PRO GRADU -TUTKIELMA

Jaana Hartikainen

Englanninkielistä oppimateriaalia differentiaalilaskennasta suomalaiseen lukioon

TAMPEREEN YLIOPISTO
Luonnontieteiden tiedekunta
Matematiikka
Syyskuu 2018

Tampereen yliopisto
Luonnontieteiden tiedekunta
HARTIKAINEN, JAANA: Englanninkielistä oppimateriaalia differentiaalilaskennasta suomalaiseen lukioon
Pro gradu -tutkielma, 90 s.
Matematiikka
Syyskuu 2018

Tiivistelmä

Tämän pro gradu -tutkielman tarkoituksena on toimia englanninkielisenä oppimateriaalina suomalaisen lukion MAA6: Derivaatta -kurssille. Tutkielma ei ole käännös mistään oppikirjasta, vaan tutkielmassa on laadittu täysin uusi oppimateriaali hyödyntäen englanninkielisiä lähdemateriaaleja sekä suomalaista vuoden 2015 opetussuunnitelmaa. Tutkielma ei ole kurssimateriaalina täysin kattava, sillä siinä ei käsitellä kurssiin kuuluvaa rationaalifunktioiden osa-aluetta. Kurssin kaksi muuta osa-aluetta, raja-arvo sekä derivaatta, käsitellään opetussuunnitelman vaatimalla tavalla. Tutkielmassa on haluttu painottaa teorian monipuolista ymmärtämistä erilaisten todistusten ja kaavojen johtamisten avulla. Materiaalin tueksi on listattu kurssin aiheita käsitteleviä ylioppilastehtäviä, mutta varsinaisia kappalekohtaisia tehtäviä tässä oppimateriaalissa ei ole. Tämän pro gradu -tutkielman pääasiallisina lähteinä on käytetty Adrian Bannerin kirjaa *The Calculus Lifesaver* sekä S.L. Salasin ja Einar Hillen kirjaa *Calculus: One and Several Variables*.

Asiasanat raja-arvo, derivaatta

Sisältö

1 Johdanto

Tämä pro gradu -tutkielma on kirjoitettu englanninkieliseksi oppimateriaaliksi raja-arvosta ja derivaatasta suomalaiseen lukioon. Suomessa on useampia englanninkielistä opetusta tarjoavia yläkouluja, mutta englanninkielistä Opetushallituksen asettamaa opetussuunnitelmaa vastaavaa lukio-opetusta ja siihen liittyvää oppimateriaalia ei vielä ole. Suomessa on joissakin kouluissa tarjolla kaksivuotinen IB Diploma Programme -opetusohjelma, joka tietyiltä osin vastaa lukion opetussuunnitelmaa, ja jossa kielten kursseja lukuunottamatta opetuskieli on englanti. Tässä, tunnetummin IB-lukioksi kutsutussa opetusohjelmassa on kuitenkin oma opetussuunnitelmansa, joka eroaa opetushallituksen asettamasta lukion opetusuunnitelmasta varsin paljon. [16]. Englanninkielisen opetuksen tarjoaminen kattaa jo lähes kaikki muut oppiasteet, joten on syytä varautua myös englanninkielisen lukio-opetuksen tuloon.

Tässä tutkielmassa opetusmateriaali alkaa raja-arvon käsittelystä ja jatkuu derivaatan käsittelyyn. Kurssin ensimmäinen osuus, rationaalifunktiot, on jätetty materiaalin ulkopuolelle. Kurssimateriaalissa oletetaan, että lukija on perehtynyt rationaalifunktion käsitteeseen ja ominaisuuksiin sillä tavalla, kun opetushallitus opetussuunnitelmassa tämän kurssin osalta velvoittaa. Teorian käsittelyssä on myös oletettu, että lukija on suorittanut matematiikan pitkän oppimäärän aiemmat kurssit MAY1-MAA5. Tutkielman oppimateriaalia voi käyttää joko itsenäisessä opiskelussa tai opettajajohtoisessa opetuksessa. Materiaalissa ei ole aihekohtaisia tehtäviä, mitä useimmissa oppikirjoissa on, joten oppimisen tueksi on haettava tehtäviä esimerkiksi eri oppikirjasarjojen kurssikirjoista. Tutkielman oppimateriaaliosuuden loppuun on listattu raja-arvoa ja derivaattaa käsitteleviä ylioppilaskokeen tehtäviä, joiden avulla oppilas voi harjoitella ja syventää osaamistaan. Vaikka aihekohtaisia tehtäviä ei ole, oppimateriaalin esimerkit on pyritty tekemään sellaisiksi, että oppilaalle tulisi käsitys siitä, millaisia erilaisia tehtäviä voidaan kurssin avulla ratkaista, ja millaisia erilaisia ratkaisuvaihtoehtoja on. Näiden esimerkkien ymmärtämisellä ja osaamisella oppilas pystyy sisäistämään esitetyn teorian ja opetussuunnitelman vaatiman matemaattisen osaamisen.

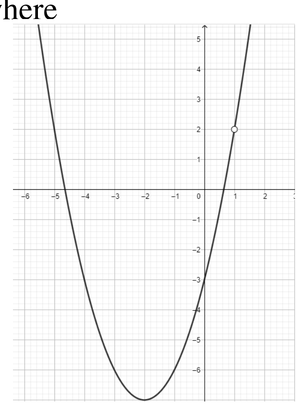
Lukion opetussuunnitelma on laadittu vuonna 2015, ja MAA6 -kurssia järjestetään uuden opetussuunnitelman mukaan ensimmäistä kertaa lukuvuonna 2017-2018 [10, s.3]. Tästä syystä suomenkielistä uutta opetussuunnitelmaa vastaavaa oppimateriaalia on vain vähän tarjolla. Oppimateriaalin ulkoasu vastaa pro gradu -tutkielmaa, eikä se ulkomuodoltaan vastaa oppikirjojen visuaalisuutta. Tämän tutkielman tarkoitus ei ole olla käännös jo olemassa olevasta materiaalista, joten tutkielmaa varten on pitänyt paneutua opetussuunnitelmaan tarkasti, jotta opetusmateriaali täyttää varmasti sen vaatimukset. Matemaattisten termien käännöksen apuna on käytetty erilaisia ja eri oppiasteille tarkoitettuja englanninkielisiä opetusmateriaaleja. Tämän pro gradu -tutkielman pääasiallisina lähteinä on käytetty Adrian Bannerin kirjan *The Calculus Lifesaver* lukuja 3-6 (s.41-126), sekä S.L. Salasin ja Einar Hillen kirjan *Calculus: One and Several Variables* lukuja 2-3 (s.47-182).

2 Oppimateriaali: Limit of a function and continuity

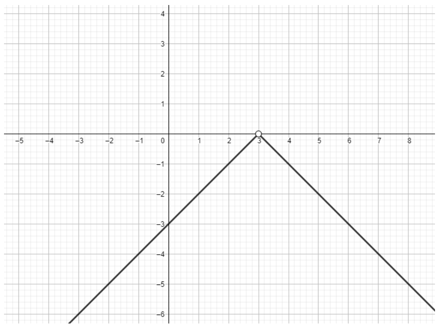
2.1 Limit of a function

Let us look at the two functions $f(x)$ and $g(x)$, where

$$f(x) = \frac{x^3 + 3x^2 - 7x + 3}{x - 1}$$



$$g(x) = \begin{cases} x - 3 & \text{when } x < 3, \\ -x + 3 & \text{when } x > 3. \end{cases}$$



Now $f(x)$ is defined on $\mathbb{R}\setminus\{1\}$ (because if $x = 1$ there would be a division by 0), and the function $g(x)$ is defined on $\mathbb{R}\setminus\{3\}$. Even though $f(x)$ and $g(x)$ are not defined at all points on $\mathbb{R}$, sometimes it is important to understand how these functions behave near these restrictions of the variable. In order to do that, we need to have some tools to help us understand what happens when x is getting close to the point 1 or the point 3.

Definition 2.1.1

Let b be a real number. The function $f(x)$ has a ***limit*** b at a point a if the value of $f(x)$ approaches b when the independent variable x gets closer to a. If $f(x)$ has a limit b at a, we write this as

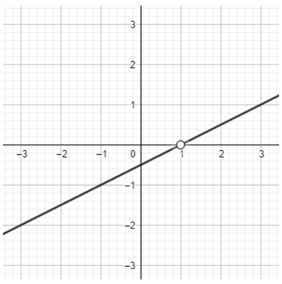
$$\lim_{x \to a} f(x) = b.$$

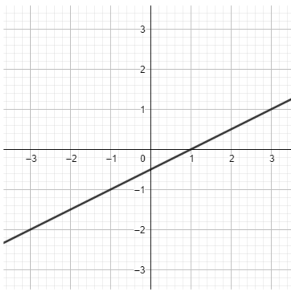
[4, p.35], [1, p.41-43] .

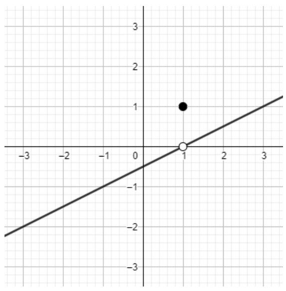
Note!

The value of function f at point $x = a$ is irrelevant, and it does not affect to the existence of the limit, nor the value of the limit. Like in previous examples, the functions are not defined at the point to which x approaches. Still, the limit might exist even though the value of f does not exist at that point. The function might also be defined and have a value of $x = a$, but it is possible that the limit $\lim_{x \to a} f(x) \neq f(a)$. For example, in all graphs below, the limit of the function is 0 when x approaches 1.

$$f(x) = \frac{1}{2}x - \frac{1}{2},\ x \neq 1 \qquad g(x) = \frac{1}{2}x - \frac{1}{2} \qquad h(x) = \begin{cases} \frac{1}{2}x - \frac{1}{2} & \text{when } x \neq 1, \\ 1 & \text{when } x = 1. \end{cases}$$







Example 1

Let us look again at the functions given at the beginning. First we have the function $g(x)$, where

$$g(x) = \begin{cases} x - 3 & \text{when } x < 3, \\ -x + 3 & \text{when } x > 3. \end{cases}$$

If we look at the graph it looks like the value of $g(x)$ is getting closer to 0 when x approaches 3. Let us do a table of values to confirm that.

x	$g(x)$	x	$g(x)$
2,8	-0,2	3,2	-0,2
2,9	-0,1	3,1	-0,1
2,99	-0,01	3,01	-0,01

It seems like the value of g approaches 0 when x approaches 3.

Answer:

$$\lim_{x \to 3} g(x) = 0.$$

Example 2

Let us then look at the graph of $f(x)$. Here

$$f(x) = \frac{x^3 + 3x^2 - 7x + 3}{x - 1}.$$

It seems that the value of $f(x)$ approaches the value 2 when x approaches the point 1. We can calculate the values of $f(x)$ as x gets closer to 1.

x	$f(x)$	x	$f(x)$
0,8	0,84	1,2	3,24
0,9	1,41	1,1	2,61
0,99	1,9401	1,01	2,0601

It looks like the value of $f(x)$ is endeed getting closer and closer to 2 as x is getting gloser to 1.

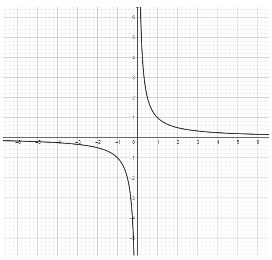
Answer:

$$\lim_{x \to 1} f(x) = 2.$$

Example 3

Let us look at the function $h(x) = \frac{1}{x}$, and the graph of this function.

We find that this function does not have a limit at $x = 0$. If we look at the graph we see that when x is negative and getting closer to 0, the value of $h(x)$ is getting smaller, but when x is positive and getting closer to 0, the value of $h(x)$ is getting bigger.



If we do the table of values, we come to the same conclusion.

x	$h(x)$	x	$h(x)$
-0,1	-010	0,1	10
-0,01	-100	0,01	100
-0,0019	-1000	0,001	1000

Answer:

$$\lim_{x\to 0} h(x) \text{ does not exist.}$$

Ways to calculate limits

However, we do need another and more accurate way of calculating limits of functions. If we look again at the fuction

$$f(x) = \frac{x^3 + 3x^2 - 7x + 3}{x - 1}$$

in Example 2, we can calculate that limit so that we do not have to rely on estimation. When we are dealing with a rational function, we can try to simplify the function. The way to do this is to find a common factor for the numerator and the denominator, and then do a contraction with that factor.

If we calculate $f(1)$, we obtain

$$f(1) = \frac{1^3 + 3 \cdot 1^2 - 7 \cdot 1 + 3}{1 - 1} = \frac{0}{0}.$$

So this tells us that both the numerator and the denominator can be divided into factors, where $(x-1)$ is one of the factors. (Polynomial $P(x)$ has a factor $(x - x_0)$, where x_0 is a root for $P(x)$).
Clearly $x - 1 = 1 \cdot (x - 1)$. We can use polynomial long division to determine the other factor for $x^3 + 3x^2 - 7x + 3$. In fact,

$$\begin{array}{rl}
 & x^2 + 4x - 3 \\ \cline{2-2}
x - 1) & x^3 + 3x^2 - 7x + 3 \\
 & -x^3 + x^2 \\ \cline{2-2}
 & 4x^2 - 7x \\
 & -4x^2 + 4x \\ \cline{2-2}
 & -3x + 3 \\
 & 3x - 3 \\ \cline{2-2}
 & 0
\end{array}$$

We get $x^3 + 3x^2 - 7x + 3 = (x-1)(x^2+4x-3)$. Now we can cancel off $(x-1)$, and then we get the exact answer as follows:

$$\lim_{x\to 1} \frac{x^3 + 3x^2 - 7x + 3}{x-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2+4x-3)}{\cancel{(x-1)}\cdot 1} = \lim_{x\to 1}(x^2+4x-3) = 2$$

Answer:

$$\lim_{x\to 1} f(x) = 2.$$

If $\lim_{x\to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)} = \frac{b}{0}$ and $b \neq 0$, then rational function $\frac{P(x)}{Q(x)}$ does not have a limit at point a.

Ways to estimate and calculate limits

$$\lim_{x\to a} f(x)$$

(1) Estimate from the graph.

(2) Estimate by calculating the table of values.

(3) Cancel off (rational functions)

- Find a common factor. If $f(a) = \frac{0}{0}$, the common factor is $(x-a)$.
- If $f(a) = \frac{b}{0}$ and $b \neq 0$, limit does not exist.

Note!

- $\lim_{x\to a} c = c$ when c is constant.
- $\lim_{x\to a} \big(f(x) + g(x)\big) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
- $\lim_{x\to a} cf(x) = c \cdot \lim_{x\to a} f(x)$ when c is constant.
- $\lim_{x\to a} \big(f(x) \cdot g(x)\big) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$.
- $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ when $\lim_{x\to a} g(x) \neq 0$.

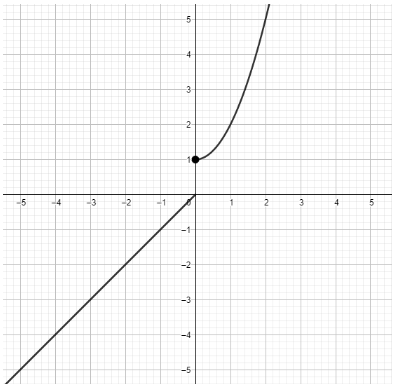
[3, p.24-27] , [4, p.38] , [11, p.57-65] .

2.2 One-sided limit

Let us start by investigating the function

$$f(x) = \begin{cases} x & \text{when } x < 0, \\ x^2 + 1 & \text{when } x \geq 0. \end{cases}$$

As we see from the graph, it seems that the limit of $f(x)$ does not exist at 0. If we make the table of values, we can confirm that the value of $f(x)$ approaches 0 when x approaches 0 from the negative side and 1 when x approaches 0 from the positive side.



x	$f(x) = x, x < 0$	$f(x) = x^2 + 1, x \geq 0$
-0,1	-0,1	
-0,01	-0,01	
-0,001	-0,001	
0,001		1,000001
0,01		1,0001
0,1		1,01

Definition 2.2.1

If the value of the function $f(x)$ approaches the value b when x approaches a from the left-hand side (with values less than a), then b is the ***left-sided limit*** of $f(x)$ at a. We write this as

$$\lim_{x \to a^-} f(x) = b.$$

If the value of the function $f(x)$ approaches the value b when x approaches a from the right-hand side (with values greater than a), then b is the ***right-sided limit*** of $f(x)$ at a. We write this as

$$\lim_{x \to a^+} f(x) = b.$$

Left-sided limit and right-sided limit are ***one-sided limits***.
[4, p.44] , [1, p.43-44] .

So in the example

$$f(x) = \begin{cases} x & \text{when } x < 0, \\ x^2 + 1 & \text{when } x \geq 0 \end{cases}$$

one-sided limits are

$$\lim_{x \to 0^-} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to 0^+} f(x) = 1.$$

Note!

The following rule is equivalent with the definition 2.2.1.

Function f has a limit $\lim_{x \to a} f(x) = b$ if and only if both the left-sided limit and the right-sided limit exist and are equal;

$$\lim_{x \to a} f(x) = b$$
$$\Leftrightarrow$$
$$\lim_{x \to a^-} f(x) = b = \lim_{x \to a^+} f(x).$$

[11, p.66] .

Example 1

Calculate one-sided limits of $f(x)$ at point 3 when $f(x)$ is as below. Does $\lim_{x \to 3} f(x)$ exist? Here

$$f(x) = \begin{cases} x^2 - 3x & \text{when } x \leq 3, \\ x - 2 & \text{when } x > 3. \end{cases}$$

Solution

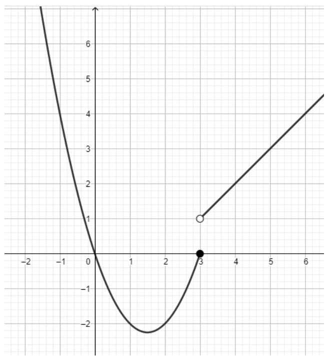
The left-sided limit:

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2 - 3x) = 3^2 - 3 \cdot 3 = 0.$$

The right-sided limit:

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x - 2) = 3 - 2 = 1.$$

We find that $\lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$, so $\lim_{x \to 3} f(x)$ does not exist.



