THE IMPORTANCE OF THE SHAPE OF ARCHES

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Abstract

The aim of the article is to show the importance of the anti-funicular geometry of the arch in tied-arch bridges. The concrete tied-arch resembles a prestressed unbonded structure when tie tendons and adjustable hanger cables are used. In the study the hanger cables are assumed vertical.

The form of the arches has been of interest to Architects and Engineers through the ages. The mathematical solution to the catenary curve was successfully solved in the late 17th century. Interest in arch bridge constructions has increased recently. The traditional arch forms of circle and parabola do not represent optimal geometry of bridge arches. The comparative calculations of a concrete tied-arch footbridge with a span length of 100 m by using different arch forms awakes the reader to notice the importance of the geometric shape of the arch.

This paper presents the formulae of circle, parabola, catenary and constant stress arch. The visual difference of the arches is demonstrated in the article by exact drawings.

An iterative application of graphical statics and vector algebra quickly results in the momentless and also constant stress shape of the arch for the permanent loads. The final arch geometry can be fine-tuned by the weights of the dimensioned cross-sections of the arch rib and stiffening girder.

Based on the study circle shape leads to the most uneconomical solution. The parabolic arch shape, and also the pure catenary arch shape, result in significant bending moments because of combined effect of weights of arch rib and stiffening girder. The final arch geometry and cross-sections are obtained as momentless arch by using the final weights.

INTRODUCTION

The development of the tied-arch bridges began after the Austrian engineer Joseph Langer received a patent in 1859 for a girder bridge stiffened with a steel arch. The first major bridge was the Ferdinands Bridge, span of 68.4 m, over the Mur River in Graz, completed in 1881, [1].

Experience and knowledge of arch bridges in the late 1970s can be read from the U.S. Department of Roads Bridge Bridges publication, [2]. The following paragraph is a direct citate from the above work.

An increase of rise decreases arch thrust inversely with the rise-to-span ratio, reducing the axial stress from dead load and live load and bending stress from temperature change. The axial tension tie, if used, is also decreased in the same way. Offsetting these effects from standpoint of economy is the increased length of arch rib. This greater length increases the quantity of steel and dead load. It also increases the buckling length in the plane of the arch and the moment magnification factor. The lengths of suspenders are increased. The total length of lateral bracing between the ribs is increased, and the wind overturning and stresses are increased. Many existing tied arches have a rise to span ration of about 0.2 (notice in here *h/l).*

These above-mentioned issues can be solved analytically and numerically, which gives the minimum for the total cost of the bridge, whereby the height can be found for the arch. Mostly optimizations have been prepared for already designed structures in finding the minimum of material quantities. This study aims to give simple tools to stakeholders, based on structural mechanics, to calculated the cost effects of the arch rise and importance of the shape before any deeper analysis.

The most optimal structure, based on the minimum quantity of load-bearing material for uniform vertical load between two points, results in a parabolic arch, [3].

The rise of the arch in the bridges is often chosen based on the aesthetic compatibility of the landscape and the arch. Fritz Leonhardt has stated: *The shallower the arch spans, the more daring these bridges look. Span/rise ratios* (notice in here *l/h*) *up to 12 are possible*, [4].

Christian Menn has stated about the height of the arches and the aesthetics of the bridges: *The ratio of arch span to rise l/h, should be chosen between 2:1 and 10:1. Reducing l/h below 2:1 result in an awkward appearance and substantially increases in construction cost*, [5]. *Economy and Elegance cannot be optimized independently of each other. For a wide range of cases, these two objectives are closely related*, [6].

Minimizing the quantity of material in the arch requires minimizing the eccentricity of the normal force on the arch, [7]. The goal is the anti-funicular shape of the structure.

For large bridge structures, the permanent load is usually the dominating load of the bridge, [8]. This study is based on the assumption that the optimal shape of the arch can be calculated according to the permanent load of the bridge.

The proportions of the effects of traffic loads and other loads on the bridge on conventional girder bridges on cable-stayed bridges are well known from the resent experience. For tied-arch bridges, similar data have not been accumulated to the same extent. The moving loads cause the bending moments in the arch and in the stiffening girder, which alternate both sides of thrust line of permanent loads as shown in Figure 1.

