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Cite as: APL Photonics 6, 060801 (2021); https://doi.org/10.1063/5.0049678
Submitted: 08 March 2021. Accepted: 18 May 2021. Published Online: 01 June 2021

Marco Ornigotti, Luca Ornigotti, and Fabio Biancalana

COLLECTIONS

Note: This paper is part of the APL Photonics Special Topic on Synthetic Gauge Field Photonics.

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APL Photonics

SPECIAL TOPIC: Perovskite Photonics
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Submitted: 8 March 2021 • Accepted: 18 May 2021 •
Published Online: 1 June 2021

Marco Ornigotti,1,a) Luca Ornigotti,2,b) and Fabio Biancalana3

AFFILIATIONS
1 Faculty of Engineering and Natural Sciences, Photonics, Tampere University, Tampere FI-33720, Finland
2 Department of Optics, Palacký University, 17. listopadu 1192/12, 711 46 Olomouc, Czech Republic
3 School of Engineering and Physical Sciences, Heriot-Watt University, Edinburgh, United Kingdom

Note: This paper is part of the APL Photonics Special Topic on Synthetic Gauge Field Photonics.

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I. INTRODUCTION

Gauge fields are ubiquitous in nature and regulate the dynamics of several different fields of physics. Perhaps the most common example of a gauge field is the electromagnetic field, whose quantum, the photon, mediates the interaction between charged matters.1 Besides electrodynamics, gauge fields make their appearance in the standard model of particle physics, for example, as a unified way to describe the interaction of matter with the fundamental forces of nature, excluding gravity.2 In quantum mechanics, the gauge invariance of the electromagnetic field is at the core of the celebrated Aharonov–Bohm effect,3 which paved the way for a deeper understanding of gauge fields in terms of fiber bundles.1–4 Moreover, non-Abelian gauge theories are also an essential ingredient to understand geometrical and Berry phases.3 In condensed matter physics, gauge fields play a crucial role in understanding long-range interactions and the emergence of collective phenomena, such as the appearance of Abrikosov vortices5 or topological states of matter.6,7 In recent years, artificial (or synthetic) gauge fields (AGFs) have started to attract considerable attention. Contrary to gauge fields, which arise from real fields or geometric connections, AGFs can be tuned ad hoc by acting on a physical system in a certain, predetermined and tunable way. Typically, the physical mechanism generating AGFs is also different, in nature, from the one generating the actual gauge field it mimics. For example, an artificial magnetic field is experienced by cold atoms in a rotating frame due to the mathematical equivalence between the Coriolis force and the Lorentz force8 or for light propagating in a waveguide with a twisted propagation direction due to the topological equivalence between the twisting and an actual magnetic field applied to the waveguide.9,10 The advent of AGFs paved the way for completely new research fields, which merge together aspects from topology and set theory with different concepts and methods from various physical disciplines, such as topological mechanics,11 topological condensed matter physics,12 topological atomic physics,13,14 and topological photonics,15,16 where both concepts from lattice gauge field
Theories and condensed matter physics were used to create novel ways to control the propagation and coupling of light in waveguiding structures, such as photonic topological insulators, non-Abelian-like dynamics in engineered waveguide lattices, AGF switching using the angular momentum of light, and topological protection.

Besides photonics, another platform that, in the last decade, represented a rich playground for testing the effects of different classes of AGFs has been graphene. Since its experimental discovery in 2004 by Novoselov et al., graphene has, in fact, attracted considerable attention not only for its peculiar band structure, its anomalous quantum Hall effect, or its minimal conductivity but also for its unexpected connection between condensed matter physics and gauge field theory. The application of strain, stress, or bending on graphene, in fact, gives rise to AGFs in the form of effective electric and magnetic fields. Moreover, out-of-plane bending of graphene flakes is analog, for the electrons in graphene, to consider their evolution on a curved background under the action of gravity.

Graphene also represents a very interesting platform for photonics, mainly for its very large nonlinear response, compared to bulk materials. Recent theoretical and experimental results, in fact, estimate the bulk-equivalent third-order nonlinear susceptibility of graphene to go from $\chi^{(3)}_{3D} \approx 10^{-15}$ m$^2$/V$^2$ in the visible region up to $\chi^{(3)}_{3D} \approx 10^{-9}$ m$^2$/V$^2$ in the terahertz region. For comparison, the bulk third-order nonlinearity of a typical silica glass is of the order of $\chi^{(3)}_{\text{glass}} \approx 10^{-23}$ m$^2$/V$^2$ (see Ref. 48). Graphene, therefore, possesses an extremely high nonlinear response, several orders of magnitude higher than that of a normal nonlinear material. A typical way to obtain such results experimentally is by means of the Z-scan technique, which is a standard characterization method for the nonlinear response of bulk materials and can be applied for graphene as well under certain assumptions (see Ref. 46 for details). The results of such measurement leads naturally to 3D susceptibility, but in the case of graphene, these numbers should be taken with a grain of salt, since the notion of bulk susceptibility for monolayer graphene does not make much sense (graphene is, indeed, a true 2D material), and they should be used only for comparison with other materials.

