)_3 \omega^1 \wedge \omega^2\big) \\
&= \mathrm{i}_r \big((E_1)_1 \mathrm{i}_{\tilde\theta}\omega^2 \wedge \omega^3 - (E_1)_1 \omega^2 \wedge \mathrm{i}_{\tilde\theta}\omega^3 \\
&\quad + (E_1)_2 \mathrm{i}_{\tilde\theta}\omega^3 \wedge \omega^1 - (E_1)_2 \omega^3 \wedge \mathrm{i}_{\tilde\theta}\omega^1 \\
&\quad + (E_1)_3 \mathrm{i}_{\tilde\theta}\omega^1 \wedge \omega^2 - (E_1)_3 \omega^1 \wedge \mathrm{i}_{\tilde\theta}\omega^2\big) \\
&= \mathrm{i}_r \Big(\big((E_1)_2 \theta^3 - (E_1)_3 \theta^2\big)\omega^1 + \big((E_1)_3 \theta^1 - (E_1)_1 \theta^3\big)\omega^2 \\
&\quad + \big((E_1)_1 \theta^2 - (E_1)_2 \theta^1\big)\omega^3\Big) \\
&= \big((E_1)_2 \theta^3 - (E_1)_3 \theta^2\big) r^1 + \big((E_1)_3 \theta^1 - (E_1)_1 \theta^3\big) r^2 \\
&\quad + \big((E_1)_1 \theta^2 - (E_1)_2 \theta^1\big) r^3 \\
&= (\boldsymbol{E}_1 \times \boldsymbol{\theta}) \cdot \boldsymbol{r} \\
&= (\boldsymbol{r} \times \boldsymbol{E}_1) \cdot \boldsymbol{\theta}. \qquad \text{(B.2)}
\end{aligned}$$

Using this in (B.1) we get

$$\mathcal{G}(-\mathrm{i}_r \star \tilde F_{12}, \tilde\theta) = \big(\boldsymbol{r} \times (\rho_2 \boldsymbol{E}_1)\big) \cdot \boldsymbol{\theta} \mathrm{d}V, \qquad \text{(B.3)}$$

so the proxy-vector field of the volume torque density has the components $\boldsymbol{r} \times (\rho_2 \boldsymbol{E}_1)$. The surface term is obtained similarly from (4.9), that is,

$$\begin{aligned}
\mathcal{G}(-\mathrm{i}_r \star \tilde f_{12}, \tilde\theta) &= \tilde\sigma_2 \wedge \mathrm{t}_2 \mathrm{i}_r \mathrm{i}_{\tilde\theta} \star E_1 \\
&= \sigma_2 \big((\boldsymbol{E}_1 \times \boldsymbol{\theta}) \cdot \boldsymbol{r}\big) \mathrm{d}A \\
&= \sigma_2 \big((\boldsymbol{r} \times \boldsymbol{E}_1) \cdot \boldsymbol{\theta}\big) \mathrm{d}A \\
&= \big(\boldsymbol{r} \times (\sigma_2 \boldsymbol{E}_1)\big) \cdot \boldsymbol{\theta} \mathrm{d}A, \qquad \text{(B.4)}
\end{aligned}$$

so the proxy-vector field of the surface torque density has the components $\boldsymbol{r} \times (\sigma_2 \boldsymbol{E}_1)$.

B.1.2 Electric polarization approach

According to (6.26) we have

$$\mathcal{G}(-\mathrm{i}_r \star \tilde F_{12}, \tilde\theta) = \tilde\rho_2^f \wedge \mathrm{i}_r \mathrm{i}_{\tilde\theta} \star E_1 + \tilde P_2 \wedge \nabla_{\star(\tilde\theta \wedge r)} E_1. \qquad \text{(B.5)}$$