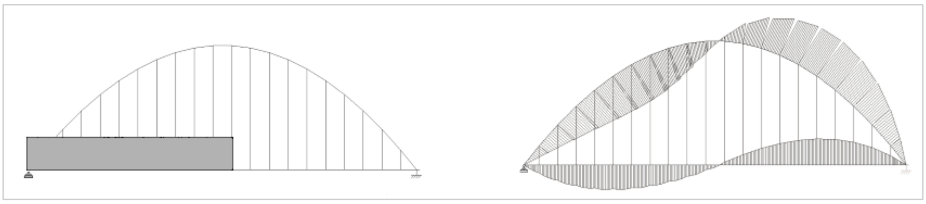


Figure 1 - Half span load and bending moments in tied-arch bridges.

Accordingly, the stresses in cross sections alternate on both side of the normal stress from the permanent loads. This may justify study the structure based on the stresses of the anti-funicular shape for permanent loads, by applying the stresses which have enough reserve for other loads and the stability of the arch.

The anti-funicular shape is solved for the plane structure by an iteration procedure according to graphical statics and vector algebra. The cost-optimal rise of the bridge is determined analytically by applying the *force length method*. An example calculation for a concrete light traffic concrete tied-arch bridge is presented in the article. The example reveals the importance of the exact shape of the arch. Effects resulting from creep and shrinkage of concrete are limited out of this study to simplify the study, as well as the arches are assumed to be incompressible.

ARCH TYPES

Figure 2 shows the arch types of parabolic, catenary, constant stress arch and circle discussed in this article. The arches, stand alone, shown below are momentless.

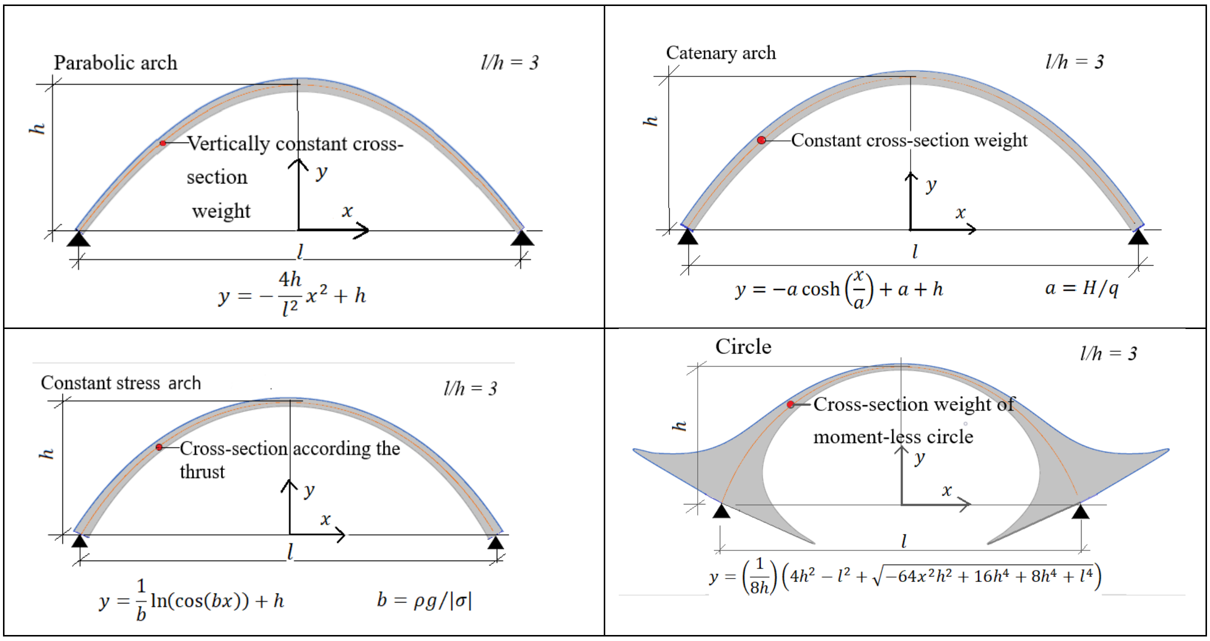


Figure 2 – Parabolic, catenary, constants stress and circular arches.

