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On the spectral and Frobenius norm of a generalized Fibonacci r-circulant matrix

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Abstract: Consider the recursion $g_0 = a$, $g_1 = b$, $g_n = g_{n-1} + g_{n-2}$, $n = 2, 3, \dots$. We compute the Frobenius norm of the r-circulant matrix corresponding to $g_0, \dots, g_{n-1}$. We also give three lower bounds (with equality conditions) for the spectral norm of this matrix. For this purpose, we present three ways to estimate the spectral norm from below in general.

Keywords: Euclidean norm, Frobenius norm, generalized Fibonacci numbers, r-circulant matrix, spectral norm

1 Introduction

Given $a, b, p, q \in \mathbb{R}$, we define the *Horadam sequence* $(h_n) = (h_n(a, b; p, q))$ via

$$h_0 = a, \quad h_1 = b,$$
$$h_n = ph_{n-1} + qh_{n-2}, \quad n = 2, 3, \dots.$$

(It is often assumed that $a, b, p, q \in \mathbb{Z}$, but real numbers apply as well.) We abbreviate

$$(u_n) = (h_n(a, b; p, 1)), \quad (g_n) = (h_n(a, b; 1, 1)),$$
$$(f_n) = (h_n(0, 1; 1, 1)), \quad (l_n) = (h_n(2, 1; 1, 1)),$$

and denote

$$\mathbf{h} = (h_0, \dots, h_{n-1}), \quad \mathbf{u} = (u_0, \dots, u_{n-1}), \quad \mathbf{g} = (g_0, \dots, g_{n-1}),$$
$$\mathbf{f} = (f_0, \dots, f_{n-1}), \quad \mathbf{l} = (l_0, \dots, l_{n-1}).$$

Throughout, we let

$$\mathbf{x} = (x_0, \dots, x_{n-1}) \in \mathbb{R}^n, \quad n \geq 2.$$

Given $r \in \mathbb{R}$, the r-circulant matrix $\mathbf{C}_r(\mathbf{x})$ is defined as

$$\mathbf{C}_r(\mathbf{x}) = \begin{pmatrix} x_0 & x_1 & \dots & x_{n-2} & x_{n-1} \\ rx_{n-1} & x_0 & \dots & x_{n-3} & x_{n-2} \\ rx_{n-2} & rx_{n-1} & \dots & x_{n-4} & x_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ rx_2 & rx_3 & \dots & x_0 & x_1 \\ rx_1 & rx_2 & \dots & rx_{n-1} & x_0 \end{pmatrix}.$$

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(If $r \in \mathbb{Z}$, the term "r-circulant" has also another meaning [4, p. 155]: each row is obtained from the preceding row by r shiftings.)

We let $\|\cdot\|_F$ stand for the Frobenius (or, equivalently, Euclidean) norm of a matrix, and $\|\cdot\|_2$ for the spectral norm (or, equivalently, the largest singular value) of a matrix likewise for the Euclidean norm of a vector.

Shen and Cen [12] presented bounds for $\|\mathbf{C}_r(\mathbf{f})\|_F$ and $\|\mathbf{C}_r(\mathbf{l})\|_F$. Chandoul [3] extended them to $\|\mathbf{C}_r(\mathbf{g})\|_F$, and Raza and Ali [11] to $\|\mathbf{C}_r(\mathbf{u})\|_F$. Our first goal is to find $\|\mathbf{C}_r(\mathbf{g})\|_F$ exactly. We will do it in Section 2.

The above-cited authors have also presented bounds for $\|\mathbf{C}_r(\mathbf{f})\|_2$, etc. In this paper, we focus on examining lower bounds. Shen and Cen [12, Theorems 1–2] proved that

$$\begin{aligned} \|\mathbf{C}_r(\mathbf{f})\|_2 &\geq \min(|r|, 1)\sqrt{f_{n-1}f_n}, \\ \|\mathbf{C}_r(\mathbf{l})\|_2 &\geq \min(|r|, 1)\sqrt{5f_{n-1}f_n} \quad \text{if } n \text{ is even}, \\ \|\mathbf{C}_r(\mathbf{l})\|_2 &\geq \min(|r|, 1)\sqrt{5f_{n-1}f_n + 4} \quad \text{if } n \text{ is odd}. \end{aligned}$$

Chandoul [3, Theorem 2.2] extended these inequalities to

$$\|\mathbf{C}_r(\mathbf{g})\|_2 \geq \min(|r|, 1)\sqrt{g_{n-1}g_n - ab + a^2}. \tag{1}$$

More generally, Raza and Ali [11, Theorem 2.1] showed that

$$\|\mathbf{C}_r(\mathbf{u})\|_2 \geq \min(|r|, 1)\sqrt{\frac{u_{n-1}u_n - ab + pa^2}{p}}.$$

They assume that $a, b \geq 0$ but say nothing about p. However, it seems that they implicitly presume that $p \geq 1$. A couple of years earlier Yazik and Taskara [13, Theorem 5] found even more general, yet a quite complicated, lower bound for $\|\mathbf{C}_r(\mathbf{h})\|_2$.

If $|r|$ is large (and n fixed), then the left-hand side of each of the above inequalities is large but the right-hand side remains constant. If $|r|$ is small, then the right-hand side is small but the left-hand side may be large (because $\mathbf{C}_r(\mathbf{g})^T\mathbf{C}_r(\mathbf{g})$ has entries without factor r). Therefore, the right-hand sides are often poor lower bounds for the left-hand sides.

In order to exceed the above results, in Section 3, we will cultivate three previously known ways to estimate $\|\mathbf{A}\|_2$ from below, where $\mathbf{A} \in \mathbb{C}^{m\times n}$. We will also give equality conditions. Because we find this topic interesting in itself, our approach is going to be more general than actually is needed. Applying the bounds so obtained, we will in Section 4 underestimate $\|\mathbf{C}_r(\mathbf{x})\|_2$, where $\mathbf{x} \in \mathbb{C}^n$. Thereafter, in Section 5, we will attain our second goal: to find three lower bounds for $\|\mathbf{C}_r(\mathbf{g})\|_2$. In Section 6, we will compare the found lower bounds with each others and with the right-hand side of (1), briefly "rhs(1)". Finally, Section 7 completes our paper with some concluding remarks.

Norms of generalized Fibonacci r-circulant matrices are widely studied. The above references are directly connected with our paper. For other references, see, e.g., [1, 2, 5, 7, 9].

2 Computation of $\|\mathbf{C}_r(\mathbf{g})\|_F$

We recall three sum formulas for the Fibonacci numbers.

Lemma 2.1. *Let $n \in \mathbb{Z}_+$. Then*

$$f_1^2 + \cdots + f_n^2 = f_nf_{n+1}, \tag{2}$$

$$f_1f_2 + \cdots + f_{n-1}f_n = f_n^2 - \eta_n, \tag{3}$$

$$f_1^2 + 2f_2^2 + \cdots + nf_n^2 = (nf_{n+1} - f_n)f_n + \eta_n, \tag{4}$$

where

$$\eta_n = \frac{1-(-1)^n}{2}. \tag{5}$$

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Proof. By induction. See also [8, Theorem 5.5] and [8, p. 90, Eqs. 57 and 60]. □

We also need two other sum formulas.