The nonlinear response of graphene in the limit of the strong magnetic field (i.e., when the magnetic length is much smaller than the wavelength of the impinging electromagnetic pulse) has also been estimated theoretically to be inversely proportional to the external magnetic field, i.e., $\chi^{(3)}_{3D} \approx 5 \times 10^{-9}/B(T)$. In the same work, moreover, the maximum intensity of the nonlinear signal has also been estimated to be linearly proportional to the applied magnetic field, i.e., $I_{\text{max}}^{(3)} \propto B$. From these results, we can see how controlling the magnetic field results in an overall increase in the intensity of the nonlinear signal. This feature, combined with the possibility of creating magnetic fields through AGFs, could be exploited to reduce the necessary pump intensity to trigger nonlinear phenomena in graphene and might lead, in the future, to a new generation of integrated nonlinear devices.

Amid the vast literature on AGFs in graphene, some works have been focusing attention on their effects on the interaction of graphene with external electromagnetic fields, pointing out how a constant external electric field can drastically modify the arrangement of Landau levels and edge states of bent graphene flakes, introducing squeezing of the Landau states in graphene, while a time-dependent electric field can induce modulations of the angular momentum transfer between light and graphene or induce coherent population transfer between different Landau levels of the valence and conduction bands of graphene. To the best of our knowledge, however, the influence of AGFs on the nonlinear response of graphene interacting with an ultrashort laser pulse has not been investigated yet.

In this work, therefore, we present a comprehensive analysis of the interaction of ultrashort light pulses with graphene in the presence of an AGF. In particular, we consider the case of strained graphene, as presented in Ref. 40, which implements an effective constant, uniform, magnetic field orthogonal to the graphene plane. The presence of an AGF induces Landau levels in both the valence and conduction bands of graphene, thus introducing selection rules on the impinging pulse polarization. Under this assumption, we study different interaction configurations for impinging pulses of different resonant frequencies, and we show that the nonlinear signal produced by electrons near the Dirac point of graphene differs substantially from the case of unbent graphene, since the presence of the AGF breaks the symmetry of graphene, thus allowing the appearance of even harmonics in the nonlinear optical response.

This work is organized as follows: In Sec. II, we briefly review how bending graphene introduces an artificial gauge field and what its consequences are on an electron in graphene. Then, we present the theoretical framework necessary to describe the interaction of ultrashort pulses with graphene in the presence of an AGF in Sec. III. Section IV is dedicated to the main results of our work, i.e., the calculation and discussion of the nonlinear response of bent graphene to an external ultrashort pulse. Finally, conclusions and future prospects are discussed in Sec. VI.

II. ARTIFICIAL GAUGE FIELD FROM BENDING

When a mechanical strain is applied to graphene, the immediate result is the appearance of an artificial gauge field, whose spatial distribution and orientation depend on the nature of the strain applied. The general expression for the induced AGF in the absence of out-of-plane modulations can then be written as

$$A^{(i)} = \pm \frac{\beta}{a} (u_{xx} - u_{yy}) \hat{x} \pm \frac{2\beta}{a} u_{xy} \hat{y},$$

where $a \approx 1.42$ Å is the carbon–carbon interatomic distance, $\beta \approx 2$ is the electron Grüneisen parameter, accounts for the strength of the strain, and $u_{	ext{max}} = (\partial_{x} u_{xx} + \partial_{y} u_{xy})$ is the strain tensor, with $u_{\text{max}}$ being the displacement vector. Note that strain also induces a scalar potential $V(x,y) \propto u_{xx} + u_{yy}$, which can be neglected anyway by choosing an appropriate gauge, where the scalar potential is set to zero. A careful choice of the strain tensor can lead to different bending and deforming geometries, corresponding to different AGFs. Among the various choices available, we choose the bending profile discussed in Ref. 40, which allows the creation of a uniform pseudomagnetic field in a rectangular graphene flake by introducing only one deformation parameter, i.e., the bending radius $R$. The explicit expression of the displacement vector that implements this geometry is then given as
where $\mathbf{A}^{(\theta)} = \pm \mathbf{B} y / a R \mathbf{k} = - B \mathbf{\hat{k}}$ and, therefore, to a pseudomagnetic field $\mathbf{B} = B \mathbf{\hat{k}}$. A schematic representation of the bent graphene flake, together with a set of experimentally realizable parameters, is given in Fig. 1. Note that for the choice of parameters as in Fig. 1, a uniform magnetic field of magnitude $B = 10$ T can be generated within the flake. In general, however, since the magnitude of the pseudomagnetic field is inversely proportional to the bending radius of the graphene flake, i.e., $B \propto 1/R$, a smaller bending radius, i.e., a bigger bending angle, will result in a higher pseudomagnetic field. It is also worth noticing that introducing a strain also has two other effects, namely, it shifts the position of the Dirac points in $k$ space by a quantity proportional to $\mathbf{A}^{(\theta)}$ and it also renders Fermi velocity anisotropic, according to the relation $v_F \equiv v_{F0} (1 - \mathbf{\hat{u}} \cdot \mathbf{u})$, where $\mathbf{u}$ is the strain tensor. However, for the purpose of this work, we assume that at the leading order in the bending radius $R$, the Fermi velocity is not affected by the deformation and it remains approximately constant.