Parabola

Parabola is the most well-known arch type used in bridges. The formula of parabolic arch in the coordinates above is:

$$y = -\frac{4h}{l^2}x^2 + h, \tag{1}$$

If the loading, carried by the arch, is uniformly distributed along x-axis, then the parabolic arch has only centric compression stress assuming the arch material incompressible. In the tied-arch bridge the permanent load consists of arch structures, tie tendons, hanger cables and stiffening girder structures. The total permanent loading of the bridge is not uniformly distributed loading, which causes bending moments to the parabolic arch.

Catenary arch

When a suspended chain is turned upside down, a correspondingly shaped compression arch is created. The equation of the catenary arch is:

$$y = -a\cosh\left(\frac{x}{a}\right) + a + h, \tag{2}$$

where the term $a = H/q$ is the horizontal force of arch divided by the cross-section weight of the arch.

There are no bending moments in the arch when the cross section of the arch has constant weight along the entire arch. The material is assumed to be incompressible with zero bending stiffness. In the catenary equation (2), for the span *l* and the height *h*, can be solved from the boundary condition (0, 0.5*l*).

Constant stress arch

Arches can be designed to have the constant compressive stress from the springing line of the arch to the crown. Achieving the constant stress requires compatibility between the shape of the arch and the load. The equation of constant stress arch, stand alone, can be written as:

$$y = \frac{1}{b}\ln(\cos(bx)) + h, \tag{3}$$

The term $b = \rho g / |\sigma|$, where ρ is the density of the arch material, g is the acceleration due to gravity and σ is the compressive stress of the arch. The arch is shown in the Figure 2 above.

Circle

The equation of a circular arch, in the same coordinate system as the arches above, is:

$$y = \left(\frac{1}{8h}\right)\left(4h^2 - l^2 + \sqrt{-64x^2h^2 + 16h^4 + 8h^2l^2 + l^4}\right), \tag{4}$$

where $h \leq l/2$. The circle arch is shown in the Figure 2 above.

MAXIMUM THEORETICAL SPAN OF ARCHES

All the momentless arch types, for their self-weights, approach each other and finally approach girder when the rise of arch l/h increases without any limit. Figure 3 shows the difference of maximum theoretic span lengths of parabolic, catenary and constant stress arches based on maximum compression stress calculated respectively as in reference for steel arches [9]. Figure 3 reveals the superiority of the constant stress arch and that the difference of maximum theoretical arch spans, according to the rise relation used in bridges from 2 to 6, is remarkable. This indicates, that when higher arches are used, the more important is the shape of the arch.

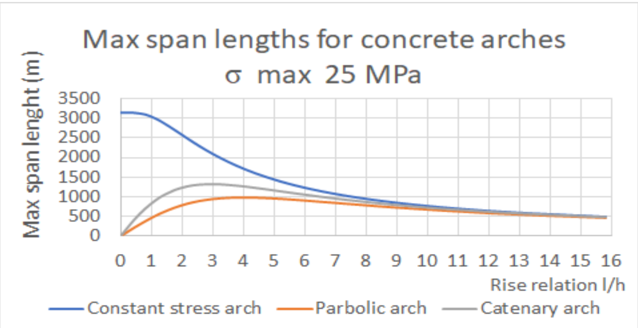


Figure 3 – Maximum spans of different types of concrete arches, compressive stress 25 MPa.

ANTI-FUNICULAR SHAPE

The anti-funicular shape defined only for the self-weight of the arches can be calculated using the formulae given above. The finding of momentless shape of the bridge arch requires iterative process, since the self-weight of the arch varies due to the shape and effects accordingly to the anti-funicular shape. Graphic static and vector algebra give a simple tool to find the shape. The drawing in Figure 4 shows the graphic static method to find the shape in vertical plane. Hanger cables are stressed to carry the deck load like the supports of a continuous girder The loads are calculated as equivalent node point loads. The thrust line is found starting from the springing line by calculating the resultant force vector path (thrust line) over the arch. In the beginning only the weight of the stiffening girder structure is needed. The arch rib, tie tendons and hangers will be dimensioned according to the forces in the respective members. Recent research has led to the programs capable to solve anti-funicular shapes also for complicated 3-dimensional structures [10].

Only a few iteration rounds (usually four to five) are needed for the acceptable convergence. Preliminary dimensioning algorithm and parameters can be applied in the dimensioning of the arch rib, tie tendons and hanger cables. The weights defined during the calculation process dictate the anti-funicular shape of the bridge. Figure 4 shows the force and weight vectors of a tied bridge.