Lemma 2.2. *Let $n \in \mathbb{Z}_+$. Then*

$$2f_1^2 + 3f_2^2 + \cdots + (n+1)f_n^2 = \big[(n+1)f_{n+1} - f_n\big]f_n + \eta_n. \tag{6}$$

Proof. By (4) and (2),

$$2f_1^2 + 3f_2^2 + \cdots + (n+1)f_n^2 = (f_1^2 + 2f_2^2 + \cdots + nf_n^2) + (f_1^2 + \cdots + f_n^2) =$$
$$nf_nf_{n+1} - f_n^2 + \eta_n + f_nf_{n+1} = (n+1)f_nf_{n+1} - f_n^2 + \eta_n,$$

and (6) follows. □

Lemma 2.3. *Let $n \in \mathbb{Z}_+$. Then*

$$f_0f_1 + 2f_1f_2 + 3f_2f_3 + \cdots + nf_{n-1}f_n = (nf_n - f_{n-1})f_n + \theta_n, \tag{7}$$

where

$$\theta_n = (-1)^n \frac{n+\eta_n}{2}. \tag{8}$$

Proof. Denote $s_n = f_0f_1 + f_1f_2 + \cdots + f_{n-1}f_n$. Then, by (3) and (2),

$$f_0f_1 + 2f_1f_2 + 3f_2f_3 + \cdots + nf_{n-1}f_n = s_n + (s_n - s_1) + (s_n - s_2) + \cdots + (s_n - s_{n-1}) =$$
$$f_n^2 + (f_n^2 - f_1^2) + (f_n^2 - f_2^2) + \cdots + (f_n^2 - f_{n-1}^2) - \eta_n + (\eta_n - \eta_1) + (\eta_n - \eta_2) + \cdots + (\eta_n - \eta_{n-1}) =$$
$$nf_n^2 - (f_1^2 + \cdots + f_{n-1}^2) - n\eta_n + \eta_1 + \eta_2 + \cdots + \eta_{n-1} = nf_n^2 - f_{n-1}f_n - n\eta_n + \eta_1 + \eta_2 + \cdots + \eta_{n-1}.$$

If n is even, then

$$-n\eta_n + \eta_1 + \eta_2 + \cdots + \eta_{n-1} = 0 + \frac{n}{2} = \theta_n.$$

If n is odd, then

$$-n\eta_n + \eta_1 + \eta_2 + \cdots + \eta_{n-1} = -n + \frac{n-1}{2} = -\frac{n+1}{2} = \theta_n,$$

and the proof is complete. □

Now, we can compute $\|\mathbf{C}_r(\mathbf{g})\|_F^2$. Applying the equation

$$g_n = af_{n-1} + bf_n, \quad n = 1, 2, \ldots, \tag{9}$$

and (2), (4), (6), (7), we have

$$\|\mathbf{C}_r(\mathbf{g})\|_F^2 = \sum_{i=0}^{n-1}(n-i)g_i^2 + \sum_{i=1}^{n-1} ir^2g_i^2 = na^2 + n\sum_{i=1}^{n-1} g_i^2 + (r^2-1)\sum_{i=1}^{n-1} ig_i^2 =$$
$$na^2 + n\sum_{i=1}^{n-1}(af_{i-1} + bf_i)^2 + (r^2-1)\sum_{i=1}^{n-1} i(af_{i-1} + bf_i)^2 =$$
$$na^2 + n\sum_{i=1}^{n-1}(a^2f_{i-1}^2 + 2abf_{i-1}f_i + b^2f_i^2) + (r^2-1)\sum_{i=1}^{n-1} i(a^2f_{i-1}^2 + 2abf_{i-1}f_i + b^2f_i^2) =$$
$$na^2 + n\Big(a^2\sum_{i=1}^{n-1} f_{i-1}^2 + 2ab\sum_{i=1}^{n-1} f_{i-1}f_i + b^2\sum_{i=1}^{n-1} f_i^2\Big) + (r^2-1)\Big(a^2\sum_{i=1}^{n-1} if_{i-1}^2 + 2ab\sum_{i=1}^{n-1} if_{i-1}f_i + b^2\sum_{i=1}^{n-1} if_i^2\Big) =$$
$$na^2 + n\big[a^2f_{n-2}f_{n-1} + 2ab(f_{n-1}^2 - \eta_{n-1}) + b^2f_{n-1}f_n\big] +$$
$$(r^2-1)\Big(a^2\big\{\big[(n-1)f_{n-1} - f_{n-2}\big]f_{n-2} + \eta_{n-2}\big\} + 2ab\big\{\big[(n-1)f_{n-1} - f_{n-2}\big]f_{n-1} + \theta_{n-1}\big\} +$$

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$$b^2\Big\{\big[(n-1)f_n - f_{n-1}\big]f_{n-1} + \eta_{n-1}\Big\}\Big) =$$
$$na^2 + n(a^2 f_{n-2}f_{n-1} + 2abf_{n-1}^2 + b^2 f_n f_{n-1}) + n(r^2-1)(a^2 f_{n-1}f_{n-2} + 2abf_{n-1}^2 + b^2 f_{n-1}f_n) +$$
$$(1-r^2)(a^2 f_{n-2}f_{n-1} + a^2 f_{n-2}^2 + 2abf_{n-1}^2 + 2abf_{n-2}f_{n-1} + b^2 f_{n-1}f_n + b^2 f_{n-1}^2)$$
$$-2nab\eta_{n-1} + (r^2-1)(a^2\eta_{n-2} + 2ab\theta_{n-1} + b^2\eta_{n-1}) =$$
$$na^2 + nr^2(a^2 f_{n-2}f_{n-1} + 2abf_{n-1}^2 + b^2 f_{n-1}f_n) + (1-r^2)(a^2 f_{n-2}f_n + 2abf_{n-1}f_n + b^2 f_{n-1}f_{n+1}) -$$
$$2nab\eta_{n-1} + (r^2-1)(a^2\eta_{n-2} + 2ab\theta_{n-1} + b^2\eta_{n-1}).$$

Furthermore,

$$a^2 f_{n-2}f_{n-1} + 2abf_{n-1}^2 + b^2 f_{n-1}f_n = af_{n-1}(af_{n-2} + bf_{n-1}) + bf_{n-1}(af_{n-1} + bf_n) = af_{n-1}g_{n-1} + bf_{n-1}g_n$$

and

$$a^2 f_{n-2}f_n + 2abf_{n-1}f_n + b^2 f_{n-1}f_{n+1} = af_n(af_{n-2} + bf_{n-1}) + bf_{n-1}(af_n + bf_{n+1}) = af_n g_{n-1} + bf_{n-1}g_{n+1}.$$

We summarize our result as follows.