Example 2

Determine whether $\lim_{x \to -1} g(x)$ exists or not, and the value of the limit if it does when

$$g(x) = \begin{cases} \frac{x^2 - 1}{2x + 2} & \text{when } x < -1, \\ 0 & \text{when } x = -1, \\ -\frac{1}{3}x - \frac{4}{3} & \text{when } x > -1. \end{cases}$$

Solution
The left-sided limit:

$$\lim_{x\to -1^-} g(x) = \lim_{x\to -1^-} \frac{x^2-1}{2x+2} = \frac{(-1)^2-1}{2\cdot(-1)+2} = \frac{0}{0}.$$

Because the replacement of x with -1 produced $\frac{0}{0}$, we know that we can divide the numerator and the denominator into factors where $x-(-1) = x+1$ is one of the factors.
We have a formula that says $(a+b)(a-b) = a^2 - b^2$, and we can apply this to the numerator (or we could use polynomial long division and divide x^2-1 with $x+1$). We get $x^2-1 = (x+1)(x-1)$. We can write the denominator as $2x+2 = 2(x+1)$. Now we can do the contraction with $x+1$, and then calculate the value of the left-sided limit. Now,

$$\lim_{x\to -1^-} g(x) = \lim_{x\to -1^-} \frac{x^2-1}{2x+2} = \lim_{x\to -1^-} \frac{\cancel{(x+1)}(x-1)}{2\cancel{(x+1)}} = \lim_{x\to -1^-} \frac{x-1}{2} = \frac{-1-1}{2} = -1.$$

The right-sided limit:

$$\lim_{x\to -1^+} g(x) = \lim_{x\to -1^+} -\frac{1}{3}x - \frac{4}{3} = -\frac{1}{3}\cdot(-1) - \frac{4}{3} = -1.$$

We get

$$\lim_{x\to -1^-} g(x) = \lim_{x\to -1^+} g(x) = -1, \text{ so}$$

$$\lim_{x\to -1} g(x) = -1.$$

This example shows us that it actually does not matter that the function $g(x)$ was defined at $x = -1$ as $g(-1) = 0$. Even though the value of $g(x)$ is 0 when x is -1, the limit of $g(x)$ is -1 when x approaches -1.

Example 3

Does the function $h(x) = \frac{|x|}{2x}$ have a limit at 0 ?
Solution
As

$$|\ x\ | = \begin{cases} -x & \text{when } x < 0, \\ x & \text{when } x \geq 0, \end{cases}$$

we can write $h(x)$ as a divided function

$$h(x) = \frac{|\ x\ |}{2x} = \begin{cases} \frac{-x}{2x} & \text{when } x < 0, \\ \frac{x}{2x} & \text{when } x > 0. \end{cases}$$

We do have to leave $x = 0$ out of the definition since the denominator equals 0 when $x = 0$. Now we can determine the left-sided and the right-sided limits as

$$\lim_{x\to 0^-} h(x) = \lim_{x\to 0^-} \frac{-x}{2x} = \lim_{x\to 0^-} \frac{-\cancel{x}}{2\cancel{x}} = \lim_{x\to 0^-} \left(-\frac{1}{2}\right) = -\frac{1}{2}$$

$$\lim_{x \to 0^+} h(x) = \lim_{x \to 0^+} \frac{x}{2x} = \lim_{x \to 0^+} \frac{\cancel{x}}{2\cancel{x}} = \lim_{x \to 0^+} \frac{1}{2} = \frac{1}{2}.$$

We find that $\lim_{x \to 0^-} h(x) \neq \lim_{x \to 0^+} h(x)$, so $h(x)$ does not have a limit at 0.

Example 4

Previously we examined the function $h(x) = \frac{1}{x}$, and we calculated that $\lim_{x \to 0} \frac{1}{x}$ does not exist. If we take a closer look, we can determine something about one-sided limits.

We know that $\frac{1}{x}$ is negative when x is negative and positive when x is positive. As x gets closer and closer to 0, the numerator 1 is divided with a number that is very close to 0. This means that $\frac{1}{x}$ is getting bigger and bigger as x gets closer and closer to 0 from the positive side. We can also see this from the graph. We can actually say that the one-sided limits of the function $h(x)$ approach positive and negative *infinities* respectively. Indeed,

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x \to 0^+} \frac{1}{x} = \infty.$$

2.3 Limits at infinity

We can also try to determine how a certain function behaves when we let x get larger and larger in either positive or in negative sense. We say that we study functions *limit at infinity*.

Limits at infinity cannot be calculated the same way as limits with real numbers. We have to decipher how that function will behave if we make x bigger or smaller. Limits at infinity are always one-sided limits, since x can only approach infinity from one direction. You cannot "go around" infinity and approach it from two opposite sides.

Ways to calculate limits at infinity

(1) Deduce from the graph.

(2) Deduce from the table of values.

(3) Deduce from the function

- Cancel off common factors to make function more simple if necessary.
- Cancel off the variable of highest degree.

Tips!

(a) If a is a real number, $r > 0$, and there is no restrictions for x^r when approaching infinity, then

$$\lim_{x\to\pm\infty} \frac{a}{x^r} = 0.$$

(b) If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree n, then

$$\lim_{x\to\pm\infty} p(x) = \lim_{x\to\pm\infty} a_n x^n.$$

Example 1

Let us look again at the function $h(x) = \frac{1}{x}$, we see that

$$\lim_{x\to\infty} \frac{1}{x} = 0, \text{ and}$$
$$\lim_{x\to-\infty} \frac{1}{x} = 0.$$

If we keep on dividing 1 with bigger and bigger numbers, we end up with a value that is very close to 0. The same happens if we divide 1 with numbers that are larger and larger in negative sense, $h(x)$ approaches 0 from the negative side. This is easy to deduce from the graph as well.

Example 2

Let us then look at the function $f(x) = -3x^3 + 4x^2 - 8x + 10$. What the tip box says is that if we want to determine

$$\lim_{x\to\infty} f(x),$$

we actually only have to look at the term of the highest degree, in this case this would be $-3x^3$. So

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} -3x^3.$$

Now we cannot write

$$\lim_{x\to\infty} -3x^3 = -3 \cdot (\infty)^3.$$

Even so, it is easy to deduce that if we keep replacing x with bigger and bigger numbers, and make them to the power of 3 we get even bigger and bigger values. If we multiply these values with -3 we get even larger numbers in negative sense. So we can say that

$$\lim_{x\to\infty} f(x) = -\infty.$$

This can be deduced from the graph as well.

Example 3

Determine

$$\lim_{x\to\infty} \frac{x^2 + 3x - 1}{3x^2 + x + 3}.$$

Solution

We find that both the numerator and the denominator approach infinity as x approaches infinity, and we cannot deduce anything from this. This is a situation where canceling off the variable of the highest degree comes in handy. So first we cancel off x^2 and get

$$\begin{aligned} &\lim_{x\to\infty} \frac{x^2 + 3x - 1}{3x^2 + x + 3} \\ &= \lim_{x\to\infty} \frac{\cancel{x^2}\left(1 + \frac{3x}{x^2} - \frac{1}{x^2}\right)}{\cancel{x^2}\left(3 + \frac{x}{x^2} + \frac{3}{x^2}\right)} \\ &= \lim_{x\to\infty} \frac{1 + \frac{3}{x} - \frac{1}{x^2}}{3 + \frac{1}{x} + \frac{3}{x^2}}. \end{aligned}$$

We know that $\frac{a}{x^n} \to 0$ with every $a \in \mathbb{R}$ when $n > 0$ and x approaches infinity, and so now we can calculate this limit as

$$\begin{aligned}
&\lim_{x\to\infty} \frac{1 + \frac{3}{x} - \frac{1}{x^2}}{3 + \frac{1}{x} + \frac{3}{x^2}} \\
&= \frac{1 + 0 - 0}{3 + 0 + 0} \\
&= \frac{1}{3}.
\end{aligned}$$

Answer:

$$\lim_{x\to\infty} \frac{x^2 + 3x - 1}{3x^2 + x + 3} = \frac{1}{3}.$$

2.4 Continuity

Definition 2.4.1

Function $f(x)$ is ***continuous*** at point a if the value $f(a)$ is the same as $\lim_{x\to a} f(x)$. In other words, function $f(x)$ is continuous at a if

$$\lim_{x\to a} f(x) = f(a).$$

[1, p.76] .

So in order to be continuous at point a, function f must

(1) be defined at a ($f(a) = b$),

(2) have a limit at a ($\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = b$),

(3) have the same value as the limit at a ($f(a) = \lim_{x\to a} f(x) = b$).

If a function is not continuous at a, it is ***discontinuous*** at a. Then a is ***a point of discontinuity***.

Example 1

Let

$$f(x) = \begin{cases} 0 & \text{when } x \leq -2, \\ \frac{x^2-4}{2} & \text{when } x > -2. \end{cases}$$

Is the function f continuous at $x = -2$?

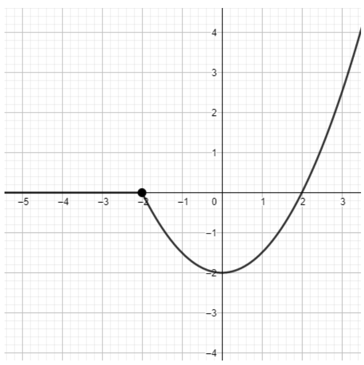
Solution

The left-sided limit:

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} 0 = 0.$$

The right-sided limit:

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \frac{x^2-4}{2} = \frac{(-2)^2-4}{2} = 0.$$



Value:

$$f(-2) = 0.$$

Now

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^+} f(x) = f(-2) = 0,$$

so the function $f(x)$ is continuous at -2.

Example 2

Set a so that the function $f(x)$ is continuous at 1 when

$$f(x) = \begin{cases} ax + 4 & \text{when } x < 1, \\ 0 & \text{when } x = 1, \\ x^2 + 3x + a & \text{when } x > 1. \end{cases}$$

Solution
We have to set a so that

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1).$$

First we notice that $f(1) = 0$, so we know that both the left-sided and the right-sided limit must be 0. We can start from the left-sided limit. Then

$$\begin{aligned} &\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (ax + 4) = a \cdot 1 + 4 = 0 \\ &\Leftrightarrow \\ &a = -4. \end{aligned}$$

So we get $a = -4$. We still have to doublecheck the right-sided limit. We see that

$$\begin{aligned} &\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2 + 3x + a) = 1^2 + 3 \cdot 1 + a = 4 + a = 0 \\ &\Leftrightarrow \\ &a = -4. \end{aligned}$$

Answer: $a = -4$.

Definition 2.4.2

- Function $f(x)$ is ***continuous over the open interval*** $]a, b[$ if and only if it is continuous at every point in $]a, b[$.
- Function $f(x)$ is ***continuous over the closed interval*** $[a, b]$ if and only if it is continuous at every point in $]a, b[$, the right-sided limit of f at a is $f(a)$ and the left-sided limit of f at b is $f(b)$.
- Function $f(x)$ is ***continuous in*** $\mathbb{R}$ if and only if it is continuous at every point in $\mathbb{R}$.

[1, p.77] .

A function that is continuous over some interval must first of all be defined at every point in that interval. Also for every point in that interval, limit and value of the function must be the same.The easy way to decipher if a certain function is continuous over an interval is to ask "can you draw the graph of that function without lifting your pen?". If not, the function is not continuous over that interval.

Steps to check continuity

Is function $f(x)$ continuous over an interval $[a, b]$?

(1) First, check that f(x) is defined at every point in $[a, b]$.

(2) Second, check that $\lim_{x \to x_0} f(x) = f(x_0)$ applies for every x_0 in $]a, b[$.

(3) Third, check that $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$.

Note!

Let f and g be continuous functions over a given interval. Then

- the function af is continuous over this interval for each real a,
- the functions $f + g$ and fg are continuous over this interval,
- the function $\frac{f}{g}$ is continuous over this interval, except for the points x_0 where $g(x_0) = 0$,

[11, p.80] .

Note!

All polynomial functions are continuous. So all functions
$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ are continuous in $\mathbb{R}$,
$(n \in \mathbb{N}, a_i \in \mathbb{R} \text{ when } i \leq n)$.

Example 3

Is $f(x)$ continuous in $\mathbb{R}$ when

$$f(x) = \frac{x}{x^2 - 2} \ ?$$

Solution

The first step was to check if $f(x)$ is defined in $\mathbb{R}$. We know that since x and $x^2 - 2$ are continuous functions in $\mathbb{R}$, $f(x)$ is continuous in $\mathbb{R}$, except if there are values for x so that the denominator equals zero. Clearly,

$$\begin{aligned} x^2 - 2 &= 0 \\ x^2 &= 2 && \Big\| \sqrt{\ } \\ x &= \pm\sqrt{2} \end{aligned}$$

Since $f(x)$ is not defined at points $-\sqrt{2}$ and $\sqrt{2}$, it can not be continuous at these points. So $f(x)$ is continuous at points x when $x \neq \pm\sqrt{2}$.

Answer: No, since $f(x)$ is not defined in the entire $\mathbb{R}$.

Example 4

Set a so that the function $f(x)$ is continuous over $[-2, 2]$ when

$$f(x) = \begin{cases} 2x^2 + 4x + 2 & \text{when } x \leq a, \\ -2x + 4 & \text{when } x > a. \end{cases}$$

Solution

We know that a is somewhere in closed interval $[-2, 2]$. We also know, that the functions $2x^2 + 4x + 2$ and $-2x + 4$ are both continuous functions. So the only point of possible discontinuity is the point a where the expression changes. We have to set a so that $\lim_{x\to a^+} f(x) = f(a) = \lim_{x\to a^-} f(x)$. We can calculate where these two expressions are equal. In fact,

$$\begin{aligned} 2x^2 + 4x + 2 &= -2x + 4 \\ 2x^2 + 6x - 2 &= 0 \\ x &= \frac{-6 \pm \sqrt{6^2 - 4 \cdot 2 \cdot (-2)}}{2 \cdot 2} \\ x &= \frac{-6 \pm \sqrt{52}}{4} \\ x &= \frac{-6 \pm 2\sqrt{13}}{4} \\ x = \frac{-3 - \sqrt{13}}{2} &\vee x = \frac{-3 + \sqrt{13}}{2} \\ x \approx -3,3 &\vee x \approx 0,3. \end{aligned}$$

Since $x \in [-2, 2]$, the first value of x does not work. Now we know, that $2x^2+4x+2 = -2x + 4$ when $x = \frac{-3+\sqrt{13}}{2}$. So if we set $a = \frac{-3+\sqrt{13}}{2}$, $f(x)$ is continuous in $[-2, 2]$.

Answer:

$$a = \frac{-3 + \sqrt{13}}{2}.$$

2.5 Intermediate value theorem (Bolzano)

Intermediate value theorem

If function $f(x)$ is continuous over a closed interval $[a, b]$, the function $f(x)$ gets any value in $[f(a), f(b)]$ at some point within the interval $[a, b]$.

Especially, if $f(a)$ and $f(b)$ are values of opposite signs, then there must be at least one root in $]a, b[$.
This means that if $f(a)$ is negative and $f(b)$ is positive, or vice versa, there must be at least one value of c in $]a, b[$ so that $f(c) = 0$.
[1, s.80-82].

This might seem irrelevant or obvious, but this theorem is really handy tool if we need to find roots for functions that we cannot calculate.

Example

Prove that the equation $x^3 - 3 = 4x^2$ has at least one solution. Find an approximation of this solution with 0,1 accuracy.

Solution

First we need to move all of the terms to the left side, so we can write that equation in terms of a function. Then, with the help of *Intermediate value theorem* we can start to narrow down the root. Clearly,

$$x^3 - 3 = 4x^2$$
$$x^3 - 4x^2 - 3 = 0.$$

Let us denote $f(x) = x^3 - 4x^2 - 3$, and find an approximation for a root. We know that as a polynomial function $f(x)$ is continuous in $\mathbb{R}$, and so we can use this theorem. We can start by giving different values for x, and hopefully we find some interval where $f(x)$ changes sign. We find

$$f(0) = -3 < 0$$
$$f(10) = 597 > 0.$$

So we know that there is a root in $]0,10[$. Basically, you can choose any values for x in this interval, and try to narrow down this interval. For example

$$
\begin{aligned}
f(5) &= 22 > 0 \\
f(4) &= -3 < 0 \\
f(4{,}5) &= 7{,}125 > 0 \\
f(4{,}2) &= 0{,}528 > 0 \\
f(4{,}1) &= -1{,}319 < 0.
\end{aligned}
$$

So we have found that the root is somewhere in an open interval $]4{,}1; 4{,}2[$. Lets count one more value for $x = 4{,}15$. We get

$$f(4{,}15) = -0{,}416625 < 0.$$

So now we know that the root is in $]4{,}15; 4{,}2[$. All of these numbers round up to 4,2, and we were asked to give an estimation with 0,1 accuracy.

Answer: Solution is approximately 4,2.

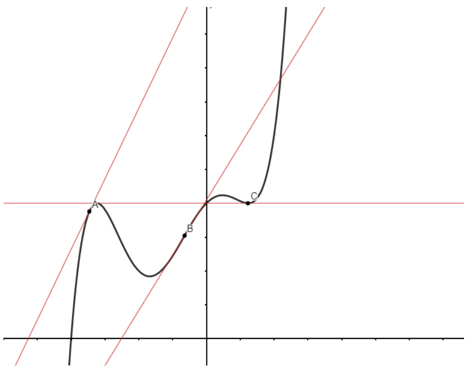
3 Oppimateriaali: Derivative

3.1 Rates of change

The rates of change of functions might seem an odd and irrelevant thing to study, but there are a lot of things that are constantly changing in our lives. If you think about for example your own life, the news or prices, you might find plenty of examples. Weather is getting hotter or colder, mortage rates are getting bigger or smaller, and price for gasoline is higher or lower than yeasterday. There are a lot of things changing constanty, and studying the speed of change at different times is relevant.

Rate of change tells us how rapidly the values of function are changing and to which direction. Graphically this means the steepness of the graph at a specific point. We can decipher this steepness by drawing a *tangent line* in point given, and deduce steepness from the slope of that tangent line. Tangent line is a line that goes through that point and is tangential to the curve, basically it only grazes the curve.

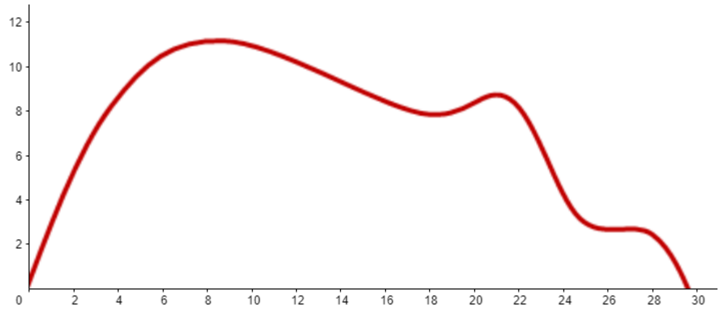
Below we see a graph of a function, and three points A, B and C in that graph. We can estimate the steepness of the curve in these points by drawing tangent lines into these points.