The dynamics of electrons in graphene in the presence of the AGF $\mathbf{A}^{(\theta)}$ can be studied by replacing the kinetic momentum $\mathbf{p}$ in the low-energy graphene Hamiltonian $\hat{\mathcal{H}} = v_F \mathbf{p} \cdot \mathbf{A}$ (with $\mathbf{A} = \sigma_1 \mathbf{\hat{x}} + \sigma_2 \mathbf{\hat{y}}$ and $\sigma_i$ being Pauli matrices) with the canonical momentum $\mathbf{P} = \mathbf{p} + e \mathbf{A}^{(\theta)}$ derived from minimal coupling of the electron field with the AGF, which results in the following Dirac equation for electrons in the vicinity of the Dirac point:

$$i \hbar \partial_t \phi = -i \hbar v_F \sigma \cdot \mathbf{P} \phi, \quad (3)$$

where $j = (x, y)$. The above equation can be solved analytically as it represents the well-known problem of a relativistic electron in a magnetic field, whose solution can be cast in terms of the Landau eigenfunctions,

$$\phi_n^\pm (x, y, \xi, \mathbf{k}) = N e^{i \left( k_0 x - \frac{\xi}{\hbar} y \right)} \left( \text{sign}(n) \phi_{\mp 1}^0 (\xi), \phi_n^0 (\xi) \right), \quad (4)$$

where $N$ is a normalization constant, which equals one for $n = 0$ and $1/\sqrt{2}$ otherwise; $\phi_n^0 (\xi)$ are harmonic oscillator eigenstates; $\xi = (y + L^2 k / L_c)$ with $L_c = \sqrt{\hbar c / e B}$ being the magnetic length (for a pseudomagnetic field of $B = 10$ T, we get $L_c \approx 8.11$ nm); and $\bar{\mathcal{E}}_n = \text{sign}(n) \hbar \omega_c \sqrt{|n|}$, with $\omega_c = v_F / L_c$ being the cyclotron frequency. The above solution is, moreover, equipped with the constraint $\phi_n^0 (\xi) = 0$. A sketch of the structure of Landau levels in graphene is depicted in Fig. 2. Note that contrary to the case of a non-relativistic electron in magnetic field, where the spacing of Landau levels is constant, i.e., Landau levels have the full structure of a harmonic oscillator, in graphene, they are not equally spaced, as the energy eigenvalue $\bar{\mathcal{E}}_n$ scales as $\sqrt{|n|}$. Moreover, due to the peculiar band structure of graphene in the vicinity of the Dirac point, two sets of Landau levels are created by the magnetic field, corresponding to the Landau levels in the valence band [associated with the negative values of the index $n$ and appearing as red lines in Fig. 2(a)] and in the conduction band [associated with the positive values of the index $n$ and appearing as blue lines in Fig. 2(a)]. The two sets of oscillator states are almost disjoint from each other, with the exception of the ground state $n = 0$ [green dot in Fig. 2(a)], which sits at exactly the Dirac point and it is common to both sets.

The creation of Landau levels in graphene introduces selection rules for the dipole-allowed transitions, i.e., an incoming photon can only excite the transition $|n_i \rangle \rightarrow |n_f \rangle$ if and only if $|n_f| = |n_i| \pm 1$, where the plus sign holds for right-handed circularly polarized photons, while the minus sign holds for left-handed circularly polarized photons only. These selection rules are a consequence of the natural