For the first term we may follow the steps of the previous subsection. For the second term we first note that the rotational virtual displacement vector field v is given in the basis as

$$
\begin{aligned}
v &= \star(\tilde{\theta} \wedge r) \\
&= \star(\theta^i e_i \wedge r^j e_j) \\
&= \star\big((\theta^2 r^3 - \theta^3 r^2) e_2 \wedge e_3 + (\theta^3 r^1 - \theta^1 r^3) e_3 \wedge e_1 \\
&\quad (\theta^1 r^2 - \theta^2 r^1) e_1 \wedge e_2\big) \\
&= (\theta^2 r^3 - \theta^3 r^2) e_1 + (\theta^3 r^1 - \theta^1 r^3) e_2 \\
&\quad (\theta^1 r^2 - \theta^2 r^1) e_3
\end{aligned}
$$

so that we have $\boldsymbol{v} = \boldsymbol{\theta} \times \boldsymbol{r}$. Using this and (A.17) in (A.15) we get

$$
\begin{aligned}
\mathcal{G}(-\mathrm{i}_r \star \tilde{F}_{12}, \tilde{\theta}) &= \Big(\big(\boldsymbol{r} \times (\rho_2^f \boldsymbol{E}_1)\big) \cdot \boldsymbol{\theta} + (\boldsymbol{P}_2 \cdot \nabla \boldsymbol{E}_1) \cdot (\boldsymbol{\theta} \times \boldsymbol{r})\Big) \mathrm{d}V \\
&= \Big(\big(\boldsymbol{r} \times (\rho_2^f \boldsymbol{E}_1)\big) \cdot \boldsymbol{\theta} + \big(\boldsymbol{r} \times (\boldsymbol{P}_2 \cdot \nabla \boldsymbol{E}_1)\big) \cdot \boldsymbol{\theta}\Big) \mathrm{d}V \\
&= \big(\boldsymbol{r} \times (\rho_2^f \boldsymbol{E}_1 + \boldsymbol{P}_2 \cdot \nabla \boldsymbol{E}_1)\big) \cdot \boldsymbol{\theta} \mathrm{d}V. \qquad \text{(B.6)}
\end{aligned}
$$

According to the couple density expression (6.23) we have

$$
\begin{aligned}
\mathcal{G}(\tilde{C}_{12}, \tilde{\theta}) &= \tilde{P}_2 \wedge \mathrm{i}_{\tilde{\theta}} \star E_1 \\
&= \big(\boldsymbol{P}_2 \cdot (\boldsymbol{E}_1 \times \boldsymbol{\theta})\big) \mathrm{d}V. \\
&= (\boldsymbol{P}_2 \times \boldsymbol{E}_1) \cdot \boldsymbol{\theta} \mathrm{d}V. \qquad \text{(B.7)}
\end{aligned}
$$

Combining (B.6) and (B.7) we observe that the proxy-vector field of the volume torque density has the components $\boldsymbol{r} \times (\rho_2^f \boldsymbol{E}_1 + \boldsymbol{P}_2 \cdot \nabla \boldsymbol{E}_1) + \boldsymbol{P}_2 \times \boldsymbol{E}_1$.

The surface term is given from (6.27) as

$$
\begin{aligned}
\mathcal{G}(-\mathrm{i}_r \star \tilde{f}_{12}, \tilde{\theta}) &= \tilde{\sigma}_2^f \wedge \mathrm{t}_2 \mathrm{i}_r \mathrm{i}_{\tilde{\theta}} \star E_1 \\
&= \big(\boldsymbol{r} \times (\sigma_2^f \boldsymbol{E}_1)\big) \cdot \boldsymbol{\theta} \mathrm{d}A, \qquad \text{(B.8)}
\end{aligned}
$$

so the proxy-vector field of the surface torque density has the components $\boldsymbol{r} \times (\sigma_2^f \boldsymbol{E}_1)$.

B.2 Magnetostatics

Here, we consider two different expressions for $-\mathrm{i}_r \star \tilde{F}_{12}$ and one for $\tilde{C}_{12}$ in magnetostatics.

