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The CES and par production techniques, income distribution and the neoclassical theory of production

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PREFACE

This doctoral thesis has been written while I have been employed by several Finnish industrial companies. Therefore it took many years to complete, although it is partly based on my licenciate`s thesis from 1980. I am greatly indebted to the referees who have commented on the manuscript. Any possible remaining errors are, of course, mine.

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I thank the Asko Group for the possibility to finish my work in my free time. The hardware and software needed in the processing of the sampling data was provided by my employer.

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Lauri Tenhunen

THE CES AND PAR PRODUCTION TECHNOLOGIES, INCOME DISTRIBUTION AND THE NEOCLASSICAL THEORY OF PRODUCTION

1. INTRODUCTION

The discussion in neoclassical production theory has very much been associated with the CES production function. After the pathbreaking article by Arrow, Chenery, Minhas and Solow[1] introducing the constant elasticity of substitution (CES) production function, the study of production theory has widely taken the form of theoretical analysis of the role of the elasticity of substitution[2], empirical estimation of the elasticity[3] and straight generalisations of the CES production function into certain classes of variable elasticity of substitution production functions[4]. On the other hand the study of production theory has been directed towards the measurement of technical change[5] and the aggregation of production functions[6].

After "the most general form" of the CES production functions was found and classified by Ruzio Sato in 1977, the discussion around pure CES technology has not been a subject of extremely active theoretical examination. The investigation has concentrated on empirical work concerning CES technology and substitution.

Active work in the area of production theory has under the last 4-5 years concentrated on the empirical examination of the productivity changes in the western economies[7]. The demand for electricity, material inputs and energy has lately been specially analysed with KLEMF production models[8].

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1 Arrow - Chenery - Minhas - Solow (1961)
2 For example Samuelson (1947), pp. 467-480 and Johansen (1972), pp. 67-72 and pp. 218-224
3 For example, Nerlove (1967), pp. 55-122 and Kmenta (1967), pp. 180-189 and Zarembka (1970), pp. 47-53
4 Among others Lu - Fletcher (1968), pp. 449-452 and Lovell (1973), pp. 676-692 and Revankar (1971), pp. 61-72 and Sato - Hoffman (1968), pp. 453-460 and Sato R. (1961), pp. 33-41. An adaptation in regional economics in Finland is made by Hirvonen - Hämäläinen - Haikala (1983)
5 For example, Kennedy - Thirlwall (1973), pp. 116-176 and Jorgenson (1966), pp. 1-17. The latest examinations in Finland are made by Karko (1988) and Summa (1986)
6 Among others Green (1964) and Sato K. (1975)
7 See, for example, Baily (1986), pp.443-451, Berndt - Fuss (1986), pp.7-29, Griliches (1986), pp.141-154, Jorgenson (1986), pp.1841-1915, Jorgenson (1988), pp.23-41 Jorgenson - Gollop - Fraumeni (1987)
8 See, for example, Dargay (1988) and Törmä (1987).

In the theoretical and empirical study of production the flexible functional forms have been increasingly used during the last years. Flexible functional forms can approximate various unknown production technologies. They place relatively few restrictions on the technology. These forms can easily be formulated to have several inputs and they can even be estimated as a linear input demand system.[1]

The traditional neoclassical theory of production, with its assumptions of costless possibilities of substitution and choice of optimal scale, is a suitable tool for the analysis of the long-run development of industrial structure at an aggregate level. In general, it is not as good for the analysis of short-run or medium-term problems of industrial structure within an industry. These can be better analysed with the putty-clay production theory assuming full substitution possibilities ex ante, but fixed factor ratios and capacity ex post optimization, leading to different vintages of capital and a gradual transformation of the structure over time. A good review and analysis of this is given by Försund and Hjalmarsson[2]. The basic ideas are given already by Salter and later developed further by Johansen[3]. Adjustment costs in the theory of industrial production are thoroughly analysed by Söderström[4].

In this work we compare CES technology to a new production technology which will be called par production technology. This will be done by using the neoclassical concepts of production theory. As the par production function is an economical generalisation of the logarithmic mean, the comparisons reveal new features in the CES technology as well. As the logarithmic mean is originally a function of two variables only, this justifies the "par" designation.