![FIG. 1. (a) Pictorial representation of a rectangular graphene flake deformed into an arc. The radii of the lower and upper edges are, respectively, $R$ and $R'$. With a graphene flake of width $W = 200$ nm, length $L = 192$ nm, and an inner and outer radii of, respectively, $R = 5L = 960$ nm and $R' = R + W = 1.16 \mu$m, the maximum achievable magnitude of the pseudomagnetic field in the central region of the flake is $B = 10$ T. (b) Flattened equivalent geometry of the bent graphene flake in panel (a). The curvature induced by the bending is replaced with an artificial gauge field, which gives rise to a uniform pseudomagnetic field $\mathbf{B} = B \mathbf{\hat{k}}$ parallel to the $z$ axis. The width $w$ and length $\ell$ of the flattened flake might be extended to infinity, without changing the essential role of the pseudomagnetic field in the interaction dynamics of the flake with an external electromagnetic pulse.](scitation.org/journal/app)

![FIG. 2. (a) Pictorial representation of the band structure of graphene in the vicinity of a Dirac point (solid cones). The presence of a pseudomagnetic field generates Landau levels in both the valence band (red lines) and conduction band (blue lines), as well as at the Dirac point itself (green point). The energies of the Landau levels created in this manner are the same in modulus for the two bands and differ only in sign, i.e., positive energies are associated with the conduction band, while negative energies with the valence band. (b) Schematic representation of the level structure induced by the pseudomagnetic field and the correspondent selection rules for the first three Landau levels in the valence and conduction bands. The green solid arrows correspond to the transition allowed for incoming left-handed circular (LHC) polarization photons, while the red dashed arrows correspond to transitions allowed for incoming right-handed circular (RHC) polarization photons.](scitation.org/journal/app)
Without any loss of generality, we can assume that the impinging case, we can entirely describe the electric field by means of its vec-
work in the Landau gauge, where the scalar potential is zero. In this potential, given by the combination of the true,
accounts for the temporal shape of the pulse so that the total vector
\[ A(x, y, t) = \frac{i e}{\hbar} A^0(y) \hat{A}_\mu \psi(x, y, t, k) \], (5)
Without any loss of generality, we can assume that the impinging electric field is linearly polarized along the x-direction and it is characterized by a central frequency \( \omega_c \). We also assume that the electric field is normally impinging on the graphene flake, i.e., the field is propagating along the \( z \) direction, as defined in Fig. 1. We then work in the Landau gauge, where the scalar potential is zero. In this case, we can entirely describe the electric field by means of its vector potential \( A(t) = -\int dt E(t) \equiv \mathcal{A}(t) \exp[-i\omega_c t] \), where \( \mathcal{A}(t) \) accounts for the temporal shape of the pulse so that the total vector potential, given by the combination of the true, \( A(t) \), and the artificial, \( A^{(0)}(y) \), gauge field, can be written as \( A(y, t) = \mathcal{A}(t) - B y \hat{k} \). In this gauge, the Dirac equation reduces to the following, manifestly covariant, compact form:
\[ \sqrt{i} \hbar \gamma^\mu \left( \partial_\mu + \frac{i e}{\hbar} A_\mu \right) \psi = 0, \quad (6) \]
where \( \mu = (0, 1, 2) \rightarrow (\gamma_0, \gamma_1, \gamma_2), \gamma_0 = \sigma^z, \gamma_1 = i \sigma^y, \) and \( \gamma_2 = -i \sigma^x. \) In general, this equation does not admit an explicit analytical solution with an arbitrary time-dependent vector potential. However, following the procedure described in Ref. 56, if we know the form of the instantaneous eigenstates for the above equation, we can then make an educated guess at the true form of the solution for any time \( t \). To this aim, we first take the Fourier transform with respect to the \( x \)-variable and operate the following phase transformation:
\[ \psi(y, t, k) = \int dk e^{ikx} e^{iG(t)c} \phi(y, t, k), \quad (7) \]
with
\[ G(t) = \frac{ev_F}{\hbar} \int_0^t dt A(\tau), \quad (8) \]
and note that Eq. (6) reduces to
\[ \left[ \frac{1}{v_F} \gamma^0 \partial_t + y^1 \left( -i k - \frac{i e B}{\hbar} y \right) + \partial_y \right] \phi = 0, \quad (9) \]
which is equivalent to Eq. (3). This indicates that the Landau eigenstates in Eq. (4) can be taken as the instantaneous eigenstates for the problem at hand and that the general solution of Eq. (6) can be written as
\[ \psi(y, t, k) = \sum_n e^{i0(t)c} \left[ c_n^+ \phi_n^*(y, k) + c_n^0 \phi_n^0(y, k) \right]. \quad (10) \]
Substituting the above ansatz into Eq. (6) leads to a set of coupled mode equations for the expansion coefficients \( c_n^\pm(t) \), which, for a linearly polarized impinging electromagnetic field, read
\[ \dot{c}_n^+ = i \hbar M^2_n \Omega(t) e^{-i\omega_n t} \left[ f_{m,m-1}^n(t) c_{m-1}^+ + f_{m,m+1}^n(t) c_{m+1}^+ \right] + \sum_{n<0} f_{m,n}^+ \phi_m^0 \sigma_x |c_n^0|, \quad (11a) \]
\[ \dot{c}_n^0 = -i \hbar M^2_n \Omega(t) e^{-i\omega_n t} \left[ f_{m,m-1}^n(t) c_{m-1}^+ + f_{m,m+1}^n(t) c_{m+1}^+ \right] + \sum_{n<0} f_{m,n}^0 \phi_m^0 \sigma_y |c_n^0|, \quad (11b) \]
where \( \Omega(t) = (ev_F/B) \dot{\mathcal{A}}(t) \) is the Rabi frequency, \( f_{m,n}^\pm(t) = e^{i(\omega_n \pm \omega_m) t} \) accounts for the eigenvalue mismatch between the states participating in the temporal evolution of \( c_n^\pm(t) \), \( \omega_m = \omega_c \sqrt{m} \) is the eigenfrequency associated with the state \( \phi_m^0 \) (note that the sign of \( \omega_m \)) has been explicitly taken care of during the calculations that lead to the above equations already, and \( \phi_m^0 \sigma_\mu \phi_m^0 \) is the dipole matrix element of graphene,\(^{10}\) whose explicit expression, in this case, is given by
\[ \phi_m^0 \sigma_\mu \phi_m^0 = \delta_{n|m} \delta_{n|m} - \delta_{n|m,m+1}. \quad (12) \]
These equations are the first result of our work. They, in fact, describe the interaction of an arbitrarily shaped linearly polarized impinging field. Note how the polarization of the field enters only in the dipole matrix element, and therefore, the above equation can be easily generalized for an arbitrary polarization by replacing \( \sigma_\mu \) in the expression of the dipole matrix element with the Pauli matrix (or combination thereof) corresponding to the impinging polarization.
We can simplify the coupled mode equations above by assuming that the impinging electromagnetic field is nearly resonant with one specific transition. Since the polarization of the impinging field is linear, rather than circular, both green and red transitions in Fig. 2(b) will be allowed once the frequency of the incoming field has been chosen to match one of the transitions between the Landau levels. A closer inspection on the structure of the selection rules depicted in Fig. 2(b), moreover, reveals that the only nontrivial dynamics that can be induced in bent graphene corresponds to an impinging field resonant with the transition \([0] \leftrightarrow [\pm 1] \), and a consequent three-level intra-band dynamics, rather than a simple two-level one, either inter-band or intra-band.
If we then now assume that the impinging field is resonant with the \([0] \leftrightarrow [\pm 1] \) transitions, i.e., \( \omega_L = \omega_1 = \omega_c \), Eqs. (11) become
\[ \frac{d}{dt} \begin{pmatrix} c_1^+ \\ c_0 \\ c_1^- \end{pmatrix} = \begin{pmatrix} 0 & \Gamma(t) & 0 \\ \Gamma(t) & 0 & 0 \\ 0 & -\Gamma(t) & 0 \end{pmatrix} \begin{pmatrix} c_1^+ \\ c_0 \\ c_1^- \end{pmatrix}, \quad (13) \]
where \( \Gamma(t) = \Omega(t) \exp(-i\omega_1 t) \). Note that the structure of the coefficient matrix in the equations above admits the existence of a dark
state, corresponding in this case to no population being at the Dirac point at any given time, i.e., \( c^0(t) = 0 \). The consequences of this and its benefits for coherent population control of graphene in the presence of an external magnetic field have been recently investigated in Ref. 52. We solve the above equations with the initial condition \( c^0(0) = 1 \), i.e., with the electron initially in the valence band, and for a vector potential described by a Gaussian pulse of duration \( \tau \) and central frequency \( \omega_0 = \omega_1 \) so that \( \Omega(t) \) becomes