Theorem 2.1. *Let $r, a, b \in \mathbb{R}$. Then*

$$\|\mathbf{C}_r(\mathbf{g})\|_F = \left[\alpha r^2 + \beta(1-r^2) + \gamma\right]^{\frac{1}{2}}, \tag{10}$$

where

$$\alpha = n(af_{n-1}g_{n-1} + bf_{n-1}g_n),$$
$$\beta = af_n g_{n-1} + bf_{n-1}g_{n+1} - a^2\eta_{n-2} - 2ab\theta_{n-1} - b^2\eta_{n-1},$$
$$\gamma = n(a^2 - 2ab\eta_{n-1}).$$

3 Underestimating $\|\mathbf{A}\|_2$

Our first approach to estimate $\|\mathbf{A}\|_2$ from below rests upon applying $\|\mathbf{A}\|_F$.

Lemma 3.1. *Let $\mathbf{A} \in \mathbb{C}^{m\times n}$. Then*

$$\|\mathbf{A}\|_2 \geq \frac{1}{\sqrt{q}}\|\mathbf{A}\|_F, \quad q = \min(m, n). \tag{11}$$

Equality is attained if and only if all singular values of $\mathbf{A}$ are equal. For $\mathbf{A} \in \mathbb{C}^{n\times n}$, an equivalent condition is that $\mathbf{A}$ is a scalar multiple of a unitary matrix.

Proof. Let $\mathbf{A}$ have singular values $\sigma_1 \geq \cdots \geq \sigma_q$. Since

$$\|\mathbf{A}\|_2^2 = \sigma_1^2, \quad \frac{1}{q}\|\mathbf{A}\|_F^2 = \frac{\sigma_1^2 + \cdots + \sigma_q^2}{q},$$

we obtain (11) with equality condition; see also [6, Problem 5.6.P23], [6, p. 594]. For the last statement, see [6, Problem 2.6.P13]. □

In the other two alternative procedures which we study in this paper, we consider the spectral norm as the largest eigenvalue $\lambda(\cdot)$ of a suitable Hermitian matrix.

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Lemma 3.2. *Given* $\mathbf{A} \in \mathbb{C}^{m\times n}$*, define*

$$\mathbf{H}_\mathbf{A} = \begin{pmatrix} \mathbf{O} & \mathbf{A} \\ \mathbf{A}^* & \mathbf{O} \end{pmatrix} \in \mathbb{C}^{(m+n)\times(m+n)}, \quad \mathbf{K}_\mathbf{A} = \mathbf{A}^*\mathbf{A} \in \mathbb{C}^{n\times n}.$$

Then

$$\|\mathbf{A}\|_2 = \lambda(\mathbf{H}_\mathbf{A}), \quad \|\mathbf{A}\|_2 = \lambda(\mathbf{K}_\mathbf{A})^{\frac{1}{2}}. \tag{12}$$

Proof. The first equation follows from [6, Theorem 7.3.3]. The second is obvious. □

Lemma 3.3. *Let* $\mathbf{M} \in \mathbb{C}^{n\times n}$ *be Hermitian. If* $\mathbf{0} \neq \mathbf{x} \in \mathbb{C}^n$*, then*

$$\lambda(\mathbf{M}) \geq \frac{\mathbf{x}^*\mathbf{M}\mathbf{x}}{\mathbf{x}^*\mathbf{x}}. \tag{13}$$

Equality is attained if and only if $\mathbf{x}$ *is an eigenvector corresponding to* $\lambda(\mathbf{M})$.

Proof. See [6, Theorem 4.2.2]. □

Throughout, we let $r_1, \dots, r_m$ (respectively $c_1, \dots, c_n$) denote the row (column) sums of $\mathbf{A} = (a_{ij}) \in \mathbb{C}^{m\times n}$. We also denote

$$\mathbf{r} = (r_1, \dots, r_m), \quad \mathbf{c} = (c_1, \dots, c_n), \quad \mathbf{1}_k = (1, \dots, 1) \in \mathbb{R}^k.$$

Theorem 3.1. *Let* $\mathbf{A} \in \mathbb{C}^{m\times n}$*. Then*

$$\|\mathbf{A}\|_2 \geq \frac{2\,|r_1 + \cdots + r_m|}{m+n}. \tag{14}$$

In particular, for $\mathbf{A} \in \mathbb{C}^{n\times n}$,

$$\|\mathbf{A}\|_2 \geq \frac{|r_1 + \cdots + r_n|}{n}. \tag{15}$$

Proof. Let us denote $\mathbf{1} = \mathbf{1}_{m+n}$ and $s = |s|\mathrm{e}^{\mathrm{i}\theta} = r_1 + \cdots + r_m$. By (12) and (13),

$$\|\mathbf{A}\|_2 = \lambda(\mathbf{H}_\mathbf{A}) \geq \frac{\mathbf{1}^*\mathbf{H}_\mathbf{A}\mathbf{1}}{\mathbf{1}^*\mathbf{1}} = \frac{1}{m+n}\Big(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{m} \bar{a}_{ji}\Big) =$$
$$\frac{1}{m+n}\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij} + \bar{a}_{ij}) = \frac{2}{m+n}\sum_{i=1}^{m}\sum_{j=1}^{n}\Re a_{ij} = \frac{2}{m+n}\Re\sum_{i=1}^{m} r_i = \frac{2}{m+n}\Re s,$$

where $\Re$ stands for the real part. Applying this to $\mathrm{e}^{-\mathrm{i}\theta}\mathbf{A}$, we obtain

$$\|\mathbf{A}\|_2 = \|\mathrm{e}^{-\mathrm{i}\theta}\mathbf{A}\|_2 \geq \frac{2}{m+n}\Re(\mathrm{e}^{-\mathrm{i}\theta}s) = \frac{2}{m+n}\Re(\mathrm{e}^{-\mathrm{i}\theta}|s|\mathrm{e}^{\mathrm{i}\theta}) = \frac{2|s|}{m+n},$$

verifying (14). □

Theorem 3.2. *If* $m \neq n$ *and* $\mathbf{A} \neq \mathbf{O}$*, then* (14) *is strict. Assuming* $m = n$ *and* $\mathbf{A} \geq \mathbf{O}$ *(entrywise), equality is attained in* (15) *if and only if*

$$r_1 = \cdots = r_n = c_1 = \cdots = c_n.$$

Proof. By Lemma 3.3, a necessary condition for equality is that $\mathbf{1}$ is an eigenvector of $\mathbf{H}_\mathbf{A}$. Since

$$\mathbf{H}_\mathbf{A}\mathbf{1} = \begin{pmatrix} \mathbf{A}\mathbf{1}_n \\ \mathbf{A}^*\mathbf{1}_m \end{pmatrix} = \begin{pmatrix} \mathbf{r} \\ \bar{\mathbf{c}} \end{pmatrix}$$