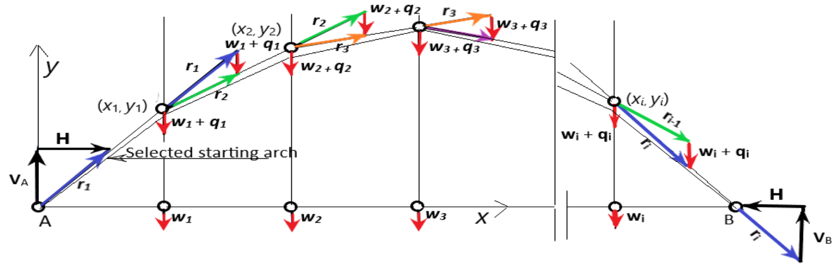


Figure 4 – Vector drawing for the calculation of the thrust line of a tied-arch bridge.

COST-OPTIMAL RISE OF TIED-ARCH BRIDGE

In the following, the optimal rise ratio of tied-arch bridge is solved analytically based on the quantities and the costs of the arch rib, tie tendons and hangers. Selections of empirical stresses to the arch rib, tie tendons and hanger cables for the permanent loads are needed as well as the unit prices respectively.

In the analytic calculation the assumed load on the bridge is uniformly distributed load, carried by hanger cables. The force lengths (the force multiplied the length of respective member) of the parabolic tied-arch bridge arch, tie tendons and hanger cables are calculated by integration as shown in equations (5), (6) and (7) and related Figure 5. When the force length is divided by the stress the method is known as *force volume method*, [11].

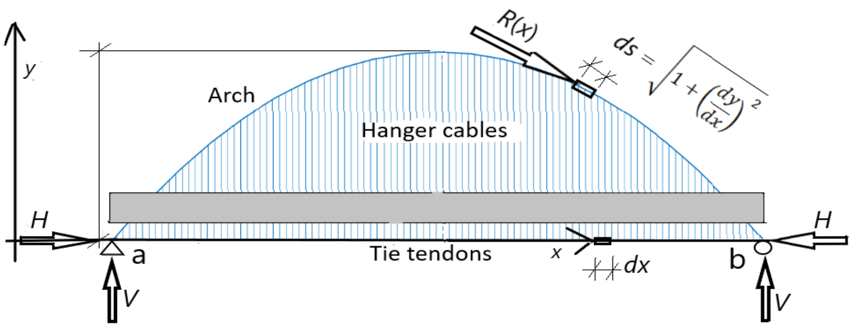


Figure 5 – Tied-arch bridge.

The force length integral f_l over the tied arch bridge can be written as follows:

$$f_l = \int_a^b R(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx + \int_a^b H dx + \int_a^b wy dx \tag{5}$$

By applying *R(x)*, *H*, *h(x)* and ds, the formula of the force length f_l of the bridge can be written as:

$$f_l = 2w_b \int_0^{\frac{l}{2}} \frac{l^2}{8h} \sqrt{1+\left(\frac{dy}{dx}\right)^2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \tag{6}$$

$$+2w_b \int_0^{\frac{l}{2}} \frac{l^2}{8h} dx + 2(w_b - w_a) \int_0^{\frac{l}{2}} \left(-\frac{4hx^2}{l^2} + h\right) dx,$$

where w_b is the estimated total uniformly distributed vertical load of the bridge and w_a is the estimated load component of the arch included in the total load w_t [MN/m].

The integration results the force length of the bridge as:

$$f_l = \left(\frac{l^2}{8h} + \frac{2h}{3}\right) w_b l + \left(\frac{l^2}{8h}\right) w_b l + \left(\frac{2h}{3}\right)(w_b - w_a) l \tag{7}$$

When the force lengths of the arch, tie and hanger cables are multiplied by the cost factors, the costs can be calculated as:

$$C_{ath} = \left(\frac{l^2}{8h} + \frac{2h}{3}\right) lC_a + \left(\frac{l^2}{8h}\right) lC_t + \left(\frac{2h}{3}\right) lC_h, \tag{8}$$

where $C_a = w_b a_c / \sigma_{c,}$ $C_t = w_b a_t \rho / \sigma_t$ and $C_h = (w_b - w_a) a_h \rho / \sigma_h$,

a_c is the unit price of concrete in the arch [€/m^3]
a_t and a_h are respective unit prices of the tie tendons and hanger cables [€/kg]
$\sigma_{c,}$ σ_t and σ_h are the stresses used [MPa] in the arch, tie tendons and hangers for permanent loads
ρ is the unit weight of steel material in tie tendons and hangers [7850 kg/m^3].