The present study has the following chief purposes:

(1) To introduce a new par production function technology.
(2) To apply the resulting ideas to the economic theory of income distribution.
(3) To compare the CES and par production function techniques both theoretically and with sampling experiments.

In chapter 2 some main aspects of neoclassical production theory are reviewed. The par technology is introduced in chapter 3. The most important characters are compared between CES and par technologies especially in chapters 4 and 5.

The Monte Carlo estimations in chapter 6 are based on the estimation methods presented in chapter 5.

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[1] See Fuss, McFadden and Mundlak (1978), pp.219-268, Considine and Mount (1984) pp.434-443
[2] Försund - Hjalmarsson (1987)
[3] Johansen (1972) and Salter (1960)
[4] Söderström (1974)

The CES and par technologies differ most in the following two features. In CES technology the ratio of the income shares is not limited and the shares vary from 0 to 1 or vice versa depending on the ratio of inputs[1]. In par technology the ratio of the income shares is finite and the limits are controlled by the distribution limit parameter(s). This is the most important feature of the par production technology. Both technologies also supply a different system for the optimisation as they demand separate forms for the income share equations. Based on the linearizations made in many situations in the study, the interesting and practical result here is that the CES production technology mainly represents the first order situations in the linearized forms of the par production technology. This condition serves as a base for discriminating the two production technologies from each other in estimation situations.

In chapter 4 we find a very interesting analogy between the CES income share equations and the statistical logistic distribution function. This leads to the dispersion parameter interpretation of the elasticity of substitution and to a special interpretation of the Burr-Hatke equation in the theory of income distribution.

In chapter 5.1 the economic formulation possibilities for the logarithmic mean in the case of several variables are presented.

Although the par production function seems to be rather complicated mathematically, it supplies very interesting economical analysing possibilities especially when we want to test whether there are real distribution limits in the empirical situation in question. In fact the par function form is good for analysing several share equation situations which are very common in economics. Using the parameterized income shares as weights the par production function can be exactly linearized in relation to the distribution limit parameter(s).

As we compare the CES and par production technologies with sampling experiments in chapter 6, the presented methods and linearized forms are tested in the study as well. The sampling experiments give us some practical pieces of information to plan possible empirical estimations when using the par production technology in examination.

The results of the study are summarized in the conclusions in chapter 7.

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[1] In the Cobb-Douglas case the income shares are constants.

2. FEATURES OF THE CES PRODUCTION TECHNOLOGY AND THE NEOCLASSICAL THEORY OF PRODUCTION

2.1 Some basic concepts of the production theory

Usually a variety of restrictions on the production function are utilized in the production theory. To represent a relatively complete catalog of assumptions that are employed in the literature, we shall first note the production function $Y=Y(X)$, where X is a real-valued, n-dimensional vector of nonnegative inputs and Y is the output[1]. These general properties can be stated as follows:[2]

1. (a) Monotonicity. If $X^*\geq X$, then $Y(X^*)\geq Y(X)$.
   (b) Strict monotonicity. If $X^*>X$, then $Y(X^*)>Y(X)$.

2. (a) Quasi-concavity. The input requirement set $V(Y)=\{X:Y(X)\geq Y\}$ is a convex set.
   (b) Concavity. $Y(\theta*X^0+(1-\theta)*X^*)\geq\theta*Y(X^0)+(1-\theta)*Y(X^*)$ for all $0\leq\theta\leq1$.

3. (a) Weak essentiality. $Y(0_n)=0$, where 0_n is the null vector.
   (b) Strict essentiality. $Y(X_1,\ldots,X_{i-i},0,X_{i+1},\ldots,X_n)=0$ for all X_i.

4. The input requirement set $V(Y)$ is closed and nonempty for all $Y>0$.

5. $Y(X)$ is finite, nonnegative, real-valued, and single valued for all nonnegative and finite X.

6. (a) $Y(X)$ is everywhere continuous; and
   (b) $Y(X)$ is everywhere twice-continuously differentiable.

In most studies the production function is assumed to be homogeneous or homothetic.[3]

Properties 1a and 1b imply that additional units of any input can never decrease the level of output, which means that the marginal productivities are positive. This assumption is almost universally maintained in production analyses.