(where $\bar{\mathbf{c}}$ is understood entrywise), this happens if and only if

$$r_1 = \cdots = r_m = \bar{c}_1 = \cdots = \bar{c}_n. \tag{16}$$

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Assume (16). Then $s = mr_1 = m\bar{c}_1$ but also $s = nc_1 = n\bar{r}_1$. Writing $r_1 = \alpha + \beta \mathrm{i}$, $c_1 = \alpha - \beta \mathrm{i}$, we therefore have

$$m\alpha = n\alpha, \quad m\beta = -n\beta.$$

So, $m = n \vee \alpha = 0$ by the first equation, and $\beta = 0$ by the second. We have now shown that a necessary condition for $\mathbf{H_A}$ to have $\mathbf{1}$ as an eigenvector is

$$(\alpha = \beta = 0) \vee \left[(m = n) \wedge (r_1 = \cdots = r_n = c_1 = \cdots = c_n \in \mathbb{R})\right].$$

If $\alpha = \beta = 0$, then the right-hand side of (14) is zero; so, to have equality in this case, necessarily $\mathbf{A} = \mathbf{O}$. Therefore, a necessary condition for equality in (14) is

$$\mathbf{A} = \mathbf{O} \vee \left[(m = n) \wedge (r_1 = \cdots = r_n = c_1 = \cdots = c_n \in \mathbb{R})\right].$$

The first claim of the theorem is thus proved.

The problem is that the corresponding eigenvalue is not necessarily $\lambda(\mathbf{H_A})$. However, there is no problem if $\mathbf{A} \geq \mathbf{O}$. Because a positive eigenvector corresponds to the Perron root [6, Theorem 8.3.4], this eigenvalue is $\lambda(\mathbf{H_A})$, and the second claim follows. □

Theorem 3.3. *Let $\mathbf{A} \in \mathbb{C}^{m \times n}$. Then*

$$\|\mathbf{A}\|_2 \geq \left(\frac{|r_1|^2 + \cdots + |r_m|^2}{n}\right)^{\frac{1}{2}}. \tag{17}$$

Assuming $\mathbf{A} \geq \mathbf{O}$ (or, more generally, $\mathbf{A}^*\mathbf{A} \geq \mathbf{O}$), equality is attained if and only if all row sums of $\mathbf{A}^*\mathbf{A}$ are equal.

Proof. Denote $\mathbf{1} = \mathbf{1}_n$; then

$$\|\mathbf{A}\|_2 \geq \frac{\|\mathbf{A1}\|_2}{\|\mathbf{1}\|_2} = \frac{\|\mathbf{r}\|_2}{\sqrt{n}},$$

verifying (17). To study equality, we have

$$\frac{\|\mathbf{A1}\|_2^2}{\|\mathbf{1}\|_2^2} = \frac{(\mathbf{A1})^*\mathbf{A1}}{\mathbf{1}^*\mathbf{1}} = \frac{\mathbf{1}^*\mathbf{A}^*\mathbf{A1}}{\mathbf{1}^*\mathbf{1}}.$$

Consequently, $\mathbf{1}$ must be an eigenvector of $\mathbf{K} = \mathbf{A}^*\mathbf{A}$ corresponding to $\lambda(\mathbf{K})$. Clearly, $\mathbf{1}$ is an eigenvector if and only if all the row sums of $\mathbf{K}$ are equal. As in the proof of Theorem 3.2, we see that the corresponding eigenvalue is $\lambda(\mathbf{K})$ if $\mathbf{K} \geq \mathbf{O}$. □

Proposition 3.4. *Let $\mathbf{A} \in \mathbb{R}^{n \times n}$. The bound* rhs(17) *is better than* rhs(15). *If $\mathbf{A} \geq \mathbf{O}$, then* rhs(17) *is better than* rhs(11), *but* rhs(11) *and* rhs(15) *are not comparable.*

Proof. Easy and omitted. □

4 Underestimating $\|\mathbf{C}_r(\mathbf{x})\|_2$

We first recall an exact expression of $\|\mathbf{C}_1(\mathbf{x})\|_2$.

Theorem 4.1. *If $\mathbf{x} \geq \mathbf{0}$, then*

$$\|\mathbf{C}_1(\mathbf{x})\|_2 = x_0 + \cdots + x_{n-1}.$$

Proof. See [10, Corollary 2]. The assumption $\mathbf{x} \geq \mathbf{0}$ can be generalized, see [10, Theorem 4]. □

Now we apply our bounds to $\|\mathbf{C}_r(\mathbf{x})\|_2$.

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Theorem 4.2. *Let $r, x_0, \dots, x_{n-1} \in \mathbb{R}$. Then*

$$\|\mathbf{C}_r(\mathbf{x})\|_2 \geq \Big(\sum_{i=0}^{n-1} x_i^2 + \frac{r^2-1}{n} \sum_{i=1}^{n-1} i x_i^2 \Big)^{\frac{1}{2}}. \tag{18}$$

Proof. Since

$$\|\mathbf{C}_r(\mathbf{x})\|_F^2 = \sum_{i=0}^{n-1} (n-i) x_i^2 + r^2 \sum_{i=1}^{n-1} i x_i^2 = n \sum_{i=0}^{n-1} x_i^2 + (r^2-1) \sum_{i=1}^{n-1} i x_i^2,$$

we have

$$\frac{\|\mathbf{C}_r(\mathbf{x})\|_F^2}{n} = \sum_{i=0}^{n-1} x_i^2 + \frac{r^2-1}{n} \sum_{i=1}^{n-1} i x_i^2,$$

and (18) follows from (11). □

Theorem 4.3. *Let $r, x_0, \dots, x_{n-1} \in \mathbb{R}$. Then*

$$\|\mathbf{C}_r(\mathbf{x})\|_2 \geq \Big| \sum_{i=0}^{n-1} x_i + \frac{r-1}{n} \sum_{i=1}^{n-1} i x_i \Big|. \tag{19}$$

Proof. The sum of entries of $\mathbf{C}_r(\mathbf{x})$ equals

$$\sum_{i=0}^{n-1} (n-i) x_i + r \sum_{i=1}^{n-1} i x_i = n \sum_{i=0}^{n-1} x_i + (r-1) \sum_{i=1}^{n-1} i x_i,$$

so (19) follows from (15). □

Theorem 4.4. *Let $r, x_0, \dots, x_{n-1} \in \mathbb{R}$. Then*

$$\|\mathbf{C}_r(\mathbf{x})\|_2 \geq \Big[\frac{1}{n} \sum_{i=0}^{n-1} \Big(\sum_{j=0}^{n-i-1} x_j + r \sum_{j=n-i}^{n-1} x_j \Big)^2 \Big]^{\frac{1}{2}}. \tag{20}$$

Proof. The $(i+1)$'st row sum of $\mathbf{C}_r(\mathbf{x})$ is

$$\sum_{j=0}^{n-i-1} x_j + r \sum_{j=n-i}^{n-1} x_j,$$

hence (17) implies (20). □

Theorem 4.5. *Equality is attained in* (18) *if and only if either*

$$r \neq \pm 1 \ \wedge\ (x_1 = \cdots = x_{n-1} = 0) \tag{21}$$

or

$$r = 1 \ \wedge\ \Big(\sum_{i=0}^{n-1} x_i x_{i-j} = 0 \Big), \quad j = 1, \dots, n-1, \tag{22}$$

or

$$r = -1 \ \wedge\ \Big(\sum_{i=0}^{j-1} x_i x_{i-j} = \sum_{i=j}^{n-1} x_i x_{i-j} \Big), \quad j = 1, \dots, n-1, \tag{23}$$

where the indices are mod n. *Assuming*

$$r, x_0, \dots, x_{n-1} \geq 0, \tag{24}$$

equality is attained in (19) *and, respectively, in* (20) *if and only if*

$$r = 1 \ \vee\ (x_1 = \cdots = x_{n-1} = 0). \tag{25}$$

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Proof. We divide the proof in three parts.