The cost optimal height of the arch can be solved from the equation

$$\frac{dC_{ath}}{dh} = \left(-\frac{l^2}{8h^2} + \frac{2}{3}\right) C_a + \left(-\frac{l^2}{8h^2}\right) C_t + \left(\frac{2}{3}\right) C_h = 0, \tag{10}$$

whereby the cost- optimal rise gets a simple formula

$$h_{opt} = l \sqrt{\frac{3(C_{a+}C_t)}{16(C_a + C_h)}}. \tag{11}$$

Small and medium size concrete arch bridge have been traditionally cast in place using scaffolding structures. The cost of scaffolding influence to the optimal height. When those costs are included, the cost optimal height can be written as:

$$h_{opt} = l \sqrt{\frac{3(C_{a+}C_t)}{16(C_a + C_h + Cs)}}, \tag{12}$$

where C_s is equal to $C_s = a_s b$, a_s is the unit price of scaffolding [€/m^3] and b is the width of the scaffoldings [m].

If the weight of arch is not included to the vertical load w_b, it can be approximated as uniform vertical load resulted from the iterative calculation rounds using the force volume of the arch above. More exact optimal height can be reached using iterative calculated anti-funicular shape using weight vectors as shown in Figure 4. Anyhow, parabolic arch can be used for preliminary cost comparison in spite of the fact it is not the final arch shape. The forces and the lengths in parabolic arch are reasonably close to the anti-funicular arch. The equation (12) has been applied in the calculation explained below.

SENSITIVITY OF THE COST-OPTIMAL ARCH HEIGHT

The following example calculation is prepared to study the cost sensitivity of the arch rise. The example bridge case is a concrete tied-arch bridge. The light traffic bridge span with single arch has the span length of 100 m and the deck weight is 0.1 MN/m. Table 1 shows an example calculation data and the cost-optimal rise and respective quantities calculated.

Table 1 – Cost-optimal height calculation data and results.

Needed input data					Results	
	Stress	Unit cost	Span	Vertical load[4]	Cost-opt. rise	Quantities
Arch[1]	8 MPa	2000 €/m^3	100 m	0.1 MN/m	27.023 m	102 m^3
Tie tendons[3]	600 MPa	20 €/kg				7655 kg
Hanger cables[3]	400 MPa	30 €/kg				3576 kg
Scaffolding		15 €/m^3				9009 m^3
[1]Unit weight 0.025 MN/m, [2]unit weight 7850 kg/m^3, [3]width 5 m, [4]without the weight of arch						

The formula (12) gives the optimum height of h_{opt} = 27.023 m, corresponding the rise relation l/h =3.70. The weight of arch rib, with preliminary assumed 8 MPa compression stress, is calculated applying the equation (7) of the force lengths divided by the stress by using four calculation rounds. The weight of arch is added to the vertical loads after each calculation round. The vertical load increases, due to arch weight from 0.1 MN/m to 0.125 MN/m after four calculation rounds. The convergence is acceptable. The cost-sensitivity in relation to variable rises are shown in Figure 6.

The calculations reveal, that the rise relation l/h between 2.6 to 4.6 yield maximum increase of 5 % in costs compared to the minimum costs. The cost of scaffolding is an important factor when deciding the arch construction methos. Without scaffolding cost the optimum rise relation l/h were 2.76.

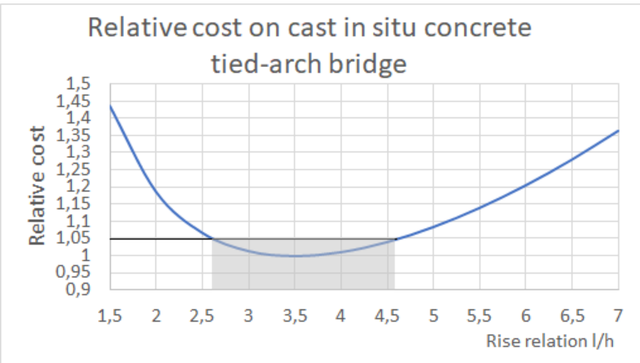


Figure 6 – Sensitivity of cost-optimal rise.