Property 2a is essentially equivalent to assuming that the law of the diminishing marginal rate of substitution holds. Property 2b states a version of the law of the diminishing marginal productivity, which means that as the utilization of a particular input rises, holding all other inputs fixed, the associated marginal increment in output must never increase.

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[1] In general, Y could be interpreted as a vector of nonnegative outputs. That is not, however, necessary here.

[2] See Chambers (1988) pp.9-14. Chambers presents a corresponding analysis in case of the unit cost function as well.

[3] A production function $Y=Y(X_1,\ldots,X_n)$ is said to be homogeneous of degree t if it has the property $\mu^t*Y=Y(\mu*X_1,\ldots,\mu*X_n)$. When $t=1$, $Y(\cdot)$ is linearly homogeneous. A production function is homothetic, if it can be written $Y=Y(J(X_1,\ldots,X_n))$, where $J(\cdot)$ is a homogeneous function and $Y(\cdot)$ a continuous, twice-differentiable, finite, nonnegative and nondecreasing function of the argument.

Property 3a means that production of a strictly positive output without the committal of scarce resources is ruled out. The property 3b implies that all inputs are essential to the production process.

Property 4 is a feasibility assumption, which implies that it is always possible to produce any positive output.

Property 5 is self-explanatory while property 6a is made to rule out discontinuous jumps in the technology. Property 6b is extensively used since it permits the use of differential calculus in the analysis.

Consider a twice differentiable quasi-concave production function

$$Y=Y(X_1,\ldots,X_n) \tag{2.1}$$

where Y is output and X_i`s $(i=1,\ldots,n)$ are the inputs. For given input prices $(P_1,\ldots,P_n)$ cost minimization for a given output level requires that

$$\frac{@\ Y}{@\ X_i} \equiv Y_i = P_i/\Gamma \qquad (i=1,\ldots,n) \tag{2.2}$$

where Γ, the Lagrangian multiplier of the constrained minimization problem, can be interpreted as marginal cost. This minimization problem yields a cost function, which gives for every vector of prices and output level, the minimum possible level of costs $C=C(P_1,\ldots,P_n,Y)$. The first order partial derivatives of C will be

$$\begin{aligned} \frac{@\ C}{@\ P_i} &= C_i(P_1,\ldots,P_n,Y) \qquad (i=1,\ldots,n) \\ \frac{@\ C}{@\ Y} &= \Gamma(P_1,\ldots,P_n,Y) \end{aligned} \tag{2.3}$$

These partial derivatives form a system whose duality relations to the relations in (2.2) and the production function will be further analysed below.

When Y is assumed to be linearly homogeneous we can write the total cost function C to be a product of the output Y and the unit cost function f as follows

$$C = Y*f \tag{2.4}$$

where f is a function of the input prices only

$$(2.5) \qquad f = f(P_1,\ldots,P_n)$$

where again P_i is the price of the i`th input.

Because of the perfect symmetry between production and cost functions, known as the duality theory[1], plus the equality between total revenue and total cost which is the equilibrium condition of a firm under pure competition, the subsequent argument on the relationship between the elasticity of substitution and the behaviour of factor shares can be conducted in terms of both production and cost functions.

Under competitive conditions, the relative share of factor X_i is given by

$$(2.6) \qquad w_i = \frac{Y_i * X_i}{Y} = \frac{f_i * P_i}{f} \qquad (i=1,\ldots,n)$$

$$Y_i = \frac{@\ Y}{@\ X_i} \qquad f_i = \frac{@\ f}{@\ P_i} \qquad \sum_{i=1}^{n} w_i = 1$$

Thus the relative share can be expressed in two alternative ways, which enables us to analyze not only the direct effect on factor shares of a change in the quantity of a factor via the change in marginal product, but also the indirect effect of a change in the price of a factor via the change in quantity demanded on that factor. This is due to the fact that the marginal effect on unit cost of a change in factor price P_i is equal to the quantity demanded of factor i.

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[1] The duality theory has been examined in depth by many writers, for example, Shephard (1953,1970), Ferguson (1979), Fuss and McFadden (ed.) (1978) and Chambers (1988) just to mention a few.