We see that for points A and B, the tangent line is rising, so the slopes of these tangent lines are positive. This means that the rate of change in points A and B is positive. For example, if this was a graph indicating price for gasoline, we could determine that the price was getting higher in these points. It also seems that the tangent line for point A is steeper than for point B, so we can deduce that the rate of change is higher in point A than in point B. The tangent line for point C seems to be horizontal, meaning that the slope is 0. This tells us that the rate of change for that point is neither positive nor negative.

Example

Mike went for a run for approximately 30 minutes. Below you see a graph indicating the speed that Mike was running at given time. Here the x axis indicates minutes Mike has run, and the y axis indicates his speed as km/h at that time.



(a) During which parts of the run Mikes speed was increasing?

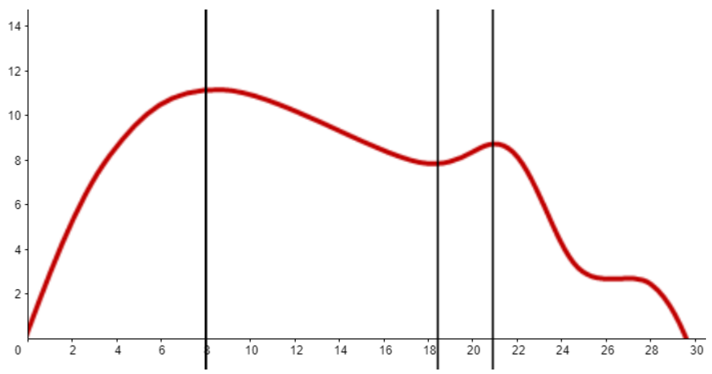
(b) In which parts was the speed decreasing?

(c) When was Mike speeding up fastest?

(d) Describe Mikes movement at 26 minutes.

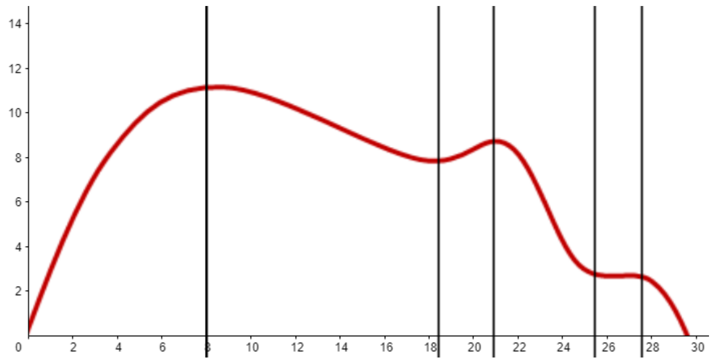
Solution

(a) If we look at the graph we see that the graph is first rising to approximately $x = 8$, meaning that Mikes speed was increasing the first 8 minutes or so. Then the graph is rising again from just over 18 minutes to approximately 21 minutes, so Mikes speed was increasing during this time as well.



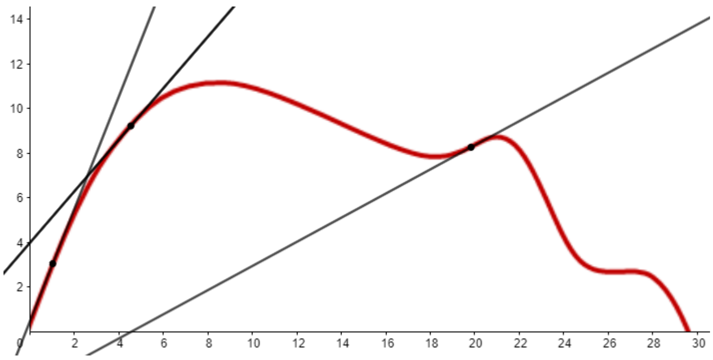
Answer: Approximately first 8 minutes, then again from approximately 18 to 21 minutes of the trip.

(b) Again we look at the graph and decipher where the graph is decreasing. It looks like the speed was decreasing approximately from 8 minutes to 18 minutes, then again from 21 minutes to 25 minutes, and finally from 28 minutes till the end of the trip.



Answer: Approximately from 8 to 18 minutes, then again approximately from 21 to 25 minutes, and finally approximately from 28 minutes to the end of the trip.

(c) If we want to know when Mike was speeding up the fastest, we have to look at the tangent lines in different points, and decipher in which point the slope of the tangent line is the steepest. We can draw tangent lines to the points in the curve where the graph seems to be rising fastest.

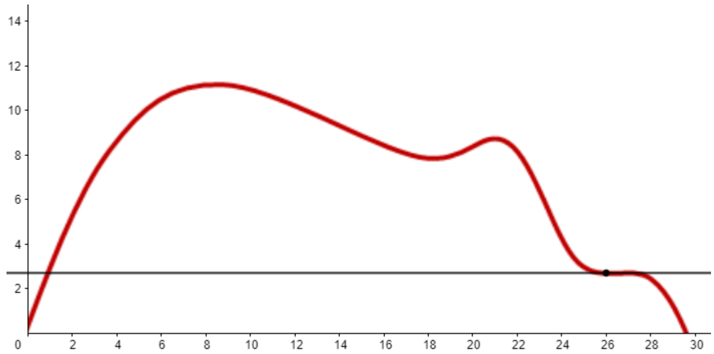


It looks like the graph is rising fastest at the beginning of the trip, approximately at first minute mark. Here the slope of the tangent line is the biggest, so the speed was increasing the fastest.

If we needed to know when Mike was slowing down fastest, we would try to find a point where the slope of the tangent line would be the smallest. Here the tangent line would be the steepest and going downwards.

Answer: Approximately at 1 minute mark.

(d) It looks like at 26 minutes Mike was running at the same speed of approximately 3km/h. Let us draw a tangent line to this point to confirm this.



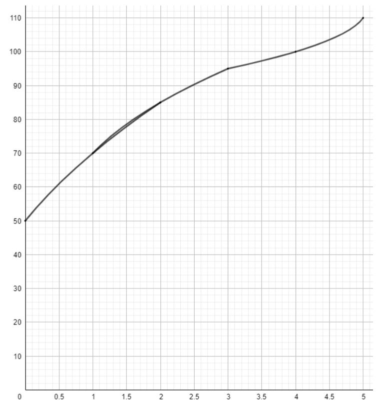
The slope of this tangent line seems to be close to 0, so Mikes speed was not increasing or decreasing at 26 minute mark.

Answer: Mike was running at constant speed, approximately 3km/h.

3.2 Average rate of change

Example

The graph below shows the height of Mona from birth to the age of five.

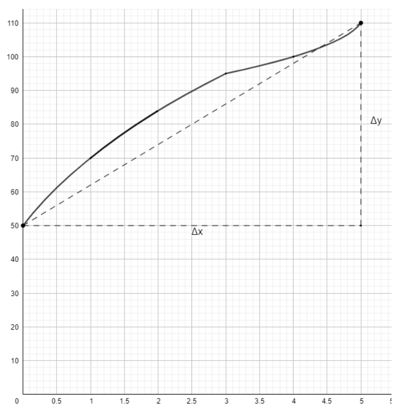


Here the input x equals Monas age in years, and y equals Monas height at that age. We already know how to determine Monas height at specific age, like for example we see that Mona was about 50cm tall when she was born, and about 85cm tall when she was 2 years old.

We can also determine the average growing speed for Mona. We know, that within the first two years of her life, Mona grew approximately 35cm. Based on this, we can say that in average, Mona grew $(85 - 50) : 2 = 17,5$cm per year. So we get the average speed by calculating the change in value of the function (change of y) and dividing that with the change of x. If we needed to know the average growing speed from 0 to 5 years of age, we would get

$$\frac{110\text{cm} - 50\text{cm}}{5\text{y} - 0\text{y}} = \frac{60\text{cm}}{5\text{y}} = 12\frac{\text{cm}}{\text{y}}.$$

As we know, kids do not grow at the same speed the entire time, and we can see this from the curvature of Monas graph as well. At some ages the growth was faster. If you mark these two points $(0, 50)$ and $(5, 110)$ and draw a straight line through them, you find that this line describes the average growth in height for Mona for first 5 years of her life. So had Mona been growing at the same speed, this straight line would be Monas graph of height.



Let us quickly remind ourselves how we form an equation $y = kx + b$ from two given points, or more specifically, how we calculate the slope k in that equation:

$$k = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{110 - 50}{5 - 0} = \frac{60}{5} = 12.$$

As we find, the average speed is the same as the slope of the line connecting the starting and the ending points.

Definition 3.2.1

The ***average rate of change*** for the function $y = f(x)$ in the interval $[a, b]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}.$$

[4, p.67]

Example

Let $f(x) = x^3 - 3x^2 - x + 3$. Calculate the average rate of change in

(a) $[-3, 0]$,

(b) $[-1, 1]$,

(c) $[0, 2]$.

Solution

(a) If we use the definition of the average rate of change, we get

$$
\begin{aligned}
\frac{\Delta y}{\Delta x} &= \frac{f(0) - f(-3)}{0 - (-3)} \\
&= \frac{0^3 - 3 \cdot 0^2 - 0 + 3 - ((-3)^3 - 3 \cdot (-3)^2 - (-3) + 3)}{3} \\
&= \frac{3 - (-48)}{3} \\
&= 17.
\end{aligned}
$$

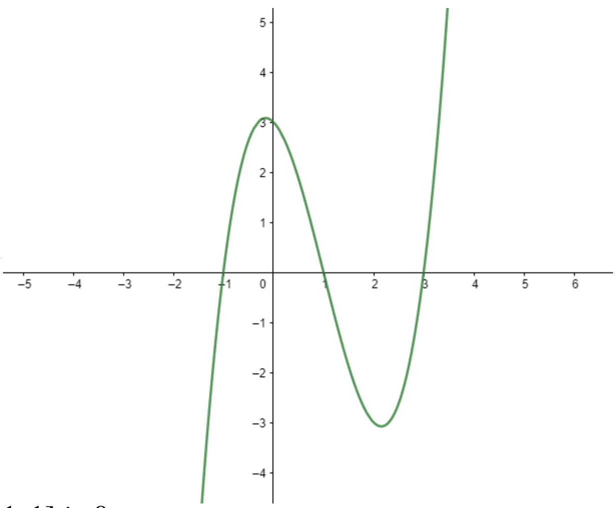
So this means that in this interval, every time x gets bigger by one, y gets bigger by 17 on average.

<u>Answer:</u> The average rate of change for $f(x)$ in $[-3, 0]$ is 17.

(b) Again we get

$$
\begin{aligned}
\frac{\Delta y}{\Delta x} &= \frac{f(1) - f(-1)}{1 - (-1)} \\
&= \frac{1^3 - 3 \cdot 1^2 - 1 + 3 - ((-1)^3 - 3 \cdot (-1)^2 - (-1) + 3)}{2} \\
&= \frac{0 - 0}{2} \\
&= 0.
\end{aligned}
$$

As we see from the graph, $f(-1) = f(1) = 0$, so if $f(x)$ were to grow at a constant speed in this interval, the graph of $f(x)$ would be a horizontal line at $y = 0$. As we can also see, the graph of $f(x)$ is not a straight line in $[-1, 1]$, so it is clear that we are dealing with an average rate of change, and not rate of change at a specific point. The graph of $f(x)$ is actually moving quite a bit in $[-1, 1]$. First the graph is rising and then decreasing.



<u>Answer:</u> The average rate of change for $f(x)$ in $[-1, 1]$ is 0.

(c) We get

$$\begin{aligned}
\frac{\Delta y}{\Delta x} &= \frac{f(2) - f(0)}{2 - 0} \\
&= \frac{2^3 - 3 \cdot 2^2 - 2 + 3 - ((0^3 - 3 \cdot 0^2 - 0 + 3)}{2} \\
&= \frac{-3 - 3}{2} \\
&= -3.
\end{aligned}$$

Answer: The average rate of change for $f(x)$ in $[0, 2]$ is -3.

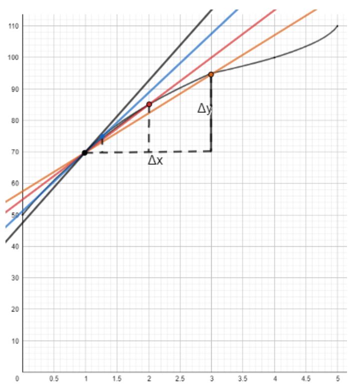
3.3 Instantaneous rate of change: Definition of the derivative

1st definition

The average rate of change is a valid way to calculate changes in speed, but it can only be used for the change over an interval, and not for a specific input. For example, let us look at Monas graph of height again. As we calculated, Monas average growing speed for the first 5 years was $12cm/year$. If we look at her first two years, we get $17,5cm/year$ for average speed. So was she growing $12cm/year$ or $17,5cm/year$ when she turned 1? We have to make the interval that includes 1 smaller and smaller to get a better answer. If we make the length of that interval as close to 0 as we can, we can calculate the *instantaneous rate of change*, or the *derivative*.

> *Definition 3.3.1*
>
> The ***instantaneous rate of change***, or the ***derivative*** of the function $f(x)$ at the point a is
>
> $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$
>
> if this limit exists.
>
> If $f'(a)$ exists, f is said to be ***differentiable*** at a.
> [4, p.76] .

Below we see again Monas graph of height.



As we see, we get different results for average growing speed if we choose different intervals. The smaller interval we take, the better estimation we get for the instantaneous rate of change. The orange, red and blue lines describe the average growing speed for the intervals shown in the graph, and the black line is the tangent line for point of $x = 1$. So if we make Δx close to zero, meaning that we move that ending point of the interval as close to the starting point as we can, we actually end up with the tangent line. We also see that as we make Δx smaller and smaller, Δy is getting smaller and smaller as well, so both of them approach 0.

Example 1

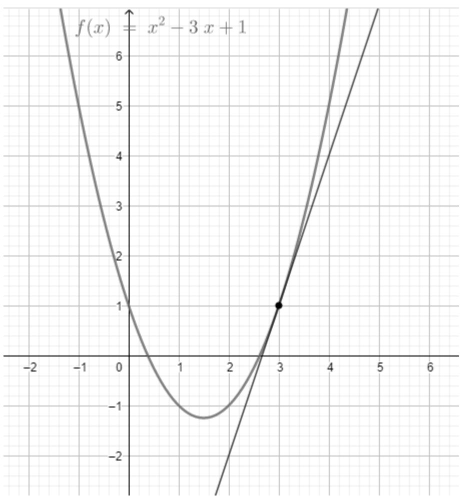
Find the derivative of $f(x) = x^2 - 3x + 1$ at $x = 3$.

Solution

According to the definition

$$\begin{aligned} f'(3) &= \lim_{x\to 3} \frac{f(x) - f(3)}{x-3} \\ &= \lim_{x\to 3} \frac{x^2 - 3x + 1 - (3^2 - 3\cdot 3 + 1)}{x-3} \\ &= \lim_{x\to 3} \frac{x^2 - 3x + 1 - 9 + 9 - 1}{x-3} \\ &= \lim_{x\to 3} \frac{x^2 - 3x}{x-3} \\ &= \lim_{x\to 3} \frac{x\cancel{(x-3)}}{\cancel{x-3}} \\ &= \lim_{x\to 3} x \\ &= 3. \end{aligned}$$

Thus $f'(3)$ exists and equals 3. If we look at the graph of $f(x)$ and the tangent line at $f(3)$, we see that the slope of the tangent line actually is 3.



<u>Answer:</u> $f'(3) = 3$.

Example 2

Find the derivative of $g(x) = \frac{1}{x}$ at $x = 2$.

Solution
With the definition of the derivative, we get

$$\begin{aligned} g'(2) &= \lim_{x\to 2} \frac{g(x) - g(2)}{x - 2} \\ &= \lim_{x\to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2} \\ &= \lim_{x\to 2} \frac{\frac{2}{2x} - \frac{x}{2x}}{x - 2} \\ &= \lim_{x\to 2} \frac{2 - x}{2x(x - 2)} \\ &= \lim_{x\to 2} \frac{-\cancel{(x - 2)}}{2x\cancel{(x - 2)}} \\ &= \lim_{x\to 2} -\frac{1}{2x} \\ &= -\frac{1}{2 \cdot 2} \\ &= -\frac{1}{4}. \end{aligned}$$

Thus $g'(2)$ exists and equals $-\frac{1}{4}$.

<u>Answer:</u> $g'(2) = -\frac{1}{4}$.

Example 3

Let $f(x) = x + \frac{1}{x}$. Determine the equation for the tangent line at $x = 4$.

Solution
Let us first determine the slope of that tangent line. The slope is the same as the derivative at that point, so with the definition of the derivative we get

$$\begin{aligned} f'(4) &= \lim_{x\to 4} \frac{f(x) - f(4)}{x - 4} \\ &= \lim_{x\to 4} \frac{\left(x + \frac{1}{x}\right) - \left(4 + \frac{1}{4}\right)}{x - 4} \\ &= \lim_{x\to 4} \frac{\frac{x^2+1}{x} - \frac{17}{4}}{x - 4} \\ &= \lim_{x\to 4} \frac{\frac{4(x^2+1)-17x}{4x}}{x - 4} \end{aligned}$$

$$= \lim_{x \to 4} \frac{4x^2 - 17x + 4}{4x(x-4)}.$$

If we replace x with 4, we get $\frac{0}{0}$, so we know that we can devide the numerator into factors, where one of the factors is $(x - 4)$. Let us do this with the polynomial long division. We get

$$\begin{array}{r|l} & 4x - 1 \\ \hline x - 4) & 4x^2 - 17x + 4 \\ & -4x^2 + 16x \\ \hline & -x + 4 \\ & x - 4 \\ \hline & 0 \end{array}$$

Now we can write $4x^2 - 17x + 4$ as $(4x - 1)(x - 4)$, and cancel off the common factor $(x - 4)$. We get

$$\begin{aligned} f'(4) &= \lim_{x \to 4} \frac{4x^2 - 17x + 4}{4x(x-4)} \\ &= \lim_{x \to 4} \frac{(4x-1)\cancel{(x-4)}}{4x\cancel{(x-4)}} \\ &= \frac{4 \cdot 4 - 1}{4 \cdot 4} \\ &= \frac{15}{16}. \end{aligned}$$

Now we know that the slope of the tangent line is $\frac{15}{16}$. We also know that the tangent line grazes the graph of the function at the point $x = 4$, meaning that the tangent line shares this point with the function. Let us calculate $f(4)$, so with this point and the slope $\frac{15}{16}$ we can determine the equation for the tangent line. In fact,

$$f(4) = 4 + \frac{1}{4} = \frac{17}{4},$$

and so the tangent line goes through the point $(4, \frac{17}{4})$. For the tangent line equation we get

$$\begin{aligned} y - y_0 &= k(x - x_0) \\ y - \frac{17}{4} &= \frac{15}{16}(x - 4) \\ y &= \frac{15}{16}x - \frac{15 \cdot 4}{16} + \frac{17}{4} \\ y &= \frac{15}{16}x + \frac{1}{2}. \end{aligned}$$

Answer: $y = \frac{15}{16}x + \frac{1}{2}$.