The cost-optimal rise of the chosen bridge depends on selected stresses and construction technology. The formulas above give a simple tool for comparison of bridge concept alternatives and their costs when selecting the bridge concept under discussion by using updated stresses and cost data.

IMPORTANCE OF THE ARCH SHAPE

Visible difference of arches

Using the iterative procedure according to graphical statics and vector algebra, the exact momentless shapes for the following type of arches shown in Figure 7 are calculated:

- as a constant stress arch with a compression of 8 MPa
- as a momentless arch for a constant cross section with a base compression of 8 MPa
- in addition, parabola, pure catenary and circle are drawn in the picture.

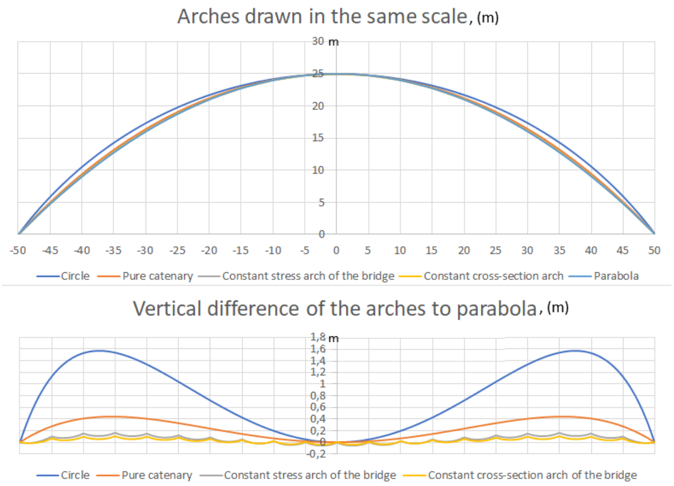


Figure 7 – Visible and actual difference of the arches.

Visibly the arches do not seem to differ from each other, but the actual differences have remarkable effects to the restrains of the arch and its dimensions and finally to the costs of the bridge. The respective comparisons of a large span concrete arch bridge can be found in reference [12].

Bending moments and stresses if the arches were built as circle, pure catenary or parabola

The following example calculations demonstrate the importance of the momentless shape of the arch. The tied-arch bridge has the same design parameters as used above. The hanger cables with 5 m spacing are used. The bridge has the rise relation of $l/h = 4$. Figure 8 shows the shape of the calculated constant stress arch by using 8 MPa stress and the respective cross-section areas of the arch rib.

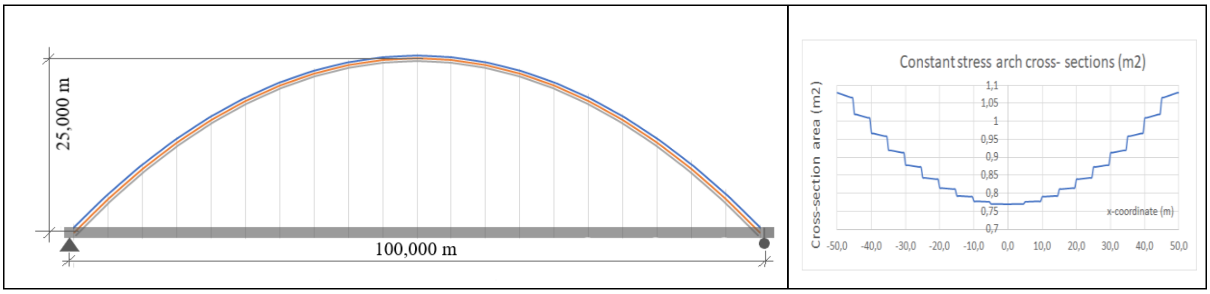
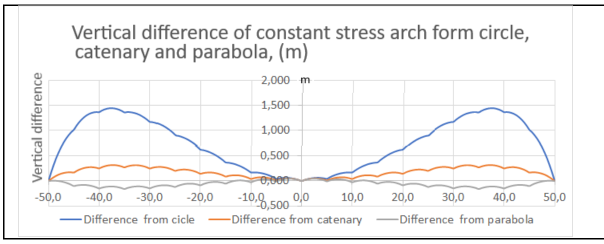


Figure 8 – Anti-funicular tied-arch and the respective cross-sections of the arch.