2.2 Changes in specific factor shares

First consider the effect of an increase in the quantity of one specific factor upon the size of the relative share of that factor. This is the problem investigated by Samuelson and the result may be stated as: The relative share of one factor increases or decreases as the quantity of that factor increases depending on whether the "Samuelson" elasticity of substitution is greater or smaller than unity, i.e.

$$(2.7) \qquad \frac{@\ w_i}{@\ X_i} \begin{array}{ll} > 0 & \text{when } \sigma_i > 1 \\ = 0 & \text{when } \sigma_i = 1 \\ < 0 & \text{when } \sigma_i < 1 \end{array}$$

where σ_i is the "Samuelson" elasticity of substitution

$$(2.8) \qquad \sigma_i = \frac{-(1-w_i)*Y_i}{Y_{ii}*X_i}$$

where Y_{ii} is the partial derivative of Y_i with respect to X_i. By the aforementioned duality theorem we can formulate the dual problem to (2.7) which is concerned about the effect of an increase in the price of X_i on w_i. The result is given by

$$(2.9) \qquad \frac{@\ w_i}{@\ P_i} \begin{array}{ll} > 0 & \text{when } \sigma_{di} > 1 \\ = 0 & \text{when } \sigma_{di} = 1 \\ < 0 & \text{when } \sigma_{di} < 1 \end{array}$$

where σ_{di} is the "dual Samuelson" elasticity of substitution of factor X_i

$$(2.10) \qquad \sigma_{di} = \frac{-(1-w_i)*f_i}{f_{ii}*P_i}$$

where f_{ii} is the partial derivative of f_i with respect to P_i. According to (2.9) the relative share of one factor increases or decreases as the price of that factor increases depending on whether the "dual Samuelson" elasticity of substitution is greater or smaller than unity.

Using the concepts of partial elasticity of complementarity and partial elasticity of substitution,
a more general result can be stated as: The relative share of one factor increases or decreases as the quantity (price) of another factor increases depending on whether the partial elasticity of complementarity (substitution) between the two factor in question is greater or smaller than unity

$$(2.11) \quad \frac{@\ w_i}{@\ X_j} > 0 \quad \text{when} \quad b_{ij} > 1$$
$$= 0 \quad \text{when} \quad b_{ij} = 1$$
$$< 0 \quad \text{when} \quad b_{ij} < 1$$

where b_{ij} is the partial elasticity of complementarity

$$(2.12) \quad b_{ij} = \frac{Y*Y_{ij}}{Y_i*Y_j} \qquad (i><j)$$

Y_{ij} notes the partial derivative of Y_i with respect to X_j. As the dual result we have

$$(2.13) \quad \frac{@\ w_i}{@\ P_j} > 0 \quad \text{when} \quad \sigma_{ij} > 1$$
$$= 0 \quad \text{when} \quad \sigma_{ij} = 1$$
$$< 0 \quad \text{when} \quad \sigma_{ij} < 1$$

where σ_{ij} is the partial elasticity of substitution between X_i and X_j

$$(2.14) \quad \sigma_{ij} = \frac{f*f_{ij}}{f_i*f_j} \qquad (i><j)$$

where f_{ij} notes the partial derivative of f_i with respect to P_j. The partial elasticity of complementarity is found by Sato and Koizumi (1971) in view of the suggestions due to Hicks (1970). It is the dual concept to the well-known definition of the Allen partial elasticity of substitution, see Allen (1938).

Two factors X_i and X_j are said to be q-complements or q-substitutes according to whether b_{ij} is positive or negative, and p-complements or p-substitutes according to whether σ_{ij} is negative or positive[1]. Noting that both b_{ij} and σ_{ij} can be either positive or negative, we can say that it is the degree of substitutability which is crucial in determining the behavior of factor shares.

Here we have to note that the Samuelson elasticities and the partial elasticities are related in the manner described in chapter 2.4 of this study.