B.2.1 Electric current approach

From (4.11) we have

$$\mathcal{G}(-\mathrm{i}_r \star \tilde{F}_{12}, \tilde{\theta}) = \tilde{J}_2 \wedge \mathrm{i}_{\star(\tilde{\theta}\wedge r)} B_1.$$

We may use (A.26) together with $\boldsymbol{v} = \boldsymbol{\theta} \times \boldsymbol{r}$. We get

$$\begin{aligned}\mathcal{G}(-\mathrm{i}_r \star \tilde{F}_{12}, \tilde{\theta}) &= (\boldsymbol{J}_2 \times \boldsymbol{B}_1) \cdot (\boldsymbol{\theta} \times \boldsymbol{r}) \mathrm{d}V \\ &= \big(\boldsymbol{r} \times (\boldsymbol{J}_2 \times \boldsymbol{B}_1)\big) \cdot \boldsymbol{\theta} \mathrm{d}V, \qquad \text{(B.9)}\end{aligned}$$

so the proxy-vector field of the volume torque density has the components $\boldsymbol{r} \times (\boldsymbol{J}_2 \times \boldsymbol{B}_1)$. The surface term is given similarly from (4.12). By using (A.30) together with $\boldsymbol{v} = \boldsymbol{\theta} \times \boldsymbol{r}$, we have

$$\begin{aligned}\mathcal{G}(-\mathrm{i}_r \star \tilde{f}_{12}, \tilde{\theta}) &= \tilde{j}_2 \wedge \mathrm{t}_2 \mathrm{i}_{\star(\tilde{\theta}\wedge r)} B_1 \\ &= (\boldsymbol{j}_2 \times \boldsymbol{B}_1) \cdot (\boldsymbol{\theta} \times \boldsymbol{r}) \mathrm{d}A \\ &= \big(\boldsymbol{r} \times (\boldsymbol{j}_2 \times \boldsymbol{B}_1)\big) \cdot \boldsymbol{\theta} \mathrm{d}A, \qquad \text{(B.10)}\end{aligned}$$

so the proxy-vector field of the surface torque density has the components $\boldsymbol{r} \times (\boldsymbol{j}_2 \times \boldsymbol{B}_1)$.

B.2.2 Magnetization approach

According to (6.40) we have

$$\mathcal{G}(-\mathrm{i}_r \star \tilde{F}_{12}, \tilde{\theta}) = \tilde{J}_2^f \wedge \mathrm{i}_{\star(\tilde{\theta}\wedge r)} B_1 + \tilde{M}_2 \wedge \nabla_{\star(\tilde{\theta}\wedge r)} B_1.$$

For the first term we follow the steps of the previous subsection. For the second term we use (A.44) and (A.46) together with $\boldsymbol{v} = \boldsymbol{\theta} \times \boldsymbol{r}$. We get

$$\begin{aligned}\mathcal{G}(-\mathrm{i}_r \star \tilde{F}_{12}, \tilde{\theta}) &= \Big(\big(\boldsymbol{r} \times (\boldsymbol{J}_2 \times \boldsymbol{B}_1)\big) \cdot \boldsymbol{\theta} + (\boldsymbol{M}_2 \cdot \nabla \boldsymbol{B}_1) \cdot (\boldsymbol{\theta} \times \boldsymbol{r})\Big) \mathrm{d}V \\ &= \Big(\big(\boldsymbol{r} \times (\boldsymbol{J}_2 \times \boldsymbol{B}_1)\big) \cdot \boldsymbol{\theta} + \big(\boldsymbol{r} \times (\boldsymbol{M}_2 \cdot \nabla \boldsymbol{B}_1)\big) \cdot \boldsymbol{\theta}\Big) \mathrm{d}V \\ &= \big(\boldsymbol{r} \times (\boldsymbol{J}_2 \times \boldsymbol{B}_1 + \boldsymbol{M}_2 \cdot \nabla \boldsymbol{B}_1)\big) \cdot \boldsymbol{\theta} \mathrm{d}V. \qquad \text{(B.11)}\end{aligned}$$