Example 4

It is important that we understand the difference between *continuity* and *differentiability*. It is possible that a function is continuous at a point but is not differentiable at that point. If function is differentiable at a point, it is also continuous at that point, but continuity does not quarantee differentiability.

Let us take a look at the function $g(x) = |x|$. Now this function can be written as

$$g(x) = \begin{cases} -x & \text{when } x < 0, \\ x & \text{when } x \geq 0. \end{cases}$$

We find that this function is continuous at $x = 0$, as

$$\begin{aligned} &\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} -x = -0 = 0, \\ &\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} x = 0, \\ &g(0) = 0. \end{aligned}$$

However, $g(x)$ is not differentiable at $x = 0$, as

$$\begin{aligned} \lim_{x \to 0^-} \frac{g(x) - g(0)}{x - 0} &= \lim_{x \to 0^-} \frac{-x - 0}{x} = -1, \\ \lim_{x \to 0^+} \frac{g(x) - g(0)}{x - 0} &= \lim_{x \to 0^+} \frac{x - 0}{x} = 1, \end{aligned}$$

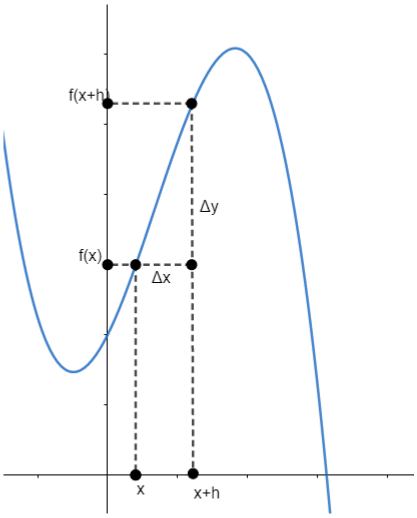
and thus

$$\lim_{x \to 0} \frac{g(x) - g(0)}{x - 0}$$

does not exist.

2nd definition

The first definition of the derivative might in some cases seem a bit difficult or complicated. We defined the derivative as a limit of $\frac{\Delta y}{\Delta x}$, where Δx approaches 0.



Before, we determined the starting point as a and the ending point as x, and the derivative of $f(a)$ was determined as limit where x approached a. Now we look at the same situation little bit differently. If we want to determine the derivative at a point x, we take x for the starting point and $x + h$ for the ending point. So where as before our Δx was $x - a$, it is now $\Delta x = (x + h) - x = h$. Still Δx approaches 0, so the limit will be for $h \to 0$.

As we see from the graph, we get $\Delta y = f(x+h) - f(x)$. So if we were to calculate the derivative this way, we would get

$$f'(x) = \lim_{h \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Definition 3.3.2

The derivative of $f(x)$ at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided that the limit exists.
[1, p.90] .

This definition 3.3.2 is equivalent with the definition 3.3.1.

Example 5 (difficult)

Find the derivative of $g(x) = \sqrt{x}$ at $x = 1$.

Solution

Let us use the second definition of the derivative. We get

$$\begin{aligned} g'(1) &= \lim_{h\to 0} \frac{g(1+h) - g(1)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{1+h} - \sqrt{1}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{1+h} - 1}{h}. \end{aligned}$$

If we replace h with 0 we get $\frac{0}{0}$. In this case we cannot cancel off anything to get forward, but we can to do the opposite. If we multiply the numerator and the dominator with $\sqrt{1+h}+1$, we can use the formula $a^2 - b^2 = (a-b)(a+b)$ to get rid of the square root. We get

$$\begin{aligned} g'(1) &= \lim_{h\to 0} \frac{\sqrt{1+h} - 1}{h} \\ &= \lim_{h\to 0} \frac{(\sqrt{1+h} - 1)(\sqrt{1+h} + 1)}{h(\sqrt{1+h} + 1)} \\ &= \lim_{h\to 0} \frac{\sqrt{1+h}^2 - 1^2}{h(\sqrt{1+h} + 1)} \\ &= \lim_{h\to 0} \frac{1 + h - 1}{h(\sqrt{1+h} + 1)} \\ &= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{1+h} + 1)} \\ &= \frac{1}{\sqrt{1+0} + 1} \\ &= \frac{1}{2}. \end{aligned}$$

<u>Answer:</u> $g'(1) = \frac{1}{2}$.

3.4 The derivative function

Example 1

Find the derivative of $f(x) = x^2$ at $x = a$.

Solution

Let us use the first definition. We get

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{x \to a} \frac{x^2 - a^2}{x - a} \\ &= \lim_{x \to a} \frac{(x + a)\cancel{(x - a)}}{\cancel{x - a}} \\ &= a + a \\ &= 2a. \end{aligned}$$

Answer: $f'(a) = 2a$.

So we know that the derivative of $f(x) = x^2$ at $x = a$ is $f'(a) = 2a$. What this means is that we can set any value for a and we get the derivative at that point. So if needed to know the derivative of $f(x)$ at 0, we could calculate this with $f'(a) = 2a$. Now $a = 0$, so we get $f'(0) = 2{\cdot}0 = 0$. We can calculate any number of values, such as

$$\begin{aligned} f'(-1) &= 2 \cdot (-1) = -2 \\ f'(1) &= 2 \cdot 1 = 2 \\ f'(-5) &= 2 \cdot (-5) = -10 \\ f'(10) &= 2 \cdot 10 = 20 \\ f'(537) &= 2 \cdot 537 = 1074 \\ f'(-\pi) &= 2 \cdot (-\pi) = -2\pi. \end{aligned}$$

It would have taken us a long time to determine all these values of the derivative by using the definition one value by one. If we set $a = x$ and consider x a variable, we get the *derivative function* $f'(x)$ of $f(x)$. We can determine the derivative function of $f(x) = x^2$ is $f'(x) = 2x$.

Usually derivative functions are determined by using the second definition of the derivative as we do in the following definition, because then we can avoid the use of

a as the derivative point.

Definition 3.4.1

The derivative function $f'(x)$ is a function for which every value of $f'(x)$ is the derivative of $f(x)$ at a point x (provided that the derivate $f'(x)$ exists). We get

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The derivative function can also be denoted with D or with $\frac{d}{dx}$ as follows:

$$D\left(f(x)\right) = \frac{d}{dx} f(x) = f'(x).$$

[1, p.90] , [4, p.86] .

Example 2

Find the derivative function of $f(x) = 2x^2 - 3x + 1$. Calculate the derivatives at $-3, 0$ and 10.

Solution

If we use the definion we get

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{2(x+h)^2 - 3(x+h) + 1 - (2x^2 - 3x + 1)}{h} \\ &= \lim_{h \to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 \cancel{-3x} - 3h \cancel{+1} \cancel{-2x^2} \cancel{+3x} \cancel{-1}}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(4x + 2h - 3)}{\cancel{h}} \\ &= 4x + 2 \cdot 0 - 3 \\ &= 4x - 3. \end{aligned}$$

Now we know that $f'(x) = 4x - 3$, and we can use this to calculate the derivatives.

In fact,

$$\begin{aligned} f'(-3) &= 4(-3) - 3 = -15, \\ f'(0) &= 4 \cdot 0 - 3 = -3, \\ f(10) &= 4 \cdot 10 - 3 = 37. \end{aligned}$$

Answer: $f'(x) = 4x - 3$, $f'(-3) = -15$, $f'(0) = -3$ and $f'(10) = 37$.

Derivatives of sum and scalar multiple

Now, as we know

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

so it is easy to determine a couple of rules to help us to determine derivative functions.

First, let us prove that $(f+g)'(x) = f'(x) + g'(x)$. We prove this by using the second definition of the derivative. We find

$$\begin{aligned} (f+g)'(x) &= \lim_{h \to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h} \\ &= \lim_{h \to 0} \frac{\big(f(x+h) + g(x+h)\big) - \big(f(x) + g(x)\big)}{h} \\ &= \lim_{h \to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h} \\ &= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right) \\ &= \underbrace{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}_{=f'(x)} + \underbrace{\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}}_{=g'(x)} \\ &= f'(x) + g'(x). \end{aligned}$$

Very similarly we can prove that $(af)'(x) = af'(x)$, where $a \in \mathbb{R}$. Again, we find

$$\begin{aligned} (af)'(x) &= \lim_{h \to 0} \frac{(af)(x+h) - (af)(x)}{h} \\ &= \lim_{h \to 0} \frac{af(x+h) - af(x)}{h} \\ &= \lim_{h \to 0} \left(a \cdot \frac{f(x+h) - f(x)}{h} \right) \\ &= a \cdot \underbrace{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}_{=f'(x)} \\ &= af'(x). \end{aligned}$$

Derivatives of sum and scalar multiple.

(a) $(f+g)'(x) = f'(x) + g'(x)$

(b) $(af)'(x) = af'(x)$

when $a \in \mathbb{R}$. [11, p.115]

Example 1

Find the derivative function $f'(x)$ when

(a) $f(x) = x^2 + x$

(b) $f(x) = 3x^2 - 8x$.

Solution

So what these derivative formulas for sum and scalar multiple tell us, is that we do not have to use the function as a whole to determine the derivative function. In point (a) we use the formula for sum, and we get

$$\begin{aligned} f'(x) &= D(x^2+x) = D(x^2) + D(x) \\ &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} + \lim_{h\to 0} \frac{(x+h)-x}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} + \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}} \\ &= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} + \lim_{h\to 0} 1 \\ &= 2x + 0 + 1 \\ &= 2x + 1. \end{aligned}$$

In point (b), we also use the formula $(af)'(x) = af'(x)$. This tells us that we can move the scalar in front of the derivative, or the limit. We also use the derivatives $D(x^2) = 2x$ and $D(x) = 1$ that we just determined. We find

$$\begin{aligned} f'(x) &= D(3x^2 - 8x) = D(3x^2) + D(-8x) = 3 \cdot D(x^2) - 8 \cdot D(x) \\ &= 3 \cdot 2x - 8 \cdot 1 = 6x - 8. \end{aligned}$$

<u>Answer:</u>

(a) $f'(x) = 2x + 1$

(b) $f'(x) = 6x - 8$.

Derivative of polynomial functions

Let us try to determine a formula for the derivatives of polynomial functions. Let us first determine the derivative function of constant function. Now let $f(x) = a$, where $a \in \mathbb{R}$. By using the definition we get

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{a-a}{h} \\ &= 0. \end{aligned}$$

So now we know that if $f(x)$ is any constant function, the derivative function is $f'(x) = 0$. If you think about this graphically, it makes sense. If you draw a graph of any constant function, the graph is a horizontal line. The derivative ment the rate of change at given point, but if the graph is a horizontal line, it is neither rising nor decreasing at any point. So the rate of change, or the slope of a tangent line is 0 at any point.

Let us then determine the derivative function of first degree polynomial functions. Let $f(x) = ax$, where $a \in \mathbb{R}$. Now

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{a(x+h) - ax}{h} \\ &= \lim_{h\to 0} \frac{\cancel{ax} + ah \cancel{- ax}}{h} \\ &= \lim_{h\to 0} a \cdot \frac{\cancel{h}}{\cancel{h}} \\ &= a. \end{aligned}$$

Now we know that $f'(x) = a$ for any $f(x) = ax$, where $a \in \mathbb{R}$. By the sum rule we then obtain $f'(x) = a$ also when $f(x) = ax + b$.

Finally, let us determine the derivative function for a polynomial function of any degree. Let $f(x) = x^n$, where $n \in \{1, 2, \ldots\}$. For this one we use the first definition of the derivative. We get the derivative function as $f'(a)$, and then we can replace a with x. We also have to use formula

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + a^2b^{n-3} + ab^{n-2} + b^{n-1}).$$

We then get

$$
\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\
&= \lim_{x\to a} \frac{x^n-a^n}{x-a} \\
&= \lim_{x\to a} \frac{\cancel{(x-a)}(x^{n-1}+x^{n-2}a+x^{n-3}a^2+\cdots+x^2a^{n-3}+xa^{n-2}+a^{n-1})}{\cancel{x-a}} \\
&= a^{n-1}+a^{n-2}\cdot a+a^{n-3}\cdot a^2+\cdots+a^2\cdot a^{n-3}+a\cdot a^{n-2}+a^{n-1} \\
&= \underbrace{a^{n-1}+a^{n-1}+\cdots+a^{n-1}+a^{n-1}}_{n \text{ times}} \\
&= na^{n-1}.
\end{aligned}
$$

Let us replace a with x, and we get $f'(x) = nx^{n-1}$ when $f(x) = x^n$ ($n \in \{1, 2, \dots\}$). We know that

$$\lim_{x\to x_0} a \cdot f(x) = a \cdot \lim_{x\to x_0} f(x)$$

so we can determine that if $f(x) = ax^n$, then $f'(x) = a \cdot nx^{n-1}$, where $a \in \mathbb{R}, n \in \{1, 2, \dots\}$.

Derivative function of polynomial functions.

(a) $D(a) = 0$

(b) $D(ax + b) = a$

(c) $D(ax^n) = nax^{n-1}$

(d) $D(a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0)$
$= D(a_nx^n) + D(a_{n-1}x^{n-1}) + \cdots + D(a_1x) + D(a_0)$
$= na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1$

$a, b \in \mathbb{R}, n \in \{1, 2, \dots\}$.
[4, p.90] .

Note!
Let $f(x) = ax^n$ and $n = 0$. Now we find $f(x) = ax^0 = a \cdot 1 = a$ and we can determine $f'(x) = D(a) = 0$ for every $x \in \mathbb{R}, a \in \mathbb{R}$. So if we have a situation where we have x^0, we are actually dealing with a constant.

Points (a)-(c) have already been proven. Point (d) actually follows straight from the points (a)-(c) and the formula $(f + g)'(x) = f'(x) + g'(x)$, but let us still prove the point (d) with the definition of the derivative. Let f be a polynomial function. We can write

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

First we have to use the definition of the derivative. Then we can rearrange the terms in the numerator, and use the formula $\lim_{h\to 0}(f+g) = \lim_{h\to 0} f + \lim_{h\to 0} g$ (page 13). Now we get

$$\begin{aligned}
D\left(f(x)\right) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{a_n(x+h)^n + a_{n-1}(x+h)^{n-1} + \cdots + a_1(x+h) + a_0 - (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)}{h} \\
&= \lim_{h\to 0} \frac{\Big(a_n(x+h)^n - a_n x^n\Big) + \Big(a_{n-1}(x+h)^{n-1} - a_{n-1}x^{n-1}\Big) + \cdots + \Big(a_1(x+h) - a_1 x\Big)}{h} \\
&= \lim_{h\to 0} \frac{a_n(x+h)^n - a_n x^n}{h} + \lim_{h\to 0} \frac{a_{n-1}(x+h)^{n-1} - a_{n-1}x^{n-1}}{h} + \cdots + \lim_{h\to 0} \frac{a_1(x+h) - a_1 x}{h} \\
&= D\left(a_n x^n\right) + D\left(a_{n-1}x^{n-1}\right) + \cdots + D\left(a_1 x\right) \\
&= n a_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1.
\end{aligned}$$

Example 1

Find the derivative function of $f(x) = \frac{1}{3}x^3 - 7x^2 + \sqrt{2}x - \pi$.

Solution

By using the formulas above we get

$$\begin{aligned}
f'(x) &= D(\frac{1}{3}x^3 - 7x^2 + \sqrt{2}x - \pi) \\
&= D(\frac{1}{3}x^3) + D(-7x^2) + D(\sqrt{2}x) + D(-\pi) \\
&= \frac{1}{3} \cdot 3x^2 - 7 \cdot 2x^1 + \sqrt{2} + 0 \\
&= x^2 - 14x + \sqrt{2}.
\end{aligned}$$

Answer:

$f'(x) = x^2 - 14x + \sqrt{2}.$

Derivative function of product and quotient

To help us determine even more derivatives easier, we need a couple furthermore rules. Here we determine the derivative rules for product and quotient functions.

Derivatives of product and quotient

(a)

$$(fg)'(x) = f(x)g'(x) + g(x)f'(x)$$

(b)

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

provided that $g(x) \neq 0$.

[11, p.116,120] .

Let us prove these formulas. First we prove the derivative formula for the product function $fg(x)$. Here we carry out this with the first definition of the derivative, so that x approaches a. We find

$$\begin{aligned}(fg)'(x) &= \lim_{x\to a}\frac{(fg)(x) - (fg)(a)}{x-a} \\ &= \lim_{x\to a}\frac{f(x)g(x) - f(a)g(a)}{x-a}.\end{aligned}$$

Here we have to utilice a little trick. If we subtrack and add $f(a)g(x)$ in the numerator, we can compose some of the terms differently. We obtain

$$\begin{aligned}&\lim_{x\to a}\frac{f(x)g(x) - f(a)g(a)}{x-a} \\ &= \lim_{x\to a}\frac{f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)}{x-a} \\ &= \lim_{x\to a}\left(\frac{(f(x) - f(a))\cdot g(x)}{x-a} + \frac{f(a)\cdot(g(x) - g(a))}{x-a}\right) \\ &= \underbrace{\lim_{x\to a}\frac{f(x) - f(a)}{x-a}}_{=f'(a)}\cdot\lim_{x\to a}g(x) + \lim_{x\to a}f(a)\cdot\underbrace{\lim_{x\to a}\frac{g(x) - g(a)}{x-a}}_{=g'(a)} \\ &= f'(a)g(a) + f(a)g'(a).\end{aligned}$$

Now, if we replace a with x, and arrange the terms in another order we get the formula $(fg)'(x) = f(x)g'(x) + g(x)f'(x)$.