1. Equality condition of (18). By Lemma 3.1, equality holds if and only if the rows of $\mathbf{C}_r(\mathbf{x})$ form a scalar multiple of an orthonormal set. In particular, their Euclidean norms must be equal. Comparing the n'th and $(n-1)$'th rows, this means that

$$r^2(x_1^2 + \cdots + x_{n-1}^2) + x_0^2 = r^2(x_2^2 + \cdots + x_{n-1}^2) + x_0^2 + x_1^2,$$

i.e., $r^2x_1^2 = x_1^2$.

First, assume $r \neq \pm 1$; then $x_1 = 0$. Comparing the $(n-1)$'th and $(n-2)$'th rows, we have

$$r^2(x_2^2 + \cdots + x_{n-1}^2) + x_0^2 = r^2(x_3^2 + \cdots + x_{n-1}^2) + x_0^2 + x_2^2,$$

i.e., $r^2x_2^2 = x_2^2$; so $x_2 = 0$. Continuing similarly, we see that necessarily $x_1 = \cdots = x_{n-1} = 0$. Since this condition is clearly sufficient, the condition (21) is verified.

Second, assume $r = \pm 1$; then the rows of $\mathbf{C}_r(\mathbf{x})$ have equal norms. Since their orthogonality condition is stated in (22) and (23), also this case is clear.

2. Equality condition of (19), *assuming* (24). By Theorem 3.2, equality holds in (15) if and only if all row sums of $\mathbf{C}_r(\mathbf{x})$ are equal. (Since $r_1 = c_n, r_2 = c_{n-1}, \ldots, r_n = c_1$, they are also equal to the column sums.) Comparing the n'th and $(n-1)$'th rows, we have

$$r(x_1 + \cdots + x_{n-1}) + x_0 = r(x_2 + \cdots + x_{n-1}) + x_0 + x_1,$$

i.e., $r = 1$ or $x_1 = 0$. Continuing as above, we obtain (25).

3. Equality condition of (20), *assuming* (24). Let d be the difference of the first and last row sum of $\mathbf{C}_r^T(\mathbf{x})\mathbf{C}_r(\mathbf{x})$. If equality holds, then $d = 0$ by Theorem 3.3. A rather extensive computation, which we omit here, shows that

$$d = (r^2 - 1)\Big(\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n}\sum_{j=1}^{i-1} x_i x_j\Big).$$

Therefore, (25) is necessary; obviously, it is a sufficient condition. □

Although rhs(11) and rhs(15) are not comparable even if $\mathbf{A} \geq \mathbf{O}$, a question arises whether they are if $\mathbf{A} = \mathbf{C}_r(\mathbf{x})$, $\mathbf{x} \geq \mathbf{0}$. The answer is negative. For example, if $\mathbf{x} = (1, 1, 1)$, then rhs(19) ≥ rhs(18) for all $r \geq -\frac{1}{2}$. On the other hand, if $\mathbf{x} = (0, 1, 0)$, then rhs(18) ≥ rhs(19) for all $r \in \mathbb{R}$.

5 Underestimating $\|\mathbf{C}_r(\mathbf{g})\|_2$

We are now ready to study $\|\mathbf{C}_r(\mathbf{g})\|_2$.

Theorem 5.1. *Let $r, a, b \in \mathbb{R}$. Then*

$$\|\mathbf{C}_r(\mathbf{g})\|_2 \geq \Big(\alpha r^2 + \beta\frac{1-r^2}{n} + \gamma\Big)^{\frac{1}{2}}, \tag{26}$$

where

$$\alpha = af_{n-1}g_{n-1} + bf_{n-1}g_n,$$
$$\beta = af_ng_{n-1} + bf_{n-1}g_{n+1} - a^2\eta_{n-2} - 2ab\theta_{n-1} - b^2\eta_{n-1},$$
$$\gamma = a^2 - 2ab\eta_{n-1},$$

and η_m and θ_m are as in (5) *and* (8), *respectively.*

Proof. The claim follows from (10), (11), and (18). □

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In applying Theorems 4.3 and 4.4 in order to estimate $\|\mathbf{C}_r(\mathbf{g})\|_2$, we also need the following three sum formulas.

Lemma 5.1. *Let $m \in \mathbb{Z}_+$. Then*

$$\sum_{i=0}^{m} g_i = g_{m+2} - b, \tag{27}$$

$$\sum_{i=1}^{m} i g_i = m g_{m+2} - g_{m+3} + a + 2b, \tag{28}$$

$$\sum_{i=0}^{m} g_i^2 = g_m g_{m+1} + a(a-b). \tag{29}$$

Proof. By induction. See also [8, p. 113, Eqs. 11, 17 and 14]. (Note that $G_i = g_{i-1}$ and that there is a typo in Eq. 17.) □

Theorem 5.2. *Let $r \in \mathbb{R}$. Then*

$$\|\mathbf{C}_r(\mathbf{g})\|_2 \geq \frac{1}{n}\left|g_{n+3} - a - (n+2)b + r(n g_{n+1} - g_{n+3} + a + 2b)\right|. \tag{30}$$

Proof. By (27) and (28),

$$\begin{aligned} n\sum_{i=0}^{n-1} g_i + (r-1)\sum_{i=0}^{n-1} i g_i &= n(g_{n+1} - b) + (r-1)\left[(n-1)g_{n+1} - g_{n+2} + a + 2b\right] = \\ &-nb + g_{n+1} + g_{n+2} - a - 2b + r\left[(n-1)g_{n+1} - g_{n+2} + a + 2b\right] = \\ &g_{n+3} - a - (n+2)b + r(n g_{n+1} - g_{n+3} + a + 2b); \end{aligned}$$

so, (19) implies (30). □

In particular, if $a, b \geq 0$, then

$$\|\mathbf{C}_1(\mathbf{g})\|_2 = g_{n+1} - b,$$

which follows also from Theorem 4.1 and (27).

Next, we apply Theorem 4.4. We denote

$$\sigma_i = g_{n-(i-1)} + \cdots + g_{n-1}, \quad \tau_i = g_0 + \cdots + g_{n-i}, \quad s_i = \sigma_i r + \tau_i,$$

where $i = 1, \ldots, n$.