According to the couple density expression (6.41) we have

$$\begin{aligned}\mathcal{G}(\tilde{C}_{12}, \tilde{\theta}) &= \tilde{M}_2 \wedge \star \mathrm{i}_{\tilde{\theta}} B_1 \\ &= \big(\boldsymbol{M}_2 \cdot (\boldsymbol{B}_1 \times \boldsymbol{\theta})\big) \mathrm{d}V. \\ &= (\boldsymbol{M}_2 \times \boldsymbol{B}_1) \cdot \boldsymbol{\theta} \mathrm{d}V. \qquad \text{(B.12)}\end{aligned}$$

Combining (B.11) and (B.12) we find that the proxy-vector field of the volume torque density has the components $\boldsymbol{r} \times (\boldsymbol{J}_2 \times \boldsymbol{B}_1 + \boldsymbol{M}_2 \cdot \nabla \boldsymbol{B}_1) + \boldsymbol{M}_2 \times \boldsymbol{B}_1$.

The surface term is given according to (6.42) as

$$\begin{aligned} \mathcal{G}(-\mathrm{i}_r \star \tilde{f}_{12}, \tilde{\theta}) &= \tilde{j}_2^f \wedge \mathrm{t}_2 \mathrm{i}_{\star(\tilde{\theta} \wedge r)} B_1 \\ &= \big(\boldsymbol{r} \times (\boldsymbol{j}_2^f \times \boldsymbol{B}_1)\big) \cdot \boldsymbol{\theta} \mathrm{d}A, \end{aligned} \tag{B.13}$$

so the proxy-vector field of the surface torque density has the components $\boldsymbol{r} \times (\boldsymbol{j}_2^f \times \boldsymbol{B}_1)$.

Appendix C

Basis representations for contact force calculation

When contact forces are determined directly from the total fields, one must include the correct surface term on the boundary of the object in question. Here, we derive basis representations of the two possible surface terms in magnetostatics. We use the conventions and notations of Appendices A and B.

C.1 Magnetization approach

According to (7.8) the surface contribution to the virtual work done on object o_2 by object o_1 is given by integrating the 2-form $\frac{\mu_0}{2} g(n,v)\mathrm{t}_2^- \tilde{M}_2 \wedge \star_s \mathrm{t}_2^- \tilde{M}_2$ over ∂o_2. For the basis representation of this we first note that the term $\mathrm{t}_2^- \tilde{M}_2 \wedge \star_s \mathrm{t}_2^- \tilde{M}_2$ may be given by using the outward unit normal n_2 as

$$\begin{aligned}
\mathrm{t}_2^- \tilde{M}_2 \wedge \star_s \mathrm{t}_2^- \tilde{M}_2 &= \mathrm{t}_2^- \tilde{M}_2 \wedge \mathrm{n}_2^- \star \tilde{M}_2 \\
&= \mathrm{t}_2^- \tilde{M}_2 \wedge \mathrm{t}^- \mathrm{i}_{n_2} \star \tilde{M}_2 \\
&= \mathrm{t}_2^- (\tilde{M}_2 \wedge \mathrm{i}_{n_2} \star \tilde{M}_2)
\end{aligned}$$

as shown in section 7.2. We may express this 2-form by using the area form $\mathrm{d}A_2 = \mathrm{t}_2 \mathrm{i}_{n_2} \mathrm{d}V$ and the unit 2-vector field σ_2 ($\star n_2$ considered as a 2-vector field on ∂o_2) as

$$\begin{aligned}
\mathrm{t}_2^- (\tilde{M}_2 \wedge \mathrm{i}_{n_2} \star \tilde{M}_2) &= \mathrm{t}_2^- (\tilde{M}_2 \wedge \mathrm{i}_{n_2} \star \tilde{M}_2)(\sigma_2) \mathrm{d}A_2 \\
&= \tilde{M}_2^- \wedge \mathrm{i}_{n_2} \star \tilde{M}_2^- (\star n_2) \mathrm{d}A_2 \\
&= \big((\tilde{M}_2^-)_i \omega^i\big) \wedge \mathrm{i}_{n_2} \star \big((\tilde{M}_2^-)_i \omega^i\big)(\star n_2) \mathrm{d}A_2. \qquad \text{(C.1)}
\end{aligned}$$