To prove the formula for quotients, we have to adopt another trick. First, let us define a new function $h(x) = \frac{f(x)}{g(x)}$. Now we have to try to determine $h'(x)$. Since $h(x) = \frac{f(x)}{g(x)}$, we get $f(x) = h(x)g(x)$, and with the derivative formula for products we get

$$f'(x) = h(x)g'(x) + g(x)h'(x).$$

Now we were after for $h'(x)$, so let us re-arrange the therms of the equation to find this $h'(x)$. We conclude

$$\begin{aligned}
f'(x) &= h(x)g'(x) + g(x)h'(x) \\
g(x)h'(x) &= f'(x) - h(x)g'(x) && \Big\| : g(x) \\
h'(x) &= \frac{f'(x)}{g(x)} - \frac{h(x)g'(x)}{g(x)} && \Big\| h(x) = \frac{f(x)}{g(x)} \\
h'(x) &= \frac{g(x) \cdot f'(x)}{(g(x))^2} - \frac{f(x)g'(x)}{(g(x))^2} \\
h'(x) &= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.
\end{aligned}$$

Now since $h'(x) = \left(\frac{f}{g}\right)'(x)$, we get

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

Example 1

Find the derivative functions of

(a) $f(x) = (x^2 + 3x - 4)(x - 8)$

(b) $g(x) = \pi(2x - 1)$

(c) $h(x) = 2x(x - 3)$.

Solution

(a) Let us use the formula for product. Now, let us assign the letters of functions in the formula as

$$(gh)'(x) = g(x)h'(x) + h(x)g'(x)$$

for clarity. Now we can see that

$$f(x) = (x^2 + 3x - 4)(x - 8) = g(x) \cdot h(x),$$

where

$$g(x) = x^2 + 3x - 4 \qquad \text{and} \qquad h(x) = x - 8.$$

For the derivative function we get

$$\begin{aligned}
f'(x) &= (gh)'(x) = g(x)h'(x) + h(x)g'(x) \\
&= (x^2 + 3x - 4) \cdot D(x - 8) + (x - 8) \cdot D(x^2 + 3x - 4) \\
&= (x^2 + 3x - 4) \cdot 1 + (x - 8)(2x + 3) \\
&= x^2 + 3x - 4 + 2x^2 + 3x - 16x - 24 \\
&= 3x^2 - 10x - 28.
\end{aligned}$$

There is actually another way of finding this derivative function. Sometimes it might be easier to first calculate the product of the functions in $f(x)$. Here we have a product of two polynomials, and so the result is also a polynomial. We find

$$\begin{aligned}
f(x) &= (x^2 + 3x - 4)(x - 8) \\
&= x^3 - 8x^2 + 3x^2 - 24x - 4x + 32 \\
&= x^3 - 5x^2 - 28x + 32.
\end{aligned}$$

Now we can determine that

$$f'(x) = 3 \cdot x^2 - 5 \cdot 2 \cdot x^1 - 28 = 3x^2 - 10x - 28$$

just by using the derivative formula for polynomials.

(b) Here we see that $g(x)$ is a product of functions π and $2x - 1$. Now $g'(x)$ would be a lot easier to determine if we do the multiplication, get $g(x) = 2\pi x - \pi$ and use the derivative formula for polynomicals (as is shown later on), but for practice let us determine $g'(x)$ by using the derivative formula for products. We get

$$\begin{aligned}
g'(x) &= (\pi) \cdot D(2x - 1) + (2x - 1) \cdot D(\pi) \\
&= \pi \cdot 2 + (2x - 1) \cdot 0 \\
&= 2\pi.
\end{aligned}$$

Again, we can check this by looking at the function as the polynomial

$$g(x) = \pi(2x - 1) = 2\pi x - \pi,$$

and determine the derivative function as

$$g'(x) = 2\pi.$$

(c) Now $h(x)$ is the product of the functions $2x$ and $x-3$, and so

$$\begin{aligned} h'(x) &= (2x)\cdot D(x-3) + (x-3)\cdot D(2x) \\ &= 2x\cdot 1 + (x-3)\cdot 2 \\ &= 2x + 2x - 6 \\ &= 4x - 6. \end{aligned}$$

This could also be calculated by the derivative formula for polynomials.

Example 2

Find the derivative functions of

(a)

$$f(x) = \frac{x^3 - x^2 + x - 1}{x - 1}$$

(b)

$$g(x) = \frac{1}{x}.$$

Solution

(a) Again, we can start by assigning the letters of functions in the formula for derivative of quontient so that it is easier to apply the formula. Let us look at the formula as

$$\left(\frac{g}{h}\right)'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2}.$$

Now $f(x) = \frac{g(x)}{h(x)}$ where $g(x) = x^3 - x^2 + x - 1$ and $h(x) = x - 1$. One thing to remember with this formula was that it does not work when the denominator equals 0. Now $x - 1 = 0$ when $x = 1$, so we have to set $x \neq 1$. Now we can use the formula and get

$$\begin{aligned} f'(x) = \left(\frac{g}{h}\right)'(x) &= \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2} \\ &= \frac{(x-1)\cdot D(x^3 - x^2 + x - 1) - (x^3 - x^2 + x - 1)\cdot D(x-1)}{(x-1)^2} \\ &= \frac{(x-1)(3x^2 - 2x + 1) - (x^3 - x^2 + x - 1)\cdot 1}{(x-1)^2} \\ &= \frac{3x^3 - 2x^2 + x - 3x^2 + 2x - 1 - x^3 + x^2 - x + 1}{(x-1)^2} \\ &= \frac{2x^3 - 4x^2 + 2x}{(x-1)^2}. \end{aligned}$$

We can simplify this little bit further. If we take $x = 1$, wee see that $2 \cdot 1^3 - 4 \cdot 1^2 + 2 \cdot 1 = 0$, so we know that we can divide $2x^3 - 4x^2 + 2x$ into factors, where $x - 1$ is one of the factors. We can find another factor by using polynomial long division. We find

$$\begin{array}{r l}
 & 2x^2 - 2x \\ \cline{2-2}
x - 1) & 2x^3 - 4x^2 + 2x \\
 & -2x^3 + 2x^2 \\ \cline{2-2}
 & -2x^2 + 2x \\
 & 2x^2 - 2x \\ \cline{2-2}
 & 0
\end{array}$$

so we can write $2x^3 - 4x^2 + 2x$ as $(x - 1)(2x^2 - 2x)$. Again we see that $2x^2 - 2x = 2x(x - 1)$, so we get

$$\begin{aligned}
f'(x) &= \frac{2x^3 - 4x^2 + 2x}{(x-1)^2} \\
&= \frac{\cancel{(x-1)}2x\cancel{(x-1)}}{\cancel{(x-1)}\cancel{(x-1)}} \\
&= 2x.
\end{aligned}$$

(b) By using the formula for the derivative of quotients, we get

$$\begin{aligned}
g'(x) &= \frac{x \cdot D(1) - 1 \cdot D(x)}{x^2} \\
&= \frac{x \cdot 0 - 1 \cdot 1}{x^2} \\
&= -\frac{1}{x^2}.
\end{aligned}$$

If we look at this derivative, we can see the similarity to the formula for derivatives of polynomials, as $\frac{1}{x} = x^{-1}$, and $-\frac{1}{x^2} = -x^{-2}$. If we looked at this and determined the derivative as it were polynomial function, we would get the same answer as

$$D\left(\frac{1}{x}\right) = D(x^{-1}) = -1 \cdot x^{-1-1} = -x^{-2} = -\frac{1}{x^2}.$$

Let us prove that we can actually use this formula for all the functions $\frac{1}{x^n}$, where $n \in \mathbb{N}$. Let $f(x) = \frac{1}{x^n} = x^{-n}, n \in \{1, 2, \dots\}$. Then

$$\begin{aligned}
f'(x) &= \frac{x^n \cdot D(1) - 1 \cdot D(x^n)}{(x^n)^2} \\
&= \frac{x^n \cdot 0 - n \cdot x^{n-1}}{x^n \cdot x^n} \\
&= -n \cdot \frac{\cancel{x^{n-1}}}{x \cdot \cancel{x^{n-1}} \cdot x^n} \\
&= -n \cdot \frac{1}{x^{n+1}} \\
&= -n \cdot x^{-(n+1)} = -nx^{-n-1}.
\end{aligned}$$

Tip!

Let $f(x) = x^n$. Now

$$f'(x) = nx^{n-1},$$

when $n \in \mathbb{Z}$.

Let us check if this works when $n = 0$. We know that $f(x) = x^0 = 1$, and $D(1) = 0$. If we use this formula, we get $f'(x) = 0 \cdot x^{-1} = 0$, so the formula works and we do not have to make any restrictions for n.

Example 3

This formula works also for all the functions $f(x) = \frac{a}{x^n}$ when $a \in \mathbb{R}$. Since we have a formula that says $D(af) = aDf$, we get

$$f'(x) = D\left(\frac{a}{x^n}\right) = D\left(a \cdot \frac{1}{x^n}\right) = a \cdot D(x^{-n}) = a \cdot (-n)x^{-n-1} = -anx^{n-1}.$$

So as long as you have a real number in the numerator (do not have any x's), you can use this formula. For example if

$$f(x) = \frac{\pi - \sqrt{2}}{x^2} = (\pi - \sqrt{2})x^{-2} \quad \text{, then} \quad f'(x) = (\pi - \sqrt{2}) \cdot (-2)x^{-3} = \frac{-2(\pi - \sqrt{2})}{x^3}.$$

Example 4

Find the derivative for

$$f(x) = \frac{x^2 + x + 1}{x}.$$

Solution

Way 1
If we look at this function as

$$f(x) = (x^2 + x + 1) \cdot \frac{1}{x},$$

we can use the formula for product. Now $f(x)$ is the product of functions $x^2 + x + 1$ and x^{-1}, and so

$$\begin{aligned} f'(x) &= (x^2 + x + 1) \cdot D(x^{-1}) + x^{-1} \cdot D(x^2 + x + 1) \\ &= (x^2 + x + 1)(-1 \cdot x^{-2}) + x^{-1}(2x + 1) \\ &= -\frac{x^2 + x + 1}{x^2} + \frac{2x + 1}{x} \\ &= \frac{-x^2 - x - 1}{x^2} + \frac{x(2x + 1)}{x \cdot x} \\ &= \frac{-x^2 - x - 1 + 2x^2 + x}{x^2} \\ &= \frac{x^2 - 1}{x^2} \\ &= \frac{\cancel{x^2}}{\cancel{x^2}} - \frac{1}{x^2} \\ &= 1 - \frac{1}{x^2}. \end{aligned}$$

Way 2
We can also distribute this function into terms, and use the formula $D(ax^n) = anx^{n-1}, n \in \mathbb{Z}$. As

$$f(x) = \frac{x^{\cancel{2}}}{\cancel{x}} + \frac{\cancel{x}}{\cancel{x}} + \frac{1}{x} = x + 1 + \frac{1}{x},$$

we get

$$f'(x) = D(x) + D(1) + D(x^{-1}) = 1 + 0 + (-1)x^{-2} = 1 - \frac{1}{x^2}.$$

Way 3
We can also determine the derivative with the formula for quotient functions. We get

$$\begin{aligned} f'(x) &= \frac{x \cdot D(x^2 + x + 1) - (x^2 + x + 1) \cdot D(x)}{x^2} \\ &= \frac{x(2x + 1) - (x^2 + x + 1)}{x^2} \\ &= \frac{x^2 - 1}{x^2} = 1 - \frac{1}{x^2}. \end{aligned}$$

3.5 Applications for the derivatives

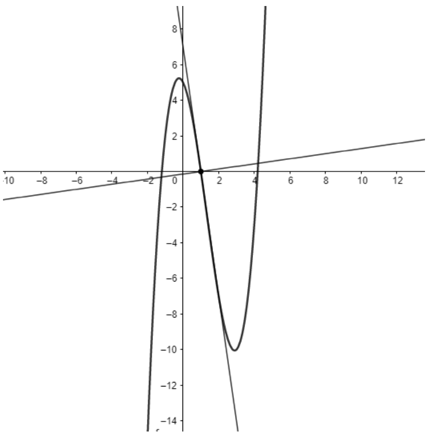
Tangent line of a curve

We started defining the derivative as the *instantaneous rate of chance*, and now it is a good time to recall this use of the derivative. We illustrated that instantaneous rate of change is actually the slope of a tangent line drawn into given point, and we know that the derivative function gives the value of the instantaneous rate of change at a given point.

Example 1

(a) Determine the equation of the tanget line for the curve $y = x^3 - 4x^2 - 2x + 5$ at $x = 1$.

(b) Determine the equation of the normal line for this tangent line at $x = 1$.



Solution

(a) So the first thing we have to do is to find the derivative at a point $x = 1$. First, let us assign the equation we have into a function, so that we can determine the derivative. Now let $f(x) = x^3 - 4x^2 - 2x + 5$. Now $f(x)$ is a polynomial function, so we can use the derivative formula for polynomial functions. For the derivative function for $f(x)$ we get

$$f'(x) = 3x^2 - 8x - 2.$$

Now we were asked for the tangent line at $x = 1$, so we can determine that the slope of this line is

$$f'(1) = 3 \cdot 1^2 - 8 \cdot 1 - 2 = -7.$$

We also know that this tangent line touches the curve of $f(x)$ at $x = 1$, so this tangent line goes through the point $(1, f(1))$. As

$$f(1) = 1^3 - 4 \cdot 1^2 - 2 \cdot 1 + 5 = 0,$$

we can form the equation of the tangent line. As $k = -7$, $x_0 = 1$ and $y_0 = 0$, we get

$$\begin{aligned} y - y_0 &= k(x - x_0) \\ y - 0 &= -7(x - 1) \\ y &= -7x + 7. \end{aligned}$$

Answer:
The tangent line for curve $y = x^3 - 4x^2 - 2x + 5$ at $x = 1$ is $y = -7x + 7$.

(b) As we know, the normal line is perpendicular to the tangent line, so the slope of the normal line is $\frac{1}{7}$ (since $\frac{1}{7} \cdot (-7) = -1$), and it goes through the same point $(1, 0)$, so we can determine the equation the same way as

$$\begin{aligned} y - y_0 &= k(x - x_0) \\ y - 0 &= \frac{1}{7}(x - 1) \\ y &= \frac{1}{7}x - \frac{1}{7}. \end{aligned}$$

Answer:
The normal line for curve $y = x^3 - 4x^2 - 2x + 5$ at $x = 1$ is $y = \frac{1}{7}x - \frac{1}{7}$.

Maximum and minimum values

First we have to define what the maximum and minimum values are. There are actually two types of maximum/minimum values: local maximum and minimum values and the absolute maximum and minimum values. Local maximum and minimum values are the values that are highest and lowest at that part of the function. Basically, if we are deciphering the local maximum value, the points next to the local maximum value point can not take on a higher value than the point of maximum value, and vice versa to the minimum. The absolute maximum and minimum values are the highest and lowest values the function takes in its domain. Maximum and minimum values are said to be *extreme values*.

Extreme values

Local max and min values:
[11, p.195-196].

(i) A function f is said to have a ***local maximum value*** $M = f(b)$ at b if $f(x) \leq M$ for all x sufficiently close to b.

(ii) A function f is said to have a ***local minimum value*** $m = f(c)$ at c if $f(x) \geq m$ for all x sufficiently close to c.

Absolute max and min values:
[11, p.203].

(i) A function f is said to have an ***absolute maximum value*** $M = f(b)$ at b if $f(x) \leq M$ for all x in the domain of f.

(ii) A function f is said to have an ***absolute minimum value*** $m = f(c)$ at c if $f(x) \geq m$ for all x in the domain of f.

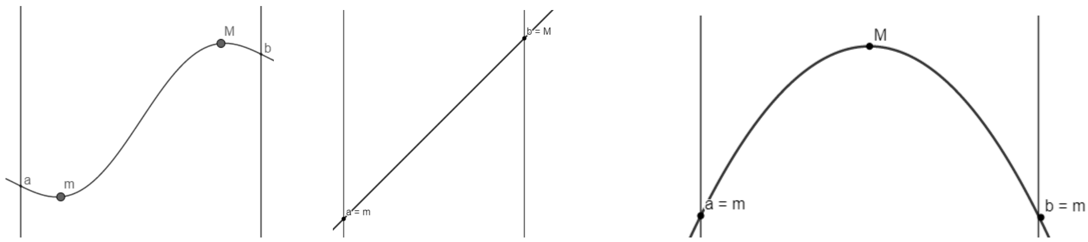
The point where function takes on maximum or minimum value is said to be ***maximum or minimum point***.

If we think about these definitions for a bit, we realize that all the absolute extreme values are also local extreme values. Since none of the points near the point of absolute extreme value can take on higher/lower value than the point of the absolute extreme value, clearly they are also local extreme values. This does not go other way around; not all of the local extreme values are absolute extreme values.

Extreme Value Theorem

If f is continuous on $[a, b]$, then f takes on both a maximum value M and a minimum value m on $[a, b]$. [11, p.97-98].

In this theorem maximum and minimum values refer to absolute maximum and minimum values. This theorem might be easy to accept if you think about it for a minute. Here we have three graphs of continuous functions on $[a, b]$.



As we see, values m and M are indicating the *absolute local values*. Obviously, this theorem qualifies for local extreme values as well, since absolute extreme values are also local extreme values, but this theorem is a lot stronger because it qualifies for absolute extreme values.

As strong as this theorem is, it only qualifies for closed intervals. If you look at the graph in the middle, it is easy to see that if this function was defined in $\mathbb{R}$, it would not have a maximum nor minimum value. For example, let $f(x) = x$. Now, there cannot be any $a \in \mathbb{R}$ for which $f(a)$ is the minimum value, since $f(a-1) = a - 1 < a = f(a)$. With the same logic we can prove that there cannot be any $b \in \mathbb{R}$ that would be point of a maximum value.

We end up with a similar problem if we have an open interval $]a, b[$. If we have the same function $f(x) = x$, the minimum value would be at the beginning of the interval, and the maximum value at the end of the interval. But as we have an open interval, we do not have a starting nor ending point. So we cannot say that $f(a) = a$ would be the minimum value, since $f(x)$ is not defined in a, even though we know that $f(x)$ can not take on a value less than a. We can only say that the minimum value of $f(x)$ *approaches* the value $\lim_{x \to a^+} f(x)$. Again, the same logic applies for the maximum value.