1. Computing τ_i and σ_i. By (27),

$$\tau_i = \sum_{j=0}^{n-i} g_j = g_{n-i+2} - b$$

and

$$\sigma_i = \sum_{j=n-(i-1)}^{n-1} g_j = \sum_{j=0}^{n-1} g_j - \sum_{j=0}^{n-i} g_j = (g_{n+1} - b) - (g_{n-i+2} - b) = g_{n+1} - g_{n-i+2}.$$

2. Computing s_i^2. Simply, observe that

$$\begin{aligned} s_i^2 &= (\sigma_i r + \tau_i)^2 = \left[(g_{n+1} - g_{n-i+2})r + g_{n-i+2} - b\right]^2 = \\ &(g_{n+1} - g_{n-i+2})^2 r^2 + 2(g_{n+1} - g_{n-i+2})(g_{n-i+2} - b)r + (g_{n-i+2} - b)^2 =: \alpha_i r^2 + \beta_i r + \gamma_i. \end{aligned}$$

3. Computing $\alpha_1 + \cdots + \alpha_n$. We have

$$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} (g_{n+1} - g_{n-i+2})^2 = \sum_{i=1}^{n} \left(g_{n+1}^2 - 2g_{n+1}g_{n-i+2} + g_{n-i+2}^2\right) =$$

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$$ng_{n+1}^2 - 2g_{n+1}\sum_{i=1}^{n} g_{n-i+2} + \sum_{i=1}^{n} g_{n-i+2}^2 = ng_{n+1}^2 - 2g_{n+1}\sum_{i=2}^{n+1} g_i + \sum_{i=2}^{n+1} g_i^2. \tag{31}$$

Since

$$\sum_{i=2}^{n+1} g_i = \sum_{i=0}^{n+1} g_i - a - b = g_{n+3} - a - 2b \tag{32}$$

by (27), and

$$\sum_{i=0}^{m} g_i^2 = g_m g_{m+1} + a(a-b)$$

by (29), we get

$$\sum_{i=2}^{n+1} g_i^2 = \sum_{i=0}^{n+1} g_i^2 - a^2 - b^2 = g_{n+1}g_{n+2} + a(a-b) - a^2 - b^2 = g_{n+1}g_{n+2} - ab - b^2. \tag{33}$$

Now we can evaluate (31):

$$\begin{aligned}\sum_{i=1}^{n} \alpha_i = ng_{n+1}^2 - 2g_{n+1}(g_{n+3} - a - 2b) + g_{n+1}g_{n+2} - ab - b^2 = \\ g_{n+1}(ng_{n+1} + g_{n+2} - 2g_{n+3} + 2a + 4b) - ab - b^2 = \\ g_{n+1}\left[ng_{n+1} + g_{n+2} - 2(g_{n+1} + g_{n+2}) + 2a + 4b\right] - ab - b^2 = \\ g_{n+1}\left[(n-2)g_{n+1} - g_{n+2} + 2a + 4b\right] - ab - b^2.\end{aligned}$$

4. Computing $\beta_1 + \cdots + \beta_n$. By (32) and (33),

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{n} \beta_i = \sum_{i=1}^{n}(g_{n+1} - g_{n-i+2})(g_{n-i+2} - b) = \\ g_{n+1}\sum_{i=1}^{n} g_{n-i+2} - \sum_{i=1}^{n} g_{n-i+2}^2 + b\sum_{i=1}^{n} g_{n-i+2} - nbg_{n+1} = \\ g_{n+1}\sum_{i=2}^{n+1} g_i - \sum_{i=2}^{n+1} g_i^2 + b\sum_{i=2}^{n+1} g_i - nbg_{n+1} = (g_{n+1} + b)\sum_{i=2}^{n+1} g_i - \sum_{i=2}^{n+1} g_i^2 - nbg_{n+1} = \\ (g_{n+1} + b)(g_{n+3} - a - 2b) - (g_{n+1}g_{n+2} - ab - b^2) - nbg_{n+1} = \\ g_{n+1}g_{n+3} - (a+2b)g_{n+1} + bg_{n+3} - ab - 2b^2 - g_{n+1}g_{n+2} + ab + b^2 - nbg_{n+1} = \\ g_{n+1}(g_{n+1} + g_{n+2}) - (a+2b)g_{n+1} + b(g_{n+1} + g_{n+2}) - g_{n+1}g_{n+2} - nbg_{n+1} - b^2 = \\ g_{n+1}^2 - \left[a + (n+1)b\right]g_{n+1} + bg_{n+2} - b^2.\end{aligned}$$

5. Computing $\gamma_1 + \cdots + \gamma_n$. Again by (32) and (33),

$$\begin{aligned}\sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n}(g_{n-i+2} - b)^2 = \sum_{i=2}^{n+1}(g_i - b)^2 = \sum_{i=2}^{n+1} g_i^2 - 2b\sum_{i=2}^{n+1} g_i + nb^2 = \\ g_{n+1}g_{n+2} - ab - b^2 - 2b(g_{n+3} - a - 2b) + nb^2 = \\ g_{n+1}g_{n+2} - 2bg_{n+3} + ab + (n+3)b^2 = \\ g_{n+1}g_{n+2} - 2b(g_{n+1} + g_{n+2}) + ab + (n+3)b^2 = \\ (g_{n+1} - 2b)(g_{n+2} - 2b) + ab + (n-1)b^2.\end{aligned}$$

6. Computing $s_1^2 + \cdots + s_n^2$. Finally,

$$\sum_{i=1}^{n} s_i^2 = \left(\sum_{i=1}^{n} \alpha_i\right)r^2 + \left(\sum_{i=1}^{n} \beta_i\right)r + \sum_{i=1}^{n} \gamma_i.$$

We have now proved the following theorem.

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Theorem 5.3. *Let $r \in \mathbb{R}$. Then*

$$\|\mathbf{C}_r(\mathbf{g})\|_2 \geq \left(\frac{\alpha r^2 + \beta r + \gamma}{n}\right)^{\frac{1}{2}}, \tag{34}$$

where

$$\alpha = g_{n+1}\left[(n-2)g_{n+1} - g_{n+2} + 2a + 4b\right] - ab - b^2,$$
$$\beta = 2\left\{g_{n+1}^2 - \left[a + (n+1)b\right]g_{n+1} + bg_{n+2} - b^2\right\},$$
$$\gamma = (g_{n+1} - 2b)(g_{n+2} - 2b) + ab + (n-1)b^2.$$

The bound (34) simplifies remarkably if $b = 0$. It is no essential restriction to take $a = 1$; then $g_0 = 1$ and $g_n = f_{n-1}$, $n = 1, 2, \ldots$. We find this result nice enough to warrant a corollary of its own.

Corollary 5.3.1. *Let $r \in \mathbb{R}$. Then $\mathbf{g} = (1, f_0, f_1, \ldots, f_{n-2})$ satisfies*

$$\|\mathbf{C}_r(\mathbf{g})\|_2 \geq \left(\frac{1}{n}\left\{g_{n+1}\left[(n-2)g_{n+1} - g_{n+2} + 2\right]r^2 + 2g_{n+1}(g_{n+1} - 1)r + g_{n+1}g_{n+2}\right\}\right)^{\frac{1}{2}}.$$

We complete this section by examining equality. Omitting the trivial case $n = 2$, we assume that $n \geq 3$.