But note that

$$
\begin{aligned}
\mathrm{i}_{n_2} \star \big((\tilde{M}_2^-)_i \omega^i\big) &= \mathrm{i}_{n_2}\big((\tilde{M}_2^-)_1 \omega^2 \wedge \omega^3 + (\tilde{M}_2^-)_2 \omega^3 \wedge \omega^1 + (\tilde{M}_2^-)_3 \omega^1 \wedge \omega^2\big) \\
&= \big((\tilde{M}_2^-)_2 (n_2)_3 - (\tilde{M}_2^-)_3 (n_2)_2\big)\omega^1 \\
&\quad + \big((\tilde{M}_2^-)_3 (n_2)_1 - (\tilde{M}_2^-)_1 (n_2)_3\big)\omega^2 \\
&\quad + \big((\tilde{M}_2^-)_1 (n_2)_2 - (\tilde{M}_2^-)_2 (n_2)_1\big)\omega^3,
\end{aligned}
$$

where the second step follows by the antiderivation property of the interior product. Using this in (C.1), we get

$$
\mathrm{t}_2^-(\tilde{M}_2 \wedge \mathrm{i}_{n_2} \star \tilde{M}_2) = \boldsymbol{M}_2^- \times (\boldsymbol{M}_2^- \times \boldsymbol{n}_2) \cdot \boldsymbol{n}_2 \mathrm{d}A_2,
$$

where $\boldsymbol{M}_2^-$ denotes the triplet of components of $\tilde{M}_2^-$. By further using the vector analysis formula $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b})$ we get

$$
\begin{aligned}
\mathrm{t}_2^-(\tilde{M}_2^- \wedge \mathrm{i}_{n_2} \star \tilde{M}_2^-) &= \big(\boldsymbol{M}_2^-(\boldsymbol{M}_2^- \cdot \boldsymbol{n}_2) - \boldsymbol{n}_2(\boldsymbol{M}_2^- \cdot \boldsymbol{M}_2^-)\big) \cdot \boldsymbol{n}_2 \mathrm{d}A_2 \\
&= -\big(\boldsymbol{M}_2^- \cdot \boldsymbol{M}_2^- - (\boldsymbol{M}_2^- \cdot \boldsymbol{n}_2)^2\big)\mathrm{d}A_2 \\
&= -\boldsymbol{M}_{2t}^- \cdot \boldsymbol{M}_{2t}^- \mathrm{d}A_2,
\end{aligned}
$$

where $\boldsymbol{M}_{2t}^- = \boldsymbol{M}_2^- - (\boldsymbol{M}_2^- \cdot \boldsymbol{n}_2)\boldsymbol{n}_2$ is the tangential component of $\boldsymbol{M}_2^-$. We thus have for the surface term $\frac{\mu_0}{2} g(n,v) \mathrm{t}_2^- \tilde{M}_2 \wedge \star_s \mathrm{t}_2^- \tilde{M}_2$ the basis representation

$$
\frac{\mu_0}{2} g(n,v) \mathrm{t}_2^- \tilde{M}_2 \wedge \star_s \mathrm{t}_2^- \tilde{M}_2 = -\frac{\mu_0}{2}(\boldsymbol{n}_2 \cdot \boldsymbol{v})(\boldsymbol{M}_{2t}^- \cdot \boldsymbol{M}_{2t}^-)\mathrm{d}A_2. \tag{C.2}
$$

This coincides with the result of Brown [14, 28].