\[
\Omega(t) = \frac{evF}{\hbar} e^{-\omega_0 t} \cos(\omega_0 t),
\]

where \( vF \) is measured in \( \text{V/m} \) and \( t_0 \) is an arbitrary temporal delay. The evolution of the expansion coefficients \( c^\pm(t) \) for an impinging pulse of amplitude \( E_0 = 10^4 \text{ V/m} \) and time duration \( \tau = 10 \text{ fs} \), corresponding to a nearly single-cycle pulse (i.e., \( \omega_1 \tau = 1.74 \)), and \( \tau = 50 \text{ fs} \), corresponding to a pulse with several optical cycles, i.e., \( \omega_1 \tau = 8.7 \), is depicted in Figs. 3(a) and 3(b), respectively. Note how a very short pulse (\( \tau = 10 \text{ fs} \)) induces an almost instantaneous change in the population of the three interested levels, accompanied by very faint Bloch–Siegert oscillations,\(^{57} \) as the interaction is too fast and the system does not have the time to adapt to it. For longer pulses, as in the case of Fig. 3(b), instead, the population dynamics appears to be more complicated, but leads, at equilibrium, to a situation in which all population returns back to the initial state \( |\pm\rangle \). In both cases, however, the dynamics always involves the ground state \( |0\rangle \) at the Dirac point.