What we do know with this theorem, is that if we have a closed interval and a continuous function in it, this function takes on maximum and minimum values within this interval. Extreme values can be either at end of the interval, or in the middle. If we look at the first and the last picture, both functions have the maximum value in the middle of the interval. If we follow the graph, we see that as we are approaching maximum point the graph is rising, and as we pass the maximum point the graph starts to decrease. If we think about this in terms of the derivative, we see that just before the maximum point the derivative is positive, since the graph is rising, and right after the maximum point the derivative is negative, since the graph is decreasing. Extreme points are points where the direction of the graph of the function changes, so the derivative for this point has to be zero.

Critical points

Possible points of extreme values are called ***critical points***. These are the following:

(1) Beginning and end points of interval (if the function is defined on a closed interval).

(2) Points where the derivative function is 0.

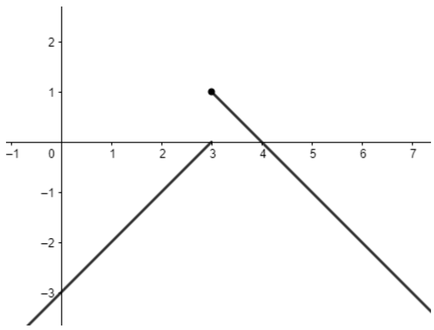
(3) Points of discontinuity or points where the function is not differentiable.

The first point is already established, as we already saw in the graph in the middle. The third point is also quite quick to establish. We can quickly think of an example of a discontinuous function that has maximum or minimum value at the point of discontinuity. For example let

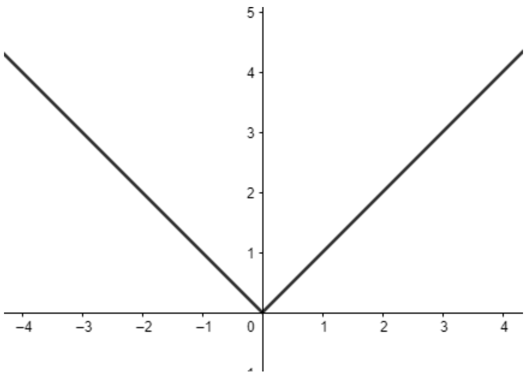
$$f(x) = \begin{cases} x - 3 & \text{when } x < 3, \\ -x + 4 & \text{when } x \geq 3. \end{cases}$$

The maximum value of $f(x)$ is 1 at point $x = 3$. As another example, if we look at the function $g(x) = |x|$, we see that this is a continuous function, but it is not differentiable at $x = 0$. Yet this point is the minimum point, where the minimum value is 0.

$$f(x) = \begin{cases} x - 3 & \text{when } x < 3, \\ -x + 4 & \text{when } x \geq 3. \end{cases}$$



$$g(x) = |x|$$



It takes a bit work to establish the second point. Let $f(x)$ be a function and a a point where f is defined, continuous and differentiable. Let a be a point of extreme value. Now since f is continuous and differentiable at this point, the first and the third point do not qualify for a. The only option is to prove that in this point a the derivative function of f is zero.

Now, if $f'(a)$ would be less than 0, the graph of the function would be decreasing through the point a, so $f(x)$ would take on smaller values at the points very next to

a at the right side of a and bigger values at the points very next to a at the left side of a. So then a would not be a point of an extreme value. Very similarly, if $f'(a) > 0$, then the graph of $f(x)$ would be rising through point a, and $f(x)$ would take on bigger values at the points very next to a on the right side of a and smaller values at the points very next to a at the left side of a. Again then a would not be a point of an extreme value. The only option left is that $f'(a) = 0$.

What this shows us is that for every single point of extreme value one or the other applies:

(1) the derivative function in that point is 0, or

(2) the derivative does not exist in that point.

Example 1

(a) Find the extreme values of $f(x) = x^2 - 3x - 1$ on interval $[-2, 4]$.

(b) Find the extreme values of $g(x) = \sqrt{2}x^4 - 4$.

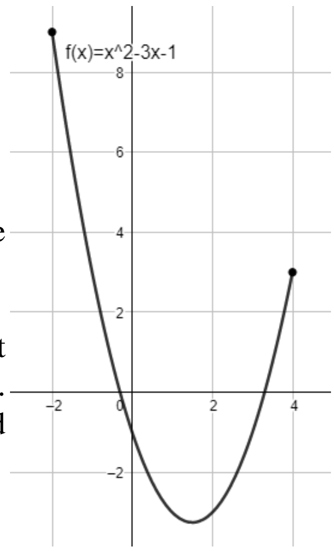
Solution

(a) First we check the ending points of the interval:

$$f(-2) = (-2)^2 - 3 \cdot (-2) - 1 = 9,$$

$$f(4) = 4^2 - 3 \cdot 4 - 1 = 3.$$

We see from the graph that these both points are local maximum points.



Then we have to determine whether there is a point in that interval where the derivative would be 0. As a polynomial function, $f(x)$ is continuous and differentiable in open interval $]-2, 4[$.

Now

$$f'(x) = 2x - 3,$$

and

$$2x - 3 = 0$$
$$x = \frac{3}{2}.$$

Now $\frac{3}{2} \in]-2, 4[$, so this point is also a possible local extreme point, and $f(\frac{3}{2}) = -\frac{13}{4}$. There is not any more possible points of extreme values. We find that $f(\frac{3}{2}) < f(4) < f(-2)$, so we know that $f(\frac{3}{2}) = -\frac{13}{4}$ is the absolute minimum, $f(-2) = 9$ is the absolute maximum and $f(4) = 3$ is a local maximum.
<u>Answer:</u>
Absolute maximum value at $x = -2$ is 9, and the absolute minimum value at $x = \frac{3}{2}$ is $-\frac{13}{4}$.

(b) Now $g(x) = \sqrt{2}x^4 - 4$ is continuous and differentiable in $\mathbb{R}$, so we only have to check the zero-points of the derivative. We find

$$g'(x) = \sqrt{2} \cdot 4x^3,$$

and

$$\begin{aligned} 4\sqrt{2}x^3 &= 0 \\ x^3 &= 0 \\ x &= 0. \end{aligned}$$

Now we know that there is a possible point of an extreme value at $x = 0$, and $g(0) = -4$. As we check any two points, one that is less than 0 and one that is greater than 0, we can tell if this point $x = 0$ is the minimum or maximum. For example

$$\begin{aligned} g(-1) &= \sqrt{2} \cdot (-1)^4 - 4 = \sqrt{2} - 4 > -4 \\ g(1) &= \sqrt{2} \cdot 1^4 - 4 = \sqrt{2} - 4 > -4. \end{aligned}$$

So $x = 0$ is a minimum point.
<u>Answer:</u>
The absolute minimum value of $g(x)$ is -4 at $x = 0$.

Note !

If you again look at the box that tells us possible points of extreme values, you see that in points 1 and 3, the functions are not differentiable in points given. This means that for all the points of extreme value in which the function is differentiable, the derivative in that point is 0. Actually what this does not mean, is that a point would automatically be a point of extreme value when the derivative equals 0.

Example 2

Determine the extreme values of $f(x) = x^3$.

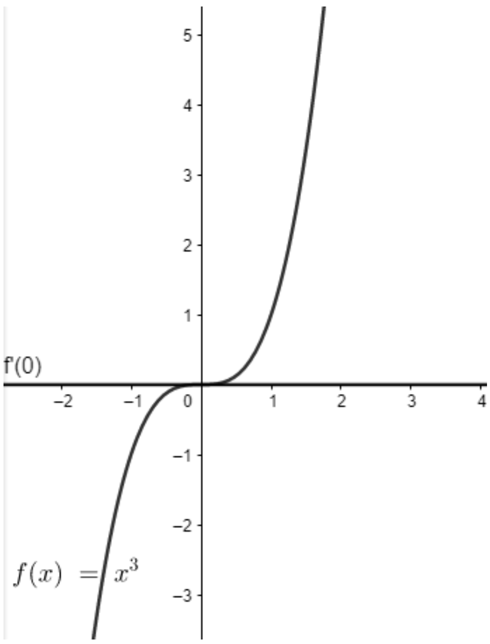
Solution

Again we have a polynomial function, so we know that it is differentiable in $\mathbb{R}$. So let us check the zero-points of the derivative. We find

$$f'(x) = 3x^2,$$

and

$$\begin{aligned} 3x^2 &= 0 \\ x &= 0. \end{aligned}$$



So $x = 0$ is the only possible point of extreme value. But if we look at the graph, it tells us that this point is neither maximum nor minimum, since $f(0) = 0$, $f(x) < 0$ for all $x < 0$, and $f(x) > 0$ for all $x > 0$.

So this proves that there can be points where the derivative is 0, but the point still is not an extreme point.

Table of signs of the derivative

Doing a table of signs is a reliable way of finding out whether the zero-points of the derivative are actually points of extreme values or not. Basically, what we do is we find out where are the possible points of extreme values, and what sign is the derivative in between. This way we find out to which direction the graph of the function is going, and that will tell us whether these points are extreme value points or not.

Example 1

Let us look at the last example, $f(x) = x^3$. We determined that the derivative is $f'(x) = 3x^2$, and the only zero-point is $x = 0$. Now, as we determined, this is the only possible point of extreme value, since there are no restrictions for the variable, and x^3 is differentiable in $\mathbb{R}$.

So the first thing is to put the possible points of extreme values in the line of numbers, in this case just the point 0.

x		0	
$f'(x)$		0	
$f(x)$		$\rightarrow$	

So the first line of this table is the line of real numbers, where you write the possible points of extreme values. The second line as you can see is the line where you mark the signs of the derivative. So far we know that at the point 0 the value of the derivative is 0, so we write this down. The third line describes the direction of the function. So as the value of the derivative is 0, we know that the function is horizontal at that point, so we mark this as $\rightarrow$.

Now we have to fill up the rest of the table. In order to do this we have to pick a number that is on the left side of 0, and another that is on the right side of 0. For example numbers -1 and 1. These numbers represent their column and will tell which direction the function is going in that column. As we determined, the derivative can change signs only in possible points of extreme values, so in zero-points of the derivative or points where that function is not differentiable. So in this case we could pick any number that is less than 0 to represent the first column, like -15367, and any number that is greater than 0 to represent the third column, like $\sqrt{2}$ if we wanted. But to keep it simple, we can choose -1 and 1. Now we calculate the value of the derivative in these points, and mark the signs in the table. We find

$$f'(-1) = 3 \cdot (-1)^2 = 3 > 0$$
$$f'(1) = 3 \cdot 1^2 = 3 > 0.$$

So the derivative is positive on both sides of 0. We know that when the derivative is positive, the graph of the function is rising, and we mark this with an arrow.

x		0	
$f'(x)$	+	0	+
$f(x)$	$\nearrow$	$\rightarrow$	$\nearrow$

Here we can see that the function is rising before 0 and after it, so 0 cannot be a point of extreme value. This type of critical point is called *saddlepoint.*

Example 2

Find the local extreme values for $f(x) = \frac{1}{x} + x$, and sketch the graph of the function with the help of the table of signs.

Solution

First we have to examine which points are the possible points of extreme values. This function is defined in $\mathbb{R}$ apart from 0, since $\frac{1}{x}$ is not defined in 0. So this is a point of intrest. Then we have to derivate the function, and find out the zero-points of the derivative. We can write the function as $f(x) = x^{-1} + x$ to make calculating of the derivative a bit simple. We get

$$\begin{aligned} f'(x) &= (-1) \cdot x^{-2} + 1 \\ &= -\frac{1}{x^2} + 1 \end{aligned}$$

and for the zero-points we get

$$\begin{aligned} -\frac{1}{x^2} + 1 &= 0 \\ \frac{1}{x^2} &= 1 \\ x^2 &= 1 \\ x &= \pm 1. \end{aligned}$$

Now we can start to construct the table of signs. We need to put the points of intrest in the table, and make sure that we remember that this function is not defined at 0.One way to do this is to mark X below 0, so that we cannot write anything there, as is done here. We can also write down the zero-points of the derivative.

x		-1		0		1	
$f'(x)$		0		X		0	
$f(x)$		$\rightarrow$		X		$\rightarrow$	

The next thing is to choose values to represent the blank columns. For example, we can choose values $-2, -\frac{1}{2}, \frac{1}{2}$ and 2. We obtain

$$
\begin{aligned}
f'(-2) &= -\frac{1}{(-2)^2} + 1 = \frac{3}{4} > 0,\\
f'(-\frac{1}{2}) &= -\frac{1}{\left(-\frac{1}{2}\right)^2} + 1 = -3 < 0,\\
f'(\frac{1}{2}) &= -\frac{1}{\left(\frac{1}{2}\right)^2} + 1 = -3 < 0,\\
f'(2) &= -\frac{1}{2^2} + 1 = \frac{3}{4} > 0.
\end{aligned}
$$

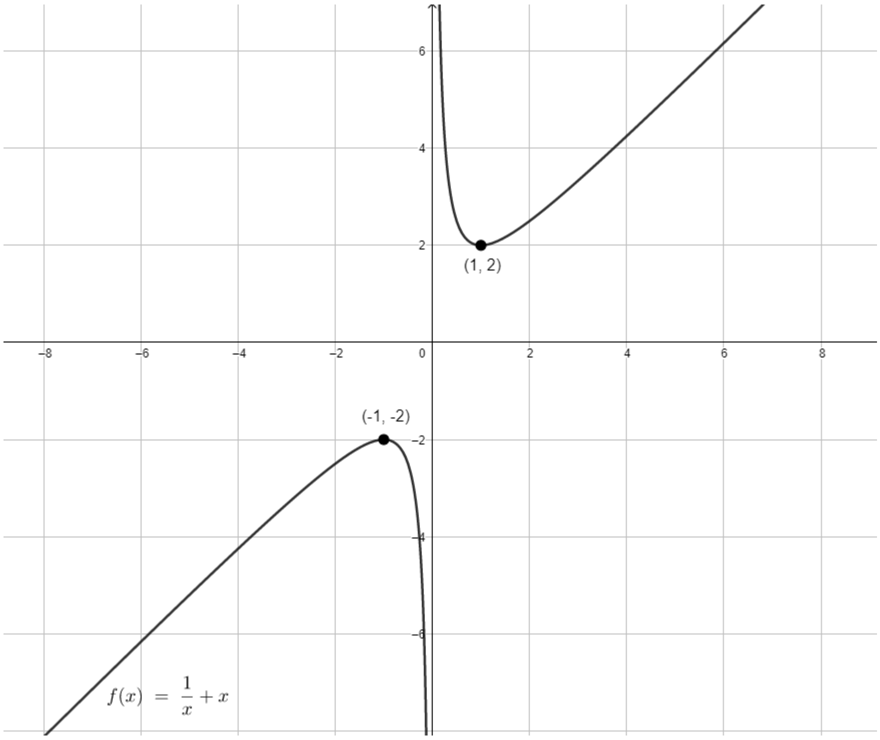
Now we can fill up the rest of the table.

x		-1		0		1	
$f'(x)$	$+$	0	$-$	X	$-$	0	$+$
$f(x)$	$\nearrow$	$\rightarrow$	$\searrow$	X	$\searrow$	$\rightarrow$	$\nearrow$

The table shows us that -1 is a point of local maximum and 1 is a point of local minimum. We still have to calculate the values of the function in these points. We find

$$
\begin{aligned}
f(-1) &= \frac{1}{-1} + (-1) = -2\\
f(1) &= \frac{1}{1} + 1 = 2.
\end{aligned}
$$

So now we know that the graph of this function is rising to the point $x = -1$, where it reaches the value -2, and then the graph starts to decrease when x approaches 0. We have to keep in mind that the graph cannot intersect the y-axis, since f is not defined at $x = 0$. Then, on the positive side of the x-axis, the graph is decreasing to the point $x = 1$, where the value of the function is 2, and then the graph starts to rise again. Now, below we have the graph of the function, and the sketch should look something like this



(1, 2)
(-1, -2)
$f(x) = \frac{1}{x} + x$

Example 3

Construct the table of signs for

$$g(x) = \frac{x^2 + 1}{x^2 + x + 1}$$

and determine local extreme values.

Solution

First we have to check whether there are any restrictions of the variable. So we check whether the denominator is 0 at any value of x. We find

$$\begin{aligned} x^2 + x + 1 &= 0 \\ x &= \frac{1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} \\ x &= \frac{1 \pm \sqrt{-3}}{2} \\ \Rightarrow &\qquad \text{no solution.} \end{aligned}$$

So there is no restrictions of the variable. The next step is to derivate the function. We can use the formula for the derivative of quotient function as

$$\begin{aligned} g'(x) &= \frac{(x^2 + x + 1) \cdot 2x - (x^2 + 1)(2x + 1)}{(x^2 + x + 1)^2} \\ &= \frac{2x^3 + 2x^2 + 2x - 2x^3 - 2x - x^2 - 1}{(x^2 + x + 1)^2} \\ &= \frac{x^2 - 1}{(x^2 + x + 1)^2}. \end{aligned}$$

And then find the zero-points of the derivative

$$\begin{aligned} \frac{x^2 - 1}{(x^2 + x + 1)^2} &= 0 \\ x^2 - 1 &= 0 \\ x &= \pm 1. \end{aligned}$$

So the table of signs begins as

x		-1		1	
$g'(x)$		0		0	
$g(x)$		$\rightarrow$		$\rightarrow$	

Then we have to find out the signs of the derivative before point -1, between points -1 and 1 and after point 1. For example we can choose values $-2, 0$ and 2. We find

$$
\begin{aligned}
g'(-2) &= \frac{(-2)^2 - 1}{((-2)^2 - 2 + 1)^2} = \frac{3}{9} > 0 \\
g'(0) &= \frac{0^2 - 1}{(0^2 + 0 + 1)^2} = -1 < 0 \\
g'(2) &= \frac{2^2 - 1}{(2^2 + 2 + 1)^2} = \frac{3}{49} > 0.
\end{aligned}
$$

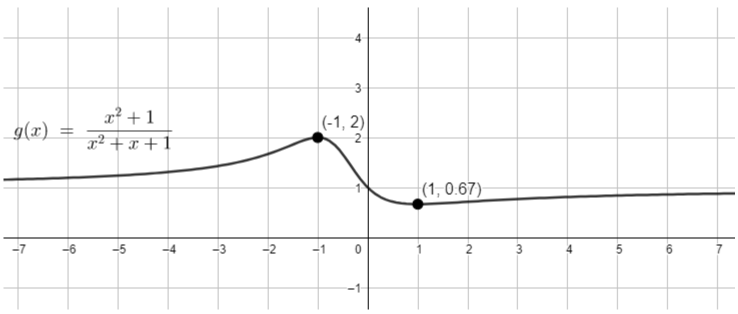
So now we can complete the table as

x		-1		1	
$g'(x)$	$+$	0	$-$	0	$+$
$g(x)$	$\nearrow$	$\rightarrow$	$\searrow$	$\rightarrow$	$\nearrow$

Now we see that point -1 is a local maximum and point 1 is a local minimum. Function values of these points are

$$
\begin{aligned}
g(-1) &= \frac{(-1)^2 + 1}{(-1)^2 - 1 + 1} = 2 \\
g(1) &= \frac{1^2 + 1}{1^2 + 1 + 1} = \frac{2}{3}.
\end{aligned}
$$

Below we see the graph of the function, and we can see that we were right. Actually, the local extreme values we found are absolute extreme values as the graph shows.