Figure 9 shows the height differences of the constant stress arch to circle, catenary, and parabola, and the corresponding bending moments of the arches for the permanent loads according to the weights of cross-sections of the constant stress arch shown in Figure 8. The bending moments are calculated using 2-pin ended arches.



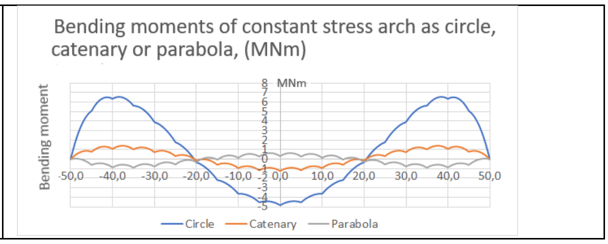


Figure 9 – Vertical differences of constant stress shape arch to circle, catenary and parabola and the respective bending moments.

Figure 10 shows the bending stresses by assuming the square cross-section of the arch for circle, catenary and parabolic arches.

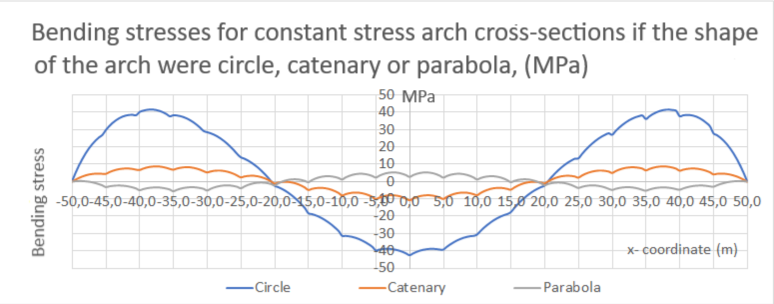


Figure 10 – Bending stresses of the constant stress arch were circle, catenary or parabola.

The constant stress arch has the compression stress of 8 MPa along the whole arch. The stresses caused by circle shape destroy the feasible design, causing stresses up to |40| MPa. The uneconomical nature of the circular arch as a bridge arch is highlighted. Parabolic arch shape increases the stresses up to |9| MPa and catenary shape to |6| MPa. There is no reason to allow these additional stresses to cause increases to the cross-section dimensions of the arch, as they can simple be eliminated by the shape.

TOLERANCES AND LIVE LOADS

In the example above, a constant stress shape has been applied to the concrete arch using compressive stress of 8 MPa. It has been assumed that the cross-sections of the arch allow, in addition, stresses from traffic loads and other design loads. The matter of choosing the arch stress requires further research.

In the previous example, the maximum eccentricities of the catenary and parabolic arches, calculated as two-pin arches are 0.274 m and 0.165 m, respectively. The corresponding bending stresses are of the order of 7.5 MPa and 5 MPa with an arch of the respective compressive force of 7 MN.

The position of the arch, with respect to the thrust line, is comparable to the position of the tendon force of the bent beam with respect to the neutral axis of the beam. The problem is that the location of the thrust line of the arch is not a visible element. In this case, attention must be paid to the position of the arch. The tolerance of the arch should therefore be the absolute dimension specified by the designer. Today's manufacturing and measurement methods can achieve remarkably accurate locations as reported in reference [13]. Attention should be paid to the construction specifications, which do not consider modern arch bridge design and construction technology. The position of the arch should be understood equally as the accuracy of the tolerance of the position of the tendon force in a girder.

The bending moment caused by the traffic load is distributed in the bridge in relation to the stiffnesses of the arch and the stiffening girder. This leads to many possibilities to optimize the cross-sections and also to vary the appearance of the bridge. The article does not explore this in more detail. This requires further research with different loads. As example, in the bridge above, the bending stresses from light traffic load can easily be of the order of 2 to 10 MPa for different stiffness and cross-sectional choices.

CONCLUSION

The cost-optimal height of the arch can be determined by a simple calculation presented in this article. The constant stress shape of the arch is often the most cost-optimal shape for the arch. The shape of arch is dictated by the loads. The anti-funicular shape can be easily iterated using graphical statics and vector algebra calculation. The construction of the arch must be carried out with remarkably precise tolerances. The usage of unbonded tie tendons and adjustable hanger cables allow precise adjustment of the forces so that shape of the anti-funicular bridge is reached and also partly allow to correct the deviations of the shape if necessary.

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