C.2 Magnetic polarization approach

According to (7.12), the surface contribution to the virtual work done on object o_2 by object o_1 is given by integrating the 2-form $\frac{1}{2\mu_0} g(n,v) \mathrm{t}_2^- M_2 \wedge \star_s \mathrm{t}_2^- M_2$ over ∂o_2. For the basis representation we first note that the term $\mathrm{t}_2^- M_2 \wedge \star_s \mathrm{t}_2^- M_2$ may be given as

$$
\begin{aligned}
\mathrm{t}_2^- M_2 \wedge \star_s \mathrm{t}_2^- M_2 &= \mathrm{t}_2^- M_2 \wedge \mathrm{n}_2^- \star M_2 \\
&= \mathrm{t}_2^- M_2 \wedge \mathrm{t}^- \mathrm{i}_{n_2} \star M_2 \\
&= \mathrm{t}_2^-(M_2 \wedge \mathrm{i}_{n_2} \star M_2).
\end{aligned}
$$

By using the area form $\mathrm{d}A_2 = \mathrm{t}_2 \mathrm{i}_{n_2} \mathrm{d}V$ and the unit 2-vector field σ_2 ($\star n_2$ considered as a 2-vector field on ∂o_2) we get

$$
\begin{aligned}
\mathrm{t}_2^-(M_2 \wedge \mathrm{i}_{n_2} \star M_2) &= \mathrm{t}_2^-(M_2 \wedge \mathrm{i}_{n_2} \star M_2)(\sigma_2)\mathrm{d}A_2 \\
&= M_2^- \wedge \mathrm{i}_{n_2} \star M_2^-(\star n_2)\mathrm{d}A_2 \\
&= (\boldsymbol{M}_2^- \cdot \boldsymbol{n}_2)^2 \mathrm{d}A_2,
\end{aligned}
$$

where we have reused the symbol $\boldsymbol{M}_2^-$ of the previous subsection for the triplet of components of M_2^-. We thus have for the surface term $\frac{1}{2\mu_0} g(n,v)\mathrm{t}_2^- M_2 \wedge \star_s \mathrm{t}_2^- M_2$ the basis representation

$$\frac{1}{2\mu_0} g(n,v)\mathrm{t}_2^- M_2 \wedge \star_s \mathrm{t}_2^- M_2 = \frac{1}{2\mu_0}(\boldsymbol{n}_2 \cdot \boldsymbol{v})(\boldsymbol{M}_2^- \cdot \boldsymbol{n}_2)^2 \mathrm{d}A_2, \tag{C.3}$$

which coincides with the result of Brown [14, 28].

Appendix D

Computation of force densities from finite element approximation of fields

The computation of electric and magnetic fields using finite element method usually leads to elementwise constant approximations for the fields [29]. In view of force calculation this is problematic because all our expressions for volume force density contain derivatives of fields. Here, I will introduce a computation method for volume force densities that avoids the problem of computing the derivatives of elementwise constant fields. I will not consider the issues of computational electromagnetics (such as weak formulations and their discretization, see [29]) but assume that the approximate fields are given on a Euclidean neighbourhood o'_{12} containing both of the objects o_1 and o_2. I will use local Cartesian coordinates on this neighbourhood, and employ the basis representations given in Appendix A.

In the following we will need to determine forces on macroscopic parts of the interacting objects o_1 and o_2. We want to exclude parts consisting of distinct pieces, and parts that contain pieces from both of the objects. Let us denote as $\mathcal{V}_1$ the set of all 3-dimensional connected submanifolds with boundaries of $o'_{12} - o_2$, and as $\mathcal{V}_2$ the set of all 3-dimensional connected submanifolds with boundaries of $o'_{12} - o_1$. Further, let us denote as $\partial\mathcal{V}_1$ the set of all 2-dimensional submanifolds of $o'_{12} - o_2$ that are boundaries of elements of $\mathcal{V}_1$. Similarly, we denote as $\partial\mathcal{V}_2$ the set of all 2-dimensional submanifolds of $o'_{12} - o_1$ that are boundaries of elements of $\mathcal{V}_2$. Since we use Cartesian coordinates on o'_{12} the elements of these sets may be taken as submanifolds of $\mathbb{R}^3$. Further, we may identify each tangent space of o'_{12} with $\mathbb{R}^3$ by first transporting tangent vectors to some reference point by keeping them constant, and then taking their Cartesian representations. Let us introduce maps