Answer:
Minimum value for g is $\frac{2}{3}$ at point 1 and maximum value for g is 2 at point -1.

Example 4

Let

$$f(x) = \begin{cases} x^2 - 2x + 2 & \text{when } -2 \le x < 2, \\ \frac{1}{3}x + \frac{4}{3} & \text{when } x \ge 2. \end{cases}$$

Find local extreme values. Are they absolute extreme values?

Solution

First we have to notice that $f(x)$ is not defined when $x < -2$. However, it is defined at point -2, so this is a possible point of extreme value. The value at this point is

$$f(-2) = (-2)^2 - 2 \cdot (-2) + 2 = 10.$$

Another point of intrest is point 2. We can first check if f is discontinuous at this point, and what the value is. We find

$$\lim_{x \to 2^-} (x^2 - 2x + 2) = 2^2 - 2 \cdot 2 + 2 = 2$$

$$f(2) = \frac{1}{3} \cdot 2 + \frac{4}{3} = 2.$$

So the function is continuous at $x = 2$, and the value of the function at point $x = 2$ is 2.

The last thing to do is to derivate the function. We have to do this in piecewise, so that first we find the derivative of $x^2 - 2x + 2$ when $-2 < x < 2$. Now -2 has to be restricted out, since this function is not differentiable at -2. The same applies when we derivate the second part $\frac{1}{3}x + \frac{4}{3}$. This derivative function only qualifies when $x > 2$. We get

$$\begin{aligned} f'(x) &= \begin{cases} D(x^2 - 2x + 2) & \text{when } -2 < x < 2, \\ D\left(\frac{1}{3}x + \frac{4}{3}\right) & \text{when } x > 2 \end{cases} \\ &= \begin{cases} 2x - 2 & \text{when } -2 < x < 2, \\ \frac{1}{3} & \text{when } x > 2. \end{cases} \end{aligned}$$

It can be shown that f does not have a derivative at point 2. We still have to calculate the zero-points of the derivative. First

$$\begin{aligned} 2x - 2 &= 0 \\ x &= 1 \qquad (\in]-2, 2[). \end{aligned}$$

As $\frac{1}{3} \neq 0$, the only zero-point of the derivative is 1. The value of the function at $x = 1$ is $f(1) = 1^2 - 2 \cdot 1 + 2 = 1$. Now we can start to assemble the table.

x	X	-2		1		2	
$f'(x)$	X	X		0		X	
$f(x)$	X			$\rightarrow$			

The X's in the table remind that f is not defined when $x < -2$, so we cannot put any values there. Also, there is X at the derivatives for points -2 and 2, since those points do not possess a derivative. The next step is to find the signs of the derivatives for columns $]-2, 1[$, $]1, 2[$ and $x > 2$. We have to keep in mind which part of the function we consider, so that we get the right signs. We find

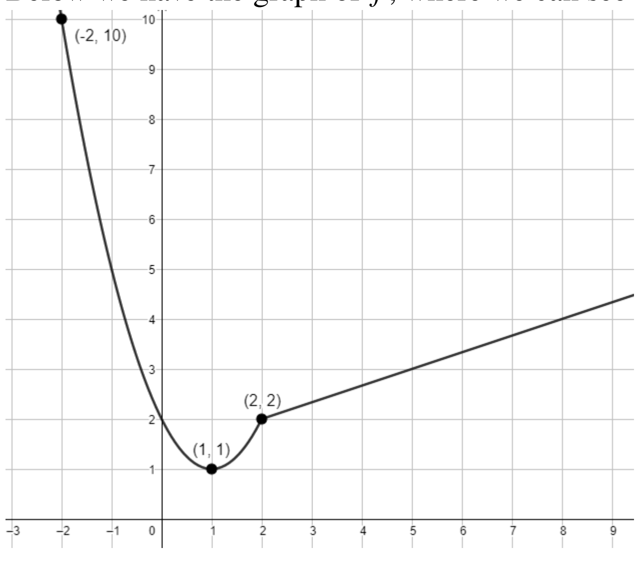
$$f'(0) = 2 \cdot 0 - 2 = -2 < 0$$
$$f'(\frac{3}{2}) = 2 \cdot \frac{3}{2} - 2 = 1 > 0$$
$$f'(3) = \frac{1}{3} > 0.$$

So now we can complete the table of signs.

x	X	-2		1		2	
$f'(x)$	X	X	$-$	0	$+$	X	$+$
$f(x)$	X		$\searrow$	$\rightarrow$	$\nearrow$		$\nearrow$

We see that the only point of local maximum is at $x = -2$. This is not absolute maximum point, since as x approaches ∞ f also approaches ∞. There is also only one point for local minimum, which is at $x = 1$. This actually is also absolute minimum. We checked that f is continuous at $x = 2$, and the derivative is positive when $x > 1$ ($x \neq 2$), so there is no way f to reach a value less than $f(1) = 1$. Point $x = 2$ is neither maximum nor minimum, since the derivative is positive on both sides of the point and the function is continuous at $x = 2$.

Below we have the graph of f, where we can see that we were correct.



Answer:
Local maximum value 10 is at the point $x = -2$ and the absolute minimum 1 is at the point $x = 1$.

Example 5

Farmer Mike needs to build up a new fence for his cows. He can only use 100m of fence, but wants to maximize the square meters. He is planning to build the fence in rectangle shape, so what are the measurements for the fence in order for the area be as large as it can?

Solution

So first things first, we know that Mike wants to build a rectangle shaped fence, and he has 100m of material. So if the sides of the rectangle are a and b, we know that $2a + 2b = 100$. We can get rid of other variable with this info, as

$$\begin{aligned} 2a + 2b &= 100 \\ a + b &= 50 \\ b &= 50 - a. \end{aligned}$$

So the sides are a and $50 - a$. Now we can make a function of the area, and find out where it gets its maximum value. The function for area is

$$f(a) = a \cdot (50 - a) = 50a - a^2.$$

We know that a cannot be negative, since it is a measurement. We also know that $50 - a$ cannot be negative for the same reason, so we get the restriction $0 \leq a \leq 50$. As a polynomial function $f(a)$ is differentiable in $]0, 50[$, so we can derivate it to find the maximum point. We get

$$f'(a) = 50 - 2a$$

and the zero-point is

$$\begin{aligned} 50 - 2a &= 0 \\ a &= 25. \end{aligned}$$

Let us write the table of values.

a	X	0		25		50	X
$f'(a)$	X	X		0		X	X
$f(a)$	X			$\rightarrow$			X

We can check the signs with $f'(10) = 50-2\cdot 10 = 30 > 3$ and $f(40) = 50 - 2 \cdot 40 = -30 < 0$. We get

	X	0		25		50	X
$f'(a)$	X	X	+	0	−	X	X
$f(a)$	X		↗	→	↘		X

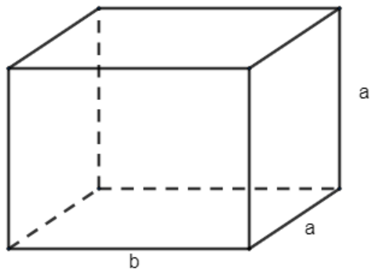
Here we see that $a = 25$ is the absolute maximum point of $f(a)$. So what we determined was that Mike should build the fence as a square, where all the sides are $25m$. Then the area of the fence is at its largest at $(25m)^2 = 625m^2$.

Answer:

The fence should be a square, where all the sides are 25 meters.

Example 6

A designer wants to desing a beautiful storage box that would have $1m^3$ volume. It would be the shape of rectangle prism, where two of the sides would be squares as in the picture below. What should the measurements be if the designer wanted to use as little of material as possible? (The box has a lid as well.)



Solution

First again, we have to get rid of the other variable. We know that the volume of the box is $V = a \cdot a \cdot b = 1m^3$, so we can use this to write b as

$$a^2 b = 1$$
$$b = \frac{1}{a^2}.$$

Now, as a and b are both measurements of a box, they both have to be greater than 0, so we do not have to worry about a being a denominator. Now we can make a function of the area of the box. We have two sides size of $a \cdot a$, two sides size of $a \cdot b$ and similar top and bottom pieces size of $a \cdot b$, so the function of the total area is

$$A(a) = 2(a^2) + 4\left(a \cdot \frac{1}{a^2}\right) = 2a^2 + \frac{4}{a} = 2a^2 + 4a^{-1}$$

Now, if we determine the derivative function of $A(a)$ and form the table of signs, we are able to find out the minimum of $A(a)$. For the derivative function we get

$$\begin{aligned} A'(a) &= 2 \cdot 2a + 4 \cdot (-1)a^{-2} \\ &= 4a - \frac{4}{a^2}, \end{aligned}$$

and for the points of value 0 we get

$$\begin{aligned} 4a - \frac{4}{a^2} &= 0 \\ 4a &= \frac{4}{a^2} && \Big\| \cdot a^2 \\ 4a^3 &= 4 && \Big\| : 4 \\ a^3 &= 1 \\ a &= 1. \end{aligned}$$

We still have to check the signs of the derivative for the intervals $]0, 1[$ and $]1, \infty[$ to form the table of signs . We find that $A'(\frac{1}{2}) = -14 < 0$ and $A'(2) = 7 > 0$, so the table on signs will become

a	X	0		1	
$A'(a)$	X	X	$-$	0	$+$
$A(a)$	X		$\searrow$	$\rightarrow$	$\nearrow$

We find that the point $a = 1$ is the minimum of $A(a)$, and $b = \frac{1}{a^2} = \frac{1}{1^2} = 1$, so the measurements of the box should be $1m \cdot 1m \cdot 1m$.

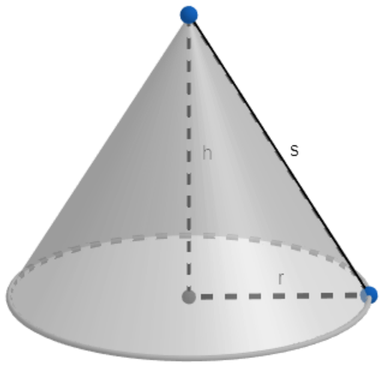
Answer:
Measurements should be $1m \cdot 1m \cdot 1m$.

Example 7 (Difficult)

A martini glass is a shape of right circular cone. The manufacturer needs the glass to hold 200ml of fluid. What are the measurements (height of the cone and radius of the circle) of the class when the manufacturer wants to use as little of material as possible?
(Hint! The value of the square root is at its smallest when the inside of the square root is at its smallest).

Solution

This problem might be a bit easier to start to solve if we think of the martini glass upside down as in the picture. We know that the volume must be 200ml, which is $0{,}2l = 0{,}2dm^3 = 200cm^3$. The formula for volume is $V = \frac{1}{3}\pi r^2 h$, so we can determine



$$\begin{aligned} 200 &= \frac{1}{3}\pi r^2 h \\ h &= \frac{200}{\frac{1}{3}\pi r^2} \\ h &= \frac{600}{\pi r^2}. \end{aligned}$$

The formula to calculate area of lateral surfice is $A = \pi r s$, so in order to make a function we have to eliminate one variable. We can determine s as an expression of r as

$$s^2 = r^2 + h^2 \qquad \Big\| h = \frac{600}{\pi r^2}$$

$$s = \sqrt{r^2 + \left(\frac{600}{\pi r^2}\right)^2} \qquad \Big\| \text{no need for } \pm\sqrt{\ldots} \text{ since } s > 0$$

$$s = \sqrt{r^2 + \frac{360000}{\pi^2} r^{-4}}.$$

We have to report the restrictions of r. First of all, r is a measurement, so it cannot be negative. It also is a factor in the denominator for h, so it can not be 0. Now we can make a function of area of lateral surfice with r as the variable, where $r > 0$. We get

$$A(r) = \pi r \sqrt{r^2 + \frac{360000}{\pi^2} r^{-4}}.$$

We can actually organize everything inside the square root to make our function little bit more simple, as

$$\begin{aligned} A(r) &= \pi r \sqrt{r^2 + \frac{360000}{\pi^2} r^{-4}} \\ &= \sqrt{\pi^2 r^2 \left(r^2 + \frac{360000}{\pi^2} r^{-4}\right)} \\ &= \sqrt{\pi^2 r^4 + 360000 r^{-2}}. \end{aligned}$$

The hint in the beginning said that the value of the square root is the lowest when the inside of the root is at its lowest. For clarity, we can set $a(r) = \pi^2 r^4 + 360000r^2$, and examine what would be the minimum value for that since this will tell us the minimum point for $A(r)$ as well. The derivative function of $a(r)$ is

$$a'(r) = \pi^2 \cdot 4r^3 + 360000 \cdot (-2)r^{-3} = 4\pi^2 r^3 - 720000r^{-3}.$$

The zero-points are

$$\begin{aligned}
4\pi^2 r^3 - 720000r^{-3} &= 0 \\
4\pi^2 r^3 &= \frac{720000}{r^3} && \Big\| \cdot r^3 \\
4\pi^2 r^6 &= 720000 && \Big\| : 4\pi^2 \\
r^6 &= \frac{720000}{4\pi^2} && \Big\| \sqrt[6]{} \\
r &= \sqrt[6]{\frac{720000}{4\pi^2}} \approx 5{,}1.
\end{aligned}$$

Let us next complete the table of values. We can choose the values 5 and 6 for r to figure out the signs of the derivative. We find

$$\begin{aligned}
a'(5) &\approx -825 < 0 \\
a'(6) &\approx 5194 > 0
\end{aligned}$$

so the table will be

	X	0		5,1	
$a'(r)$	X	X	$-$	0	+
$a(r)$	X		$\searrow$	$\rightarrow$	$\nearrow$

So this point $r = 5{,}1$ is actually the minimum point of $a(r)$, so it also is the minimum point of $A(r)$. So we know that the radius of the circle needs to be 5,1cm. We still need to calculate the height of the cone. We get

$$h = \frac{600}{\pi r^2} \approx 7{,}2,$$

so the height of the cone should be 7,2cm.

Answer:
The height of the cone should be 7,2cm and the radius of the circle 5,1cm.

4 Oppimateriaali: Sample Problems from previous Matriculation Examinations for Practice

The derivative function is a major component in Finnish upper secondary education in mathematics, so there are quite a few exam questions in the Matriculation Examinations tests that require using the derivative. Unfortunately, it is not possible to run Matriculation Examination tests in English yet, so these questions are in Finnish. Here are some exam questions for practice and to give an example on what type of problems the test might have. These are my own translations of the exam questions. You can find the original exam questions and previous math Matriculation Examination tests in Finnish from for example Matikkamatskut webpage ([8]), and you can find the correct solutions from there as well. These exam questions are all for advanced math tuition.

Fall 2017 assignment 6

The radius of a circular sector is 3 and the angle is α. The sector is bent into a lateral surface of a right circular cone. What needs to be the accurate value of the angle α in order to the volume to be as large as it can?

Similar assignments:

- Spring 2017 assignment 7
- Spring 2016 assignment 11
- Spring 2015 assignment 9
- Fall 2008 assignment 9

Fall 2016 assignment 2b

Calculate the value of the derivate function of

$$f(x) = \frac{x}{2} + \frac{2}{x} + 1$$

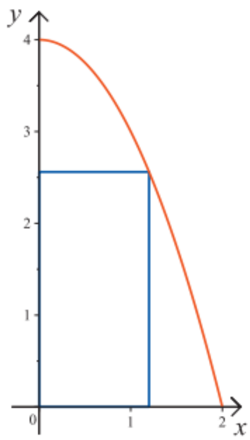
at $x = 2$.

Similar assignments:

- Fall 2013 assignment 2a
- Spring 2009 assignment 4
- Fall 2008 assignment 2a

Fall 2016 assignment 4

A rectangle has one corner at the origin and two sides at the positive x and y axes. The fourth corner is at the parabola $y = 4 - x^2$ at $x \geq 0, y \geq 0$. Determine the accurate value of the largest possible area for the rectangle.



Similar assignments:

- Fall 2015 assignment 7
- Fall 2014 assignment 6
- Fall 2010 assignment 10
- Spring 2010 assignment 7
- Fall 2009 assignment 9

Fall 2016 assignment 9.2

Let us examine the limit

$$\lim_{x \to 2} \frac{x^n - 60x - 8}{x^2 - 4}$$

with different values for exponent $n = 1, 2, 3, \ldots$.

(a) Show that the limit exists when $n = 7$.

(b) Show that the limit does not exist when $n \neq 7$.

Similar assignments:

- Spring 2015 assignment 13
- Fall 2013 assignment 12
- Fall 2011 assignment 6a

Fall 2013 assignment 4

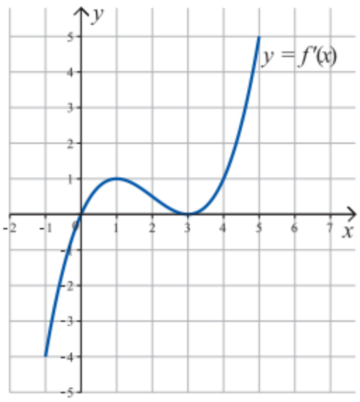
Determine the value for the parameter k for the curves $y = kx^2$ and $y = k(x-2)^2$ so that the tangential lines drawn into the intersection of these curves are perpendicular.

Similar assignments:

- Spring 2009 assignment 7

Spring 2016 assignment 4

In the picture we have a graph of the derivative function $f'(x)$ of a function $f(x)$ over the interval $-1 < x < 5$.



(a) Determine the zero-points of the derivative function $f'(x)$ based on the graph.

(b) Determine the intervals where the function $f(x)$ is increasing based on the graph.

(c) Determine the points of local extreme values of function $f(x)$ and the types of them based on the graph.