Our simulations, moreover, have been conducted at \( T = 0 \text{ K} \), i.e., the distribution of carriers in the valence and conduction bands has not been explicitly taken into account. However, as has been discussed by one of the authors in previous publications,\(^{58,59} \) the impact of temperature on the nonlinear response of graphene is negligible.

**IV. NONLINEAR SIGNAL AND DIRAC CURRENT**

The nonlinear response of graphene can be estimated by evaluating, as a function of frequency, the intensity of the nonlinear radiation emitted by it as a consequence of the interaction with an impinging, time-dependent electric field, i.e.,\(^{60} \)

\[
I(\omega) = |\omega \tilde{J}(\omega)|^2, \tag{15}
\]

where \( I(\omega) \) is the spectrum of the emitted radiation and \( \tilde{J}(\omega) \) is the Fourier transform of the Dirac current \( j(x, y, t) = \bar{\psi}(x, y, t)^\dagger \psi(x, y, t) + \psi(x, y, t)^\dagger\bar{\psi}(x, y, t) \) (where the last equality holds because of our definition of gamma matrices given in Sec. III).

Since we are only interested in the temporal features of the current, we can integrate it with respect to the transverse space to obtain

\[
J(t) = \int d^2 R j(x, y, t) = \int d^2 R (\sigma \cdot R)[\bar{\psi}(x, y, t)]^2. \tag{16}
\]

Note that since the expansion coefficients \( c^\pm(t) \) appearing in the definition of \( \psi(x, y, t) \) given by Eq. (10) only depend on time and that the solution along the x-direction can be expressed in terms of plane waves, we can safely perform the integration only over the Landau eigenstates and exploit their orthogonality relation to compute the spatial integral in Eq. (16). In doing this, moreover, the resulting time-dependent current \( J(t) \) will have the same functional form for both the case of a finite graphene flake undergoing bending and an infinite sheet of graphene subjected to a uniform magnetic field orthogonal to the graphene plane. The only difference between the two cases will then be an overall multiplicative constant, which will account for the actual arrangement of the system. Since this constant does not change the overall functional of \( J(t) \), and neither impacts the overall form of its Fourier transform, we then treat it as a simple normalization factor and scale everything to it.

If we then substitute Eq. (10) into the above expression and limit ourselves to the case of the impinging pulse being resonant with \( |0\rangle \leftrightarrow |\pm\rangle \), the x- and y-components of the integrated current have the following explicit form:

\[
\begin{align*}
J_x(t) &= c^0(t) [c^-(t)^* e^{-i\omega_0 t} - c^+(t)^* e^{i\omega_0 t}] + \text{c.c.}, \tag{17a} \\
J_y(t) &= i c^0(t) [c^-(t) e^{-i\omega_0 t} - c^+(t) e^{i\omega_0 t}] + \text{c.c.}. \tag{17b}
\end{align*}
\]
FIG. 4. Temporal evolution (a) and Fourier transform (b) of the components of the Dirac current $J_x(t)$ (red solid line) and $J_y(t)$ (blue dashed line) for an impinging pulse of duration $\tau = 20$ fs. As can be seen from panel (a), despite the fact that the impinging electric field is polarized along the $x$-direction, a nonzero current is also generated in the $y$-direction. In particular, panel (b) reveals how while $\tilde{J}_y(\omega)$ contains essentially only a peak at $\omega = \omega_1$, $\tilde{J}_x(\omega)$ has a richer structure. The extra peaks in $\tilde{J}_y(\omega)$ are localized around the fundamental frequency $\omega_1$ and are the source of half-integer harmonics seen in Figs. 5 and 6. For these plots, $\omega_L = \omega_1 = 174$ THz, $E_0 = 10^7$ V/m, and $v_F = c/300$ m/s have been used to reproduce results compatible with Fig. 3.

The temporal evolution of the Dirac current, as well as its Fourier transform, is shown in Fig. 4 for the case of an impinging pulse with $\tau = 10$ fs. Note how, despite the fact that the impinging electric field is polarized along the $x$-direction, a nonzero component of the current along the $y$-direction arises. This is a consequence of the broken centrosymmetry, induced by the artificial gauge field. Since the current enters with its whole vectorial character in the definition of the nonlinear signal as given by Eq. (15), this has significant consequences on its spectrum. In fact, for the case of a short pulse as the one used in Fig. 4, the shape of the nonlinear signal is almost entirely determined by $J_x(\omega)$, while for longer pulse widths, the interplay between the current components $J_\mu(t)$ becomes more prominent, giving rise to a richer structure of $I(\omega)$.