$$FORCE : \mathcal{V}_1 \cup \mathcal{V}_2 \to \mathbb{R}^3,$$
$$STRESS : \partial\mathcal{V}_1 \cup \partial\mathcal{V}_2 \to \mathbb{R}^3,$$

such that

$$FORCE(V) = STRESS(\partial V) \quad \text{for all } V \in \mathcal{V}_1 \cup \mathcal{V}_2. \tag{D.1}$$

Since the map $FORCE$ cannot be directly realized by using the element-wise constant field quantities, we focus in the following on the realization of $STRESS$. Once $STRESS$ is obtained from the fields $FORCE$ is given according to (D.1).

The value of $STRESS$ on an arbitrary element $\partial V \in \partial\mathcal{V}_1 \cup \partial\mathcal{V}_2$ is obtained by integrating $\boldsymbol{T}_{12} + \boldsymbol{T}_{21}$ over ∂V, that is,

$$STRESS(\partial V) = \int_{\partial V} (\boldsymbol{T}_{12} + \boldsymbol{T}_{21}) \cdot \boldsymbol{n} \mathrm{d}A. \tag{D.2}$$

By using (D.1) and (D.2), we get

$$FORCE(V) = \int_{\partial V} (\boldsymbol{T}_{12} + \boldsymbol{T}_{21}) \cdot \boldsymbol{n} \mathrm{d}A, \tag{D.3}$$

where $V \in \mathcal{V}_1 \cup \mathcal{V}_2$. In practice $\boldsymbol{T}_{12} + \boldsymbol{T}_{21}$ is determined from the total fields so that

$$FORCE(V) = \int_{\partial V} (\boldsymbol{T} - \boldsymbol{T}_{11} - \boldsymbol{T}_{22}) \cdot \boldsymbol{n} \mathrm{d}A, \tag{D.4}$$

where $V \in \mathcal{V}_1 \cup \mathcal{V}_2$. All quantities in the integal may be given in terms of fields as shown in Appendix A. Note that no derivatives of fields occur in (D.4).

To use the above in connection with the finite element method, we first assume simplicial finite element mesh (triangulation) for o'_{12}. The approximate fields are assumed constant in each element (tetrahedron) of this mesh. We will also need a *dual mesh* obtained by using a *barycentric* subdivision of the original *primal mesh* (see [29]). Instead of the tetrahedrons of the barycentric subdivision the elements of the dual mesh are 3-dimensional polyhedrons. To each node (vertex) of the primal mesh there corresponds such a polyhedron; it is obtained by taking the union of those tetrahedrons of the barycentric subdivision that contain the selected node as a vertex. This correspondence clearly goes to the opposite direction as well.

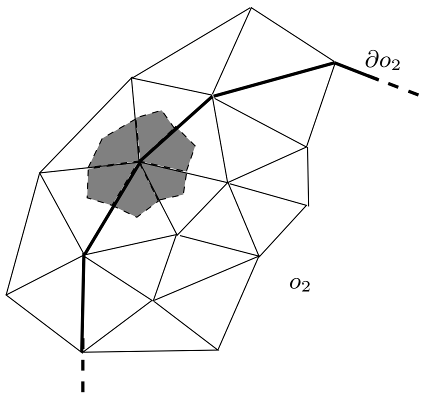

Figure D.1: An element of the dual mesh corresponding to a node that resides on the boundary of o_2. Also in this case the integration of $\boldsymbol{T}_{12} + \boldsymbol{T}_{21}$ is performed over the boundary of the polyhedral element (shown in gray). However, the force on the element comes from part of the element only. In the given analysis the geometric elements of the figure are 3-dimensional.