The equality conditions of (30) and (34) follow from those of (19) and (20), i.e., from (25); then $r, x_0, \ldots, x_{n-1} \geq 0$ is assumed. So, assuming $r, a, b \geq 0$, equality is attained in (30) and (34) if and only if

$$r = 1 \vee (a = b = 0).$$

To show this, (25) states that $r = 1 \vee (g_1 = \cdots = g_{n-1} = 0)$. In particular, $b = 0$, but then also $a = 0$, since otherwise $g_2 > 0$.

Also equality in (26) is easily settled under the assumption $r, a, b \geq 0$. We show that it is attained if and only if

$$a = b = 0. \tag{35}$$

For $r \neq 1$, applying (21) and continuing as above yields (35). For $r = 1$, equality holds by (22) if and only if

$$\sum_{i=0}^{n-1} g_i g_{i-j} = 0, \quad j = 1, \ldots, n-1. \tag{36}$$

(Remember that indices are mod n.) Since $g_0, \ldots, g_{n-1} \geq 0$, this happens if and only if $g_i > 0$ for at most one i. But if $a > 0$ or $b > 0$, then $g_2, \ldots, g_{n-1} > 0$; we hence obtain (35) again.

However, we meet a problem. Contrary to (19) and (20), the equality condition of (18) is stated for all $r, x_0, \ldots, x_{n-1}$, without assuming nonnegativity. Therefore, we must omit this assumption in studying equality. Then (36) may hold also for some a and b, at least one of them being nonzero. Also (23), applied to $(a, b, g_2, \ldots, g_{n-1})$, may be valid.

Let us consider $n = 3$. The scalar product of any two rows of $\mathbf{C}_1(a, b, a+b)$ is

$$a(a+b) + ab + b(a+b) = b^2 + 3ab + a^2,$$

and that of $\mathbf{C}_{-1}(a, b, a+b)$ is either

$$-a(a+b) + ab + b(a+b) = b^2 + ab - a^2$$

or its opposite. For $r = \pm 1$, we can therefore characterize equality in (26) as follows: Choose a arbitrarily. Equality holds if and only if $a = 0$ or

$$r = 1 \wedge b = \frac{-3 \pm \sqrt{5}}{2}\, a$$

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or

$$r = -1 \wedge b = \frac{-1 \pm \sqrt{5}}{2} a.$$

For $n = 4$, the situation changes. It is easy to see that the only matrix $\mathbf{C}_1(a, b, a+b, a+2b)$ and, respectively, $\mathbf{C}_{-1}(a, b, a+b, a+2b)$ with orthogonal rows is the zero matrix; so, we get nothing but $a = b = 0$. It is very likely that the same holds for all $n \geq 4$, since there are many equations but only two unknowns. In other words, it is very likely that, for $n \geq 4$, the only $\mathbf{C}_{\pm 1}(\mathbf{g})$ with orthogonal rows is the zero matrix. Unfortunately, our attempts to prove this seem to lead to complicated calculations.

6 Comparison between (1), (26), (30), and (34)

We have four lower bounds under comparison. Some of them are comparable in general, some not.

Proposition 6.1. *If* $r, a, b \in \mathbb{R}$, *then* rhs(26) $\geq$ rhs(1) *and* rhs(34) $\geq$ rhs(30). *If* $r, a, b \geq 0$, *then* rhs(34) $\geq$ rhs(26).

Proof. The bound rhs(1) is obtained by using a lower bound for $\|\mathbf{C}_r(\mathbf{g}\|_2$, while rhs(26) is obtained by using its exact value; hence, the first inequality follows. The second and third follow from Proposition 3.4. □

We already saw that the bounds rhs(18) and rhs(19) are not comparable even if $r \geq 0$ and $\mathbf{x} \geq \mathbf{0}$, but what about rhs(26) and rhs(30)? We conjecture that rhs(30) $\geq$ rhs(26) if $r, a, b \geq 0$.

We consider two examples. For brevity, we denote

$$\mathbf{A}_r = \mathbf{C}_r(\mathbf{g}).$$

For convenience, we examine the squares of the bounds, denoting

$$x_r = \|\mathbf{A}_r\|_2^2, \quad y_r = \text{rhs}(1)^2, \quad z_r^{(1)} = \text{rhs}(26)^2, \quad z_r^{(2)} = \text{rhs}(30)^2, \quad z_r^{(3)} = \text{rhs}(34)^2.$$

Since all bounds are nonnegative, we can do so.

Example 1. Let

$$\mathbf{A}_r = \mathbf{C}_r(f_0, \dots, f_7) = \mathbf{C}_r(0, 1, 1, 2, 3, 5, 8, 13).$$

The squared bounds to be compared are

$$\begin{aligned}
y_r &= \min(r^2, 1) f_7 f_8 = 273 \min(r^2, 1), \\
z_r^{(1)} &= f_7 f_8 r^2 + \frac{1 - r^2}{8}(f_7 f_9 - 1) = \frac{1743 r^2 + 441}{8}, \\
z_r^{(2)} &= \left\{ \frac{1}{8} \left[f_{11} - 10 + (8 f_9 - f_{11} + 2) r \right] \right\}^2 = \left(\frac{185 r + 79}{8} \right)^2, \\
z_r^{(3)} &= \left[f_9 (6 f_9 - f_{10} + 4) - 1 \right] r^2 + 2(f_9^2 - 9 f_9 + f_{10} - 1) r + (f_9 - 2)(f_{10} - 2) + 7 \\
&= 5201 r^2 + 1808 r + 1703.
\end{aligned}$$

Consider first $r \geq 1$. We have (with integer precision)

$$\begin{gathered}
x_1 = z_1^{(3)} = z_1^{(2)} = 1089, \quad z_1^{(1)} = 273, \\
x_2 = 3480, \quad z_2^{(3)} = 3265, \quad z_2^{(2)} = 3150, \quad z_2^{(1)} = 927, \\
x_9 = 64603, \quad z_9^{(3)} = 54907, \quad z_9^{(2)} = 47524, \quad z_9^{(1)} = 17703, \\
x_{50} = 1968010, \quad z_{50}^{(3)} = 1636825, \quad z_{50}^{(2)} = 1359848, \quad z_{50}^{(1)} = 544743.
\end{gathered}$$

In all these cases, $y_r = 273$. The bound $z_r^{(3)}$ is good, and $z_r^{(2)}$ is almost as good. The bound $z_r^{(1)}$ is not good but much better than y_r, except for the case $r = 1$, when they are equal.