To prove this, in Fig. 5, we plot the nonlinear signal defined in Eq. (15) for different values of the impinging pulse width, i.e., $\tau = 10$ fs [panel (a)], $\tau = 50$ fs [panel (b)], and $\tau = 100$ fs [panel (c)]. As can be seen, for very short pulses [panel (a)], a set of equally spaced harmonics appear, with spacing $n\omega_1/2$. This can be explained by noticing that the impinging pulse sees a three-level system with equally spaced levels, as $|\pm 1\rangle$ have the same distance in energy from $|0\rangle$. This allows us to approximate the square-root behavior of the Landau ladder to the more traditional harmonic oscillator behavior, which, in turn, gives the equally spaced peaks in Fig. 5(a), with spacing $\omega_1/2$.

This is the second result of our work. For short enough pulses, the local structure of Landau levels around the Dirac point can be approximated with that of a traditional harmonic oscillator, and thus

FIG. 5. Nonlinear signal, as defined in Eq. (15), as a function of the normalized frequency $\omega/\omega_1$, for different values of the pulse length, i.e., (a) $\tau = 20$ fs (correspondent to a fluence of $F = 2.65$ mJ/m$^2$), (b) $\tau = 50$ fs (correspondent to a fluence of $F = 6.64$ mJ/m$^2$), and (c) $\tau = 100$ fs (correspondent to a fluence of $F = 13.27$ mJ/m$^2$). The red dashed line in each panel represents the position of the fundamental frequency $\omega_L = \omega_1$. Note that the peak that should correspond to $\omega = \omega_1$ is indeed slightly blueshifted due to the different weight of the x- and y-components of the Dirac current, as can be seen in Fig. 4(b). This effect only manifests significantly for short pulses and tends to disappear for longer ones, as can be seen by comparing panels (a) and (c) above. For these plots, $\omega_L = \omega_1 = 174$ THz, $E_0 = 10^7$ V/m, and $v_F = c/300$ m/s have been used to reproduce results compatible with Fig. 3.
the corresponding nonlinear signal contains all the integer and half-integer harmonics of the pulse carrier frequency \( \omega_1 \).

For longer pulse durations, on the other hand, the nonlinear signal shows a richer and complicated spectrum. From Figs. 5(b) and 5(c), in fact, it is possible to see how the spectrum broadens, with respect to the situation depicted in Fig. 5(a), and higher harmonics appear, containing both even and odd contributions. Note, moreover, how in both cases of Figs. 5(b) and 5(c), although the highest intensity is reached for the seventh [panel (b)] and the 11th [panel (c)] harmonic, higher ones still have a considerable intensity. For example, for long pulses [panel (c)], the 20th can be generated with a significant intensity.

If we recall that \( \omega_1 = 174 \text{ THz} \), \( \omega = 20\omega_1 \) would correspond to radiation in the visible region, with a wavelength of approximately 540 nm. Moreover, harmonics as high as the 25th and 26th can also be generated (although with a small intensity). This corresponds to the blue side of the visible spectrum, as \( \lambda_{26h} \approx 433 \text{ nm} \). The nonlinear signal in Fig. 5(c), therefore, spans the whole spectrum between THz and the visible region, and could also be pushed into the near UV. This suggests that artificial gauge fields in graphene could be used as a mean to efficiently convert signals within these spectral regions.

V. EFFECT OF THE PSEUDOMAGNETIC FIELD ON THE NONLINEAR SIGNAL

We now briefly discuss what the effect of the pseudomagnetic field is on the nonlinear signal. To do that, we focus our attention on the case of a \( \tau = 50 \text{ fs} \) pulse and repeat our simulations using different values of the pseudomagnetic field. This, in practice, would correspond to a larger or smaller bending radius (and, consequently, bending angle) for smaller or greater values of the pseudomagnetic field, respectively (see Fig. 1).

We show the results of these simulations in Fig. 6, where the nonlinear signal is plotted against three different values of the pseudomagnetic field, namely, \( B = 2 \text{ T} \) [Fig. 6(a)], \( B = 5 \text{ T} \) [Fig. 6(b)], and \( B = 15 \text{ T} \) [Fig. 6(c)].

As can be seen, for large values of \( B \) [Fig. 6(c)] with respect to those employed in Fig. 5, half-integer harmonics centered around \( \omega/\omega_1 \approx 2–3 \) and \( \omega/\omega_1 \approx 6–8 \) distinctively appear. For \( B = 2 \text{ T} \), on the other hand, a very efficient transfer of energy between the impinging field at \( \omega_c = \omega_1 \) and its 20th harmonics takes place, which means that in the presence of small pseudomagnetic fields (i.e., large bending angles), graphene behaves as a very efficient frequency converter between the impinging field oscillating at \( \omega_c = 174 \text{ THz} \) and its 20th harmonics, which sits well-within the visible region at about \( \lambda = 540 \text{ nm} \).