Let us denote as $\mathcal{N}^*$ the set of elements (3-dimensional polyhedrons) of the dual mesh. Because objects o_1 and o_2 are separated by free space it must be that the elements of $\mathcal{N}^*$ belong to $\mathcal{V}_1 \cup \mathcal{V}_2$. Now, the values of $FORCE$ may be evaluated for arbitrary $V \in \mathcal{N}^*$ by using (D.4) since there is never a discontinuity of $\boldsymbol{T} - \boldsymbol{T}_{11} - \boldsymbol{T}_{22}$ over ∂V. This is because $\boldsymbol{T} - \boldsymbol{T}_{11} - \boldsymbol{T}_{22}$ is constant in each tetrahedron of the primal mesh. Finally, given an element $V \in \mathcal{N}^*$ that intersects object o_2, for instance, the force density on the intersection is obtained by dividing $FORCE(V)$ by the volume of the intersection. The intersection is needed to deal with elements of the dual mesh corresponding to those nodes of the primal mesh that reside on the boundary of one of the objects, see Figure D.1.

There is an easy method of evaluating the force value on any $V \in \mathcal{N}^*$ that is obtained by using an insight into the finite element method. For this, let us first express the virtual work done on $V \in \mathcal{N}^*$ as

$$FORCE(V) \cdot \boldsymbol{v} = \int_{\partial V} \big(\boldsymbol{v} \cdot (\boldsymbol{T} - \boldsymbol{T}_{11} - \boldsymbol{T}_{22})\big) \cdot \boldsymbol{n} \mathrm{d}A. \tag{D.5}$$

Then, since $\boldsymbol{v} \cdot (\boldsymbol{T} - \boldsymbol{T}_{11} - \boldsymbol{T}_{22})$ is constant in each element of the primal mesh, it follows from standard finite element analysis that the above integration may be given by using the *nodal function* λ of the node dual to V. (This is the piecewise affine function whose value is 1 at the node dual to V, and 0 at all other nodes of the primal mesh.) We have

$$\int\limits_{\partial V} \big(\boldsymbol{v} \cdot (\boldsymbol{T} - \boldsymbol{T}_{11} - \boldsymbol{T}_{22})\big) \cdot \boldsymbol{n} \mathrm{d}A = -\int\limits_{o'_{12}} \big(\boldsymbol{v} \cdot (\boldsymbol{T} - \boldsymbol{T}_{11} - \boldsymbol{T}_{22})\big) \cdot \nabla\lambda \mathrm{d}V$$
$$= -\int\limits_{o'_{12}} \boldsymbol{v} \cdot \big((\boldsymbol{T} - \boldsymbol{T}_{11} - \boldsymbol{T}_{22}) \cdot \nabla\lambda\big) \mathrm{d}V, \tag{D.6}$$

because $\boldsymbol{v} \cdot (\boldsymbol{T} - \boldsymbol{T}_{11} - \boldsymbol{T}_{22})$ is constant in each element of the primal mesh, and because we integrate over the boundary of an element of the barycentric dual, see [29] (pp. 99-103). We thus have

$$FORCE(V) = -\int\limits_{o'_{12}} (\boldsymbol{T} - \boldsymbol{T}_{11} - \boldsymbol{T}_{22}) \cdot \nabla\lambda \mathrm{d}V,$$

where λ is the nodal function of the node dual to $V \in \mathcal{N}^*$. Since λ is supported in the cluster of primal tetrahedra around its node the integration above reduces to integration over the cluster.

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Tampereen teknillinen yliopisto
PL 527
33101 Tampere

Tampere University of Technology
P.O.B. 527
FI-33101 Tampere, Finland

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