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Next, consider $r \le -1$. Again $y_r = 273$. Resembling the above, the results are

$$\begin{gathered}
x_{-1} = 744, \quad z_{-1}^{(3)} = 637, \quad z_{-1}^{(2)} = 176, \quad z_{-1}^{(1)} = 273, \\
x_{-2} = 3015, \quad z_{-2}^{(3)} = 2361, \quad z_{-2}^{(2)} = 1323, \quad z_{-2}^{(1)} = 927, \\
x_{-9} = 62791, \quad z_{-9}^{(3)} = 50839, \quad z_{-9}^{(2)} = 39303, \quad z_{-9}^{(1)} = 17703, \\
x_{-50} = 1958018, \quad z_{-50}^{(3)} = 1614225, \quad z_{-50}^{(2)} = 1314176, \quad z_{-50}^{(1)} = 544743.
\end{gathered}$$

If $|r| < 1$, the z_r-bounds do not perform as well as above but anyway much better than y_r. We obtain (with four-digit precision)

$$\begin{gathered}
x_{0.5} = 569.2, \quad z_{0.5}^{(3)} = 488.4, \quad z_{0.5}^{(2)} = 459.6, \quad z_{0.5}^{(1)} = 109.6, \quad y_{0.5} = 68.25, \\
x_{0.1} = 446.3, \quad z_{0.1}^{(3)} = 242, \quad z_{0.1}^{(2)} = 148.5, \quad z_{0.1}^{(1)} = 57.30, \quad y_{0.1} = 2.730, \\
x_{0} = 438.6, \quad z_{0}^{(3)} = 212.9, \quad z_{0}^{(2)} = 97.52, \quad z_{0}^{(1)} = 55.13, \quad y_{0} = 0, \\
x_{-0.1} = 437.3, \quad z_{-0.1}^{(3)} = 196.8, \quad z_{-0.1}^{(2)} = 57.19, \quad z_{-0.1}^{(1)} = 57.30, \quad y_{-0.1} = 2.730, \\
x_{-0.5} = 494, \quad z_{-0.5}^{(3)} = 262.4, \quad z_{-0.5}^{(2)} = 2.848, \quad z_{-0.5}^{(1)} = 109.6, \quad y_{-0.5} = 68.25.
\end{gathered}$$

In general, if $\mathbf{A}_r \ge \mathbf{O}$ or $\mathbf{A}_r \le \mathbf{O}$ (and $\mathbf{A}_r \ne \mathbf{O}$), the comparison seems to give rather similar results as above. But if $\mathbf{A}_r$ has entries of opposite signs, they may be different.

Example 2. Let

$$\mathbf{A}_r = \mathbf{C}_r(30, -19, 11, -8, 3, -5, -2, -7).$$

The squared bounds are

$$\begin{gathered}
y_r = 1533 \min(r^2, 1), \quad z_r^{(1)} = \frac{1323r^2 + 10941}{8}, \\
z_r^{(2)} = \left(\frac{119 - 95r}{8}\right)^2, \quad z_r^{(3)} = \frac{1601r^2 - 3772r + 2243}{8}.
\end{gathered}$$

If $|r| \ge 1$, then $y_r = 1533$, while

$$\begin{gathered}
x_1 = 6561, \quad z_1^{(3)} = z_1^{(2)} = 9, \quad z_1^{(1)} = 1533, \\
x_2 = 10295, \quad z_2^{(3)} = 137.9, \quad z_2^{(2)} = 78.77, \quad z_2^{(1)} = 2029, \\
x_9 = 91821, \quad z_9^{(3)} = 12247, \quad z_9^{(2)} = 8464, \quad z_9^{(1)} = 14763, \\
x_{50} = 2528004, \quad z_{50}^{(3)} = 477018, \quad z_{50}^{(2)} = 335096, \quad z_{50}^{(1)} = 414805,
\end{gathered}$$

$$\begin{gathered}
x_{-1} = 3626, \quad z_{-1}^{(3)} = 952, \quad z_{-1}^{(2)} = 715.6, \quad z_{-1}^{(1)} = 1533, \\
x_{-2} = 6310, \quad z_{-2}^{(3)} = 2024, \quad z_{-2}^{(2)} = 1492, \quad z_{-2}^{(1)} = 2029, \\
x_{-9} = 78373, \quad z_{-9}^{(3)} = 20734, \quad z_{-9}^{(2)} = 14823, \quad z_{-9}^{(1)} = 14763, \\
x_{-50} = 2454997, \quad z_{-50}^{(3)} = 524167, \quad z_{-50}^{(2)} = 370424, \quad z_{-50}^{(1)} = 414805.
\end{gathered}$$

The z_r-bounds are not comparable, and none of them estimate well. The bound y_r, although being the poorest one in most cases, is better than some of the z_r-bounds if $r = \pm 1$ or $r = \pm 2$.

As to $|r| < 1$, we have

$$\begin{gathered}
x_{0.5} = 5362, \quad z_{0.5}^{(3)} = 94.66, \quad z_{0.5}^{(2)} = 79.88, \quad z_{0.5}^{(1)} = 1409, \quad y_{0.5} = 383.3, \\
x_{0.1} = 4674, \quad z_{0.1}^{(3)} = 235.2, \quad z_{0.1}^{(2)} = 187.3, \quad z_{0.1}^{(1)} = 1369, \quad y_{0.1} = 15.33, \\
x_{0} = 4533, \quad z_{0}^{(3)} = 280.4, \quad z_{0}^{(2)} = 221.3, \quad z_{0}^{(1)} = 1368, \quad y_{0} = 0, \\
x_{-0.1} = 4404, \quad z_{-0.1}^{(3)} = 329.5, \quad z_{-0.1}^{(2)} = 258, \quad z_{-0.1}^{(1)} = 1369, \quad y_{-0.1} = 15.33, \\
x_{-0.5} = 3984, \quad z_{-0.5}^{(3)} = 566.2, \quad z_{-0.5}^{(2)} = 433.2 \quad z_{-0.5}^{(1)} = 1409, \quad y_{-0.5} = 383.3.
\end{gathered}$$

Now, $z_r^{(1)}$ is the best one but not especially good.

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7 Concluding remarks

We have above computed $\|\mathbf{C}_r(\mathbf{x})\|_F$ exactly and given three lower bounds for $\|\mathbf{C}_r(\mathbf{x})\|_2$. In particular, we have studied the case $\mathbf{x} = \mathbf{g}$, improving several previously known results. If $a, b \geq 0$ (or $a, b \leq 0$), then our bounds appear to be quite good.

Extending (9) to

$$h_n = aq\phi_{n-1} + b\phi_n, \quad n = 1, 2, \dots,$$

where

$$\phi_0 = 0, \quad \phi_1 = 1,$$
$$\phi_n = p\phi_{n-1} + q\phi_{n-2}, \quad n = 2, 3, \dots,$$

we could find $\|\mathbf{C}_r(\mathbf{u})\|_F$ and lower bounds for $\|\mathbf{C}_r(\mathbf{u})\|_2$, but the calculations turned out to become more complicated. More generally, we could do this even for $\|\mathbf{C}_r(\mathbf{h})\|_F$ and $\|\mathbf{C}_r(\mathbf{h})\|_2$, but the calculations would become even more complicated. Therefore, we did not pursue this issue any further.

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