VI. CONCLUSION AND OUTLOOK

In conclusion, we have investigated the nonlinear response of a bent graphene flake to an impinging, linearly polarized, time-dependent electromagnetic field. We have shown that the AGF induced by bending and the consequent appearance of Landau levels significantly modify the harmonic signal generated by graphene. In particular, we have shown that for the case of an impinging field resonant with the transition \( |0\rangle \leftrightarrow |\pm 1\rangle \), i.e., with the Landau ladder in the vicinity of the Dirac point, the nonlinear signal shows a spectrum containing integer, as well as non-integer harmonics of the pulse carrier frequency. In particular, for ultrashort pulses, the nonlinear signal contains integer harmonics \( \omega = n\omega_1 \) as well as half-integer harmonics \( \omega = (n + 1/2)\omega_1 \), while as the pulse gets longer, a more complicated scenario arises, with a nonlinear signal containing up to the 26th harmonics in its spectrum. Finally, we have shown that the pseudomagnetic field induced by the AGF breaks the central symmetry typical of graphene, thus allowing the emission of even harmonics of the impinging pulse carrier frequency. The magnitude of the pseudomagnetic field can be controlled by changing the radius of curvature (or, equivalently, the bending angle) of the graphene flake. By doing so, we observe, for “small” magnetic fields, the almost resonant transfer of energy between the impinging electromagnetic pulse at \( \omega = \omega_1 \) and its 20th harmonics. For an impinging pulse centered around 174 THz, as in our case, the 20th harmonics lies well-within the visible spectrum, around \( \lambda = 540 \text{ nm} \). This effect,

![Fig. 6](https://example.com/fig6.png)

**Fig. 6.** Nonlinear signal corresponding to an impinging electromagnetic pulse of duration \( \tau = 50 \text{ fs} \) for different values of the pseudomagnetic field, namely, (a) \( B = 2 \text{ T} \), (b) \( B = 5 \text{ T} \), and (c) \( B = 15 \text{ T} \). For low pseudomagnetic fields, we observe an almost resonant transfer of energy between the impinging field and its 20th harmonics [panel (a)], or an almost equal distribution of energy of the impinging pulse across its 10th–15th harmonics [panel (b)]. For high values of the pseudomagnetic field [panel (c)], instead, we observe the appearance of half-integer harmonics centered around \( \omega/\omega_1 \approx 2–3 \) and \( \omega/\omega_1 \approx 6–8 \). For these plots, \( \omega_L = \omega_1 = 174 \text{ THz} \), \( E_0 = 10^7 \text{ V/m} \) (correspondent to a fluence of \( F = 6.64 \text{ mJ/m}^2 \)), and \( v_F = c/300 \text{ m/s} \) have been used to reproduce results compatible with Fig. 5.
controllable by controlling the magnitude of the pseudomagnetic field, could potentially pave the way for novel graphene-based photonic devices, such as THz-to-visible frequency converters and frequency generators.

In this manuscript, we have focused our attention on the effect of AGFs on the nonlinear optical response of graphene to an ultra-short pulse impinging upon it at zero temperature. However, our model can be readily extended to account for a finite temperature using the more standard Dirac–Bloch model, which will give us the possibility to include other competing effects, such as dephasing, relaxation dynamics, and Coulomb interactions.

In this work, we have not explored the role of the valley degree of freedom in the nonlinear response of strained graphene. It is known, however, that introducing such a strain can break the valley symmetry.\cite{6,7} This could then possibly lead to a valley-contrasting nonlinear optical response, which could have a significant impact on valleytronic applications. To account for this, one would need to explicitly take spin–orbit coupling into account such that the combination of AGF and spin–orbit coupling will lift the pseudospin degeneracy in both bulk and edge states, leading to time reversal symmetry breaking\cite{8,9,10} and, ultimately, to a valley-dependent nonlinear response.

The interplay between AGFs and spin–orbit coupling and their effect on the nonlinear response of strained graphene, together with the extension of our analysis to other 2D materials, such as transition metal dichalcogenides (where we can directly probe the effect of Coulomb interactions on their nonlinear response in the presence of AGFs), and the extension of our formalism to 3D Dirac materials, such as topological semimetals,\cite{11,12,13} will be the subject of our next research.

ACKNOWLEDGMENTS

M.O. acknowledges support from the Academy of Finland Flagship Programme, Photonics research and Innovation (PREIN), decision 320165.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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