Similar assignments:

- Spring 2014 assignment 2

Fall 2012 assignment 5

Determine the extreme values of the polynomial $f(x) = x^3 - 6x^2 - 15x + 2$ in the interval $[2, 6]$.

Similar assignments:

- Spring 2011 assignment 5

Spring 2011 assignment 11

Determine a so that the function

$$f(x) = \begin{cases} ax^2 & \text{when } x \leq -1, \\ \frac{x^2}{1+x^2} & \text{when } x > -1, \end{cases}$$

is continuous on $\mathbb{R}$.

(a) Is the function $f(x)$ then differentiable on $\mathbb{R}$?

(b) Calculate $\lim_{x\to\infty} f(x)$.

Similar assignments:

- Fall 2008 assignment 12

5 Oppimateriaalin arviointi

5.1 Oppimateriaalin arviointi suhteessa Opetushallituksen asettamaan lukion opetussuunnitelmaan

Suomalainen lukio-opetus seuraa tarkasti Opetushallituksen antamaa lukion opetussuunnitelmaa kaikessa opetuksessa, ja oppimateriaalien on noudatettava tätä tarkkaa linjaa. Opetusmateriaali on yksi tärkeimmistä ellei jopa tärkein opettajan työväline opetuksessa, ja hyvä opetusmateriaali takaa opetussuunnitelman noudattamisen opetuksessa. Ylioppilaskokeiden tehtävät laaditaan sillä oletuksella, että opetus on noudattanut opetussuunnitelmaa, ja kokeessa menestyminen edellyttää pätevää opetusta ja oppimateriaalia. Siksi oppimateriaalin tarkasteleminen juuri opetussuunnitelman näkökulmasta on olennainen osa oppimateriaalin arviointia.

Tässä pro gradu -tutkielmassa ei ole käsitelty koko Derivaatta-kurssia. Lukion opetussuunnitelmassa on listattu Derivaatta-kurssin keskeisiksi osa-alueiksi rationaaliyhtälöt ja -epäyhtälöt, funktion raja-arvo ja jatkuvuus sekä derivaatta polynomifunktioiden osalta, tulon ja osamäärän derivoiminen, polynomifunktion kulun tutkiminen sekä ääriarvojen määrittäminen [10, s.133]. Tässä tutkielmassa ei ole käsitelty rationaaliyhtälöiden osa-aluetta, vaan opetusmateriaali alkaa funktion raja-arvon käsittelystä. Oppimateriaalin arvioinnissa käsitellään opetussuunnitelman asettamia vaatimuksia siltä osin, kun ne käsittelevät tutkielmassa olevia kurssin osa-alueita. Tässä tutkielmassa on oletettu, että oppilas osaa käsitellä rationaaliyhtälöitä ja -epäyhtälöitä opetussuunnitelman vaatimalla tasolla.

Opetushallitus on päivittänyt lukion opetussuunnitelman vuonna 2015, ja uutta opetussuunnitelmaa on noudatettu opetuksessa 1.8.2016 alkaen. Uuden opetussuunnitelman käyttöönotto tapahtuu vuosiasteittain siten, että lukio-opintonsa syksyllä 2016 aloittavilla oppilailla otetaan käyttöön uusi opetussuunnitelma, mutta kakkos- ja kolmosvuoden opinnot noudattavat vielä vanhaa opetussuunnitelmaa. Uusi opetussuunnitelma siirtyy siten käytettäväksi toisen vuoden opinnoissa vasta syksyllä 2017. [10, s. 3]. Derivaatta-kurssi kuuluu pitkän matematiikan oppimäärän pakollisiin kursseihin, ja se on järjestysluvultaan kuudes pitkän matematiikan kurssi. Tämän kurssin katsotaan kuuluvan toisen vuoden opintoihin, ja siten tällä kurssilla on noudatettu uutta opetussuunnitelmaa vasta 1.8.2017 alkaen. Voimassa oleva opetussuunnitelma on siis verrattain uusi, ja uutta opetussuunnitelmaa noudattavia kirjasarjoja ei vielä tutkielman kirjoitushetkellä ollut kovin montaa.

Opetushallituksen opetussuunnitelmassa on Derivaatta-kurssin tavoitteeksi asetettu, että "opiskelija omaksuu havainnollisen käsityksen funktion raja-arvosta, jatkuvuudesta ja derivaatasta"[10, s.133]. Tässä työssä on pyritty havainnollistamaan raja-arvoa, jatkuvuutta ja derivaattaa graafisten esimerkkien ja kuvaajien avulla. Tutkielmassa on pyritty pohtimaan uusia käsitteitä ja teorioita kuvaajia tutkimalla, sekä on selitetty mitä, käsite tarkoittaa kuvaajan kannalta. Esimerkiksi derivaatan määritelmää on havainnollistettu kuvaajien avulla, sekä on käsitelty, mitä funktion kuvaajasta voi päätellä derivaatan arvojen avulla. Esimerkkien avulla on pyritty lisäämään oppilaan ymmärrystä derivaatan teorian sekä kuvaajasta havaittujen käytännön sovellusten välillä. Myös raja-arvoa ja jatkuvuutta on pyritty havainnollistamaan erilaisten kuvaajien avulla. Esimerkiksi jatkuvuuden tutkiminen siten, voiko funktion kuvaajan piirtää nostamatta kynää paperista, saattaa helpottaa oppilaita ymmärtämään jatkuvuuden käsitteen paremmin, vaikka tietysti kynällä kuvaajan piirtäminen ei ole matemaattisesti pätevä perustelu jatkuvuudelle. Oppilaan on usein kuitenkin helpompi ymmärtää teoria sitten, kun opetettavasta asiasta on havainnollinen käsitys.

Kurssin osaamistavoitteiksi on listattu myös yksinkertaisten funktioiden derivaattojen määrittäminen sekä polynomifunktion kulun tutkiminen ja ääriarvojen määrittäminen [10, s.133]. Opetussuunnitelmassa käytetty ilmaus "yksinkertaisten funktioiden määrittäminen" jättää paljon tulkinnan varaa, sillä "yksinkertainen" tarkoittaa monelle hyvin eri asiaa. Kurssin keskeisten sisältöjen sekä Derivaatta-kurssia seuraavien kurssien sisältöjen avulla voidaan tulkita, että tässä kontekstissa "yksinkertaisella" tarkoitetaan polynomifunktioita. Oletettavasti tähän tavoitteeseen kuuluu myös tulon ja osamäärän derivoimiskaavojen käytön osaaminen, sillä niiden käyttöä ei muissa tavoitteissa mainita. Tutkielmassa on pyritty käymään läpi esimerkkien muodossa mahdollisimman monta erilaista derivoimistehtävää, jotta oppilaalla olisi käsitys millaisia tehtäviä voi tulla vastaan ja miten tällaisissa tehtävissä toimitaan. Esimerkeissä on käyty läpi paljon välivaiheita ja ratkaisuissa on selitetty sanallisesti mitä tehdään ja miksi, jotta oppilaan olisi helppo seurata tehtävän etenemistä. Tutkielmassa on pyritty myös huomioimaan mahdolliset erityistapaukset. Tutkielmassa käydään tarkasti läpi kulkukaavion tekeminen sekä ääriarvojen määrittäminen teorian ja esimerkkien avulla, kuten opetussuunnitelma tavoitteissa vaatii. Tutkielmassa on listattu myös erilaisia kurssin aihealueita hyödyntäviä ylioppilastehtäviä, millä on pyritty havainnollistamaan asetettujen kurssin tavoitteiden tärkeyttä. Esimerkiksi kulkukaavioiden teko on olennainen osa ratkaisua useassa vastaan tulevassa ylioppilastehtävässä. Tavoitteissa mainitaan vielä erikseen rationaalifunktion suurimman ja pienimmän arvon määrittäminen [10, s.133], mutta rationaalifunktioiden käsittelyn osaamisella, osamäärän derivoimiskaavan hallinnalla sekä ääriarvojen määrittämisen osaamisella tämä tavoite täyttyy.

Kurssin tavoitteissa on vielä listattu teknisten apuvälineiden käytön hallitseminen kurssin osa-alueiden tutkimisessa [10, s.133]. Tätä tavoitetta ei tässä materiaalissa ole käsitelty. Uusi opetussuunnitelma painottaa teknisten apuvälineiden käytön lisäämistä kaikilla kursseilla, ja tämän tavoitteen toteutuminen jää enemmän opettajan harteille. Matematiikan ylioppilaskirjoitukset ovat siirtyneet muiden oppiaineiden tavoin sähköisiksi, ja oppilaiden on osattava Abitti-ohjelmiston käyttö ylioppilaskoetta varten. Kuitenkin matematiikan ylioppilaskokeet rakentuvat ainakin vielä toistaiseksi siten, että A-osa ylioppilaskokeesta on kaikille pakollinen ja se on suoritettava ilman laskinta ja muita teknisiä apuvälineitä. Myös näissä tehtävissä on ollut kurssin osa-alueiden osaamista vaativia tehtäviä, joten derivoimiseen tai raja-arvojen määrittämiseen apuohjelmien ja laskimen avulla ei voi täysin tukeutua. Tämän tavoitteen toteutumiseksi olisi hyvä suorittaa luvussa 4 annetut ylioppilaskokeen tehtävät niiden ohjelmistojen avulla jotka ylioppilaskirjoituksissa on tällä hetkellä käytössä. Apuohjelmien käytössä on myös olennaista ymmärtää tehtävän vaatima teoria, jotta apuohjelmia osaa käyttää oikein. Siksi on tärkeää oppia kurssin asiat ensin ilman teknisiä apuvälineitä, ja laajentaa osaamista sen jälkeen opettajajohtoisesti apuohjelmia hyödyntäen.

Lukion opetussuunnitelmassa on listattu myös yleiset tavoitteet koko matematiikan pitkän oppimäärän opetukseen. Yleisesti pitkän matematiikan oppimäärän tavoitteena on antaa matemaattiset valmiudet kolmannen asteen jatko-opintoihin sekä matemaattinen yleissivistys [10, s.131]. Tämä pitää sisällään niin käsitteistöä, teoriaa kuin soveltamista. Tutkielmassa on pyritty juuri tähän lähestymistapaan. Tutkielmassa on pyritty todistamaan ja perustelemaan annetut kaavat, jotta oppilaat ymmärtäisivät, miten esimerkiksi erilaisia kaavoja voidaan johtaa. Opiskelijoiden ei pidä ottaa annettua tietoa sellaisenaan, vaan todistuksilla on pyritty herättämään oppilaat ajattelemaan ja kyseenalaistamaan saamaansa tietoa, ja vaatimaan päteviä perusteluja. Sitten kun uusi asia ja käsitteistö on opittu ja teoria oikeaksi todistettu, voidaan luotettavasti lähteä soveltamaan opittua niin erilaisissa tehtävissä kuin arkipäivän tilanteissakin. Opetussuunnitelman tavoitteisiin on myös kirjattu matematiikan kielen ymmärtäminen, matemaattisen tiedon esittäminen sekä matemaattisen tekstin ymmärtäminen [10, s.131]. Tutkielmassa esitetyt todistukset tähtäävät myös tämän tavoitteen toteutumiseen, sillä matematiikan kieli on samaa niin englanniksi kuin suomeksikin.

5.2 Oppimateriaali opettajan silmin ja arvio englannin kielestä

Opetusmateriaaliksi tarkoitettua tutkielmaa on tarpeen arvioida myös sen suhteen, miten se soveltuu käytännön opetustyöhön. Tutkielma on kirjoitettu siten, että oppilaalla olisi mahdollisuus käyttää opetusmateriaalia ilman opettajan opetusta rinnalla. Tutkielmassa annetut määritelmät, teoriat ja lauseet on pyritty kirjoittamaan mahdollisimman selkeästi ja useiden välivaiheiden kautta, jotta oppilaan olisi helpompaa seurata materiaalia itsenäisesti, ja oppilas ymmärtäisi teorian opitun asian takana. Tutkielmaa on helppo käyttää myös opettajajohtoisen opetuksen oppimateriaalina. Halutessaan opettaja voi ottaa opetukseen erilaisen näkökulman, jolloin oppilaalla on kaksi erilaista lähestymistapaa aiheeseen, ja opetus on moniulotteisempaa. Opettaja voi halutessaan myös jättää tutkielmasta pois haluamiaan osa-alueita. Esimerkiksi jos opettaja haluaa jättää derivointikaavojen todistukset oppilaiden suoritettavaksi itsenäisesti, ne voidaan helposti jättää sivuun. Opetushallituksen asettaman opetussuunnitelman mukaan oppilailta ei vaadita kykyä johtaa itsenäisesti derivointikaavoja [10, s.133]. Tutkielmaan on kuitenkin haluttu laittaa todistukset, jotta opettajalle jätetään mahdollisuus käyttää niitä oppilaiden tason mukaan. Oppilaiden on hyvä myös oppia erilaisia todistus- ja johtamistekniikoita tulevaisuuden varalta, vaikka niitä ei tämän kurssin opetussuunnitelmassa vaaditakaan.

Opetusmateriaaleissa on hyvin usein opetettavan teorian lisäksi teoriaa tukevia tehtäviä. Tässä työssä ei kuitenkaan kappalekohtaisia tehtäviä ole, vaan ainoat itsenäisesti suoritettavat tehtävät on materiaalin loppuun listatut ylioppilastutkintotehtävät. Tässä tutkielmassa haluttiin painottaa opetettavaa teoriaa, sillä tehtävien lisääminen olisi vähentänyt teorian osuutta. Raja-arvo ja derivaatta ovat aiheita, joissa tehtävät ovat pääsääntöisesti selkeitä sisällöltään, ja siten opettajan on helppo ottaa olemassa olevista kurssikirjoista tehtäviä, ja kääntää ne englannin kielelle. Esimerkiksi Hähkiöniemen ym. Juuri -kirjasarjan kirja Derivaatta [4] vastaa opetussuunnitelman mukaista opetusta, ja siitä löytyy paljon monipuolisia tehtäviä. Tehtävät ovat selkeämpinä ja lyhyempinä helpompia kääntää englannin kielelle kuin opetettava teoria, joten siksi tässä tutkielmassa keskityttiin teorian käsittelemiseen kattavasti. Esitetyissä esimerkeissä on myös pyritty käymään läpi erilaisia tehtävätyyppejä, jotta opettaja pystyisi helpommin luomaan oppilaille soveltuvia tehtäviä.

Tutkielmassa on pyritty käyttämään mahdollisimman luonnollista englannin kieltä, ja pyritty samanlaiseen ilmaisuun kuin englanninkielisten maiden matematiikan opetuksessa opetettavan teorian vastatessa suomalaista opetussuunnitelmaa. Valitettavasti Suomessa ei vielä ole opetussuunnitelman mukaista englanninkielistä lukiota, eikä ylioppilastutkintoa pysty suorittamaan englannin kielellä. On kuitenkin hyvin mahdollista, että tulevaisuudessa tällaiselle opetukselle on tarvetta, ja toivottavasti silloin tätä tutkielmaa voidaan hyödyntää. Englanninkieliseen ilmaisuun on otettu viitteitä englanninkielisistä lähdemateriaaleista ([1], [11]) sekä useista englanninkielisistä ulkomaisista oppimateriaaleista ja julkasuista ([7], [13], [14], [15]).

Lähteet

[1] Banner, Adrian. *The Calculus Lifesaver*. Princeton, New Jersey: Princeton University Press. 2007.

[2] Halmetoja, Markku, Häkkinen, Kaija, Merikoski, Jorma, Piippola, Lauri, Silfverberg, Harry, Tossavainen, Timo, Laurinolli, Teuvo, Sankilampi. Timo. *Matematiikan taito 7, Derivaatta*. 1. painos. Helsinki: WSOY Oppimateriaalit Oy. 2006.

[3] Hautajärvi, Tarmo, Ottelin, Jukka, Wallin-Jaakkola, Leena. *Laudatur 7, Derivaatta*. 1. painos. Keuruu: Otavan Kirjapaino Oy. 2006.

[4] Hähkiöniemi, Markus, Juhala, Satu, Juutinen, Petri, Louhikallio-Fomin, Sari, Luoma-aho, Erkki, Raittila, Terhi, Tikka, Tommi. *Juuri, Derivaatta*. 1. painos. Keuruu: Otavan Kirjapaino Oy. 2016.

[5] Jäppinen, Paavo, Kupiainen, Alpo, Räsänen, Matti. *Lukion Calculus 4, Derivaatta*. 1. painos. Keuruu: Otavan Kirjapaino Oy. 2006.

[6] Kangasaho, Jukka, Mäkinen, Jukka, Oikkonen, Juha, Paasonen, Johannes, Salmela, Maija, Tahvanainen, Jorma. *Pitkä matematiikka, Derivaatta*. 1. painos. Helsinki: WSOY Oppimateriaalit Oy. 2006.

[7] Khan Academy *Calculus*. First year, all content.
https://khanacademy.org/math/calculus-home, viitattu 26.2.2018.

[8] Aitlahti, Ville, *Matikkamatskut*.
https://www.matikkamatskut.com/yo-kokeita-ratkaisuineen.html, viitattu 13.8.2018.

[9] Merikoski, Jorma, Sankilampi, Timo, Laurinolli, Teuvo. *Matematiikan taito 6-7, Differentiaalilaskenta 1-2*. 4. painos. Porvoo: Ws Bookwell Oy. 2000.

[10] Opetushallitus, *Lukion opetussuunnitelman perusteet 2015*. Helsinki: Next Print Oy. 2015.

[11] Salas, S.L., Hille, Einar. *Calculus. One and Several Variables.* 6. painos. John Wiley and Sons, Inc. 1990.

[12] Wikipedia, *Rational function*.
https://en.wikipedia.org/wiki/Rational_function, viitattu 26.2.2018.

[13] Wikipedia, *Limit of a function*.
https://en.wikipedia.org/wiki/Limit_of_a_function, viitattu 26.2.2018.

[14] Wikipedia, *Derivative*. https://en.wikipedia.org/wiki/Derivative, viitattu 27.3.2018.

[15] Wikipedia, *Differential calculus*.
$https://en.wikipedia.org/wiki/Differential\_calculus$, viitattu 27.3.2018.

[16] Wikipedia, *International Baccalaureate*.
$https://fi.wikipedia.org/wiki/International\_Baccalaureate$, viitattu 10.9.2018.

[17] Ylioppilastutkintolautakunta, *Matriculation Examination*.
$https://www.ylioppilastutkinto.fi/en/matriculation-examination$, viitattu 13.8.2018.