---

[1] See Hicks (1970), p.289-96

2.3 Changes in the ratio of relative factor shares

The effect of a change of the ratio of inputs on the ratio of relative factor shares can be stated with the help of direct and shadow elasticity of substitution as follows

$$
(2.15)\qquad
\begin{aligned}
\frac{@\ (w_i/w_j)}{@\ (X_i/X_j)} &> 0 \quad \text{when} \quad d_{ij} > 1 \\
&= 0 \quad \text{when} \quad d_{ij} = 1 \\
&< 0 \quad \text{when} \quad d_{ij} < 1 \\
\frac{@\ (w_i/w_j)}{@\ (P_i/P_j)} &> 0 \quad \text{when} \quad s_{ij} > 1 \\
&= 0 \quad \text{when} \quad s_{ij} = 1 \\
&< 0 \quad \text{when} \quad s_{ij} < 1
\end{aligned}
$$

where the direct elasticity of substitution d_{ij} and the shadow elasticity of substitution s_{ij} are defined as

$$
(2.16)\qquad
\begin{aligned}
d_{ij} &= \frac{@\ \ln(w_i/w_j)}{@\ \ln(Y_j/Y_i)} \qquad (i><j) \\
s_{ij} &= \frac{@\ \ln(P_i/P_j)}{@\ \ln(f_j/f_i)}
\end{aligned}
$$

The direct elasticity concept is a straightforward extension of the original Joan Robinson definition (1934) made by Sato and Koizumi[1]. The shadow elasticity derives from McFadden[2].

Sato and Koizumi have defined elasticities called group and composite substitution elasticities in relation to the distributive shares[3]. The serviceability of these seems not to be specially relevant.

Recently Blackorby and Russell have criticized the use of the Allen elasticities in production analyses and in the analyses of distributive shares. According to their result the so called Morishima elasticity of substitution, defined by[4]

$$
(2.17)\qquad m_{ij} = \frac{P_i * f_{ij}}{f_j} - \frac{P_i * f_{ii}}{f_i}
$$

is the only relevant and convenient concept at least in CES type of production analyses.

---

[1] Sato and Koizumi, pp.484-489
[2] McFadden (1963), pp.78-83
[3] Sato and Koizumi, pp.484-489
[4] Blackorby and Russell (1989), pp.882-888. The Morishima elasticity is first described in Morishima (1967), pp.144-150. See also Blackorby and Russell (1981), pp.147-158.

2.4 Elasticities of substitution and complementarity: The general case

The duality relation between the Joan Robinson`s elasticity of substitution[1] and John Hicks` elasticity of complementarity[2] was formalized for the n factor case by Sato and Koizumi[3] in 1973. This chapter is based on the analysis of Moshe Syrquin and Gideon Hollender who generalized the Sato and Koizumi analysis to the case of non-homothetic technology[4].

With only two inputs and a linearly homogeneous production function, the two mentioned definitions of substitution and complementarity elasticities are equivalent. In non-homothetic technology there exists a type of duality and an interesting decomposition into substitution and scale effects appears.

The (Allen) partial elasticity of substitution can be expressed in terms of all elasticities of complementarity as follows

$$(2.18) \qquad \sigma_{ij} = \frac{\epsilon}{\theta_i * \theta_j} * \frac{B_{ij}}{|B|}$$

and the dual concept, Hicks` partial elasticity of complementarity can be expressed in terms of all partial elasticities of substitution and an additional term which measures the scale or output effect on marginal cost

$$(2.19) \qquad b_{ij} = \frac{\epsilon}{\theta_i * \theta_j} * \frac{Z_{ij}}{|Z|} - \frac{@ \ln \Gamma}{@ \ln Y}$$

in which $\theta_i = Y_i * X_i / Y$ is the output elasticity of X_i and $\epsilon = \Sigma\theta_i = C/\Gamma * Y$ is the scale elasticity. In case of constant returns to scale we have $\epsilon = 1$.

B_{ij} is the cofactor of matrix B and Z_{ij} is the cofactor of matrix Z defined below:

---

1 J.Robinson (1934)
2 J.R.Hicks (1964,1st ed.1932) and Hicks (1970), pp.289-96
3 Sato R. and T.Koizumi, pp.44-56
4 M.Syrquin and G.Hollender, pp.515-519

$$B = \begin{bmatrix} b_{11} & \cdots & b_{1n} & 1 \\ \vdots & & \vdots & \vdots \\ b_{n1} & \cdots & b_{nn} & 1 \\ 1 & \cdots & 1 & 0 \end{bmatrix}$$

(2.20)

$$Z = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1n} & \delta_1 \\ \vdots & & \vdots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} & \delta_n \\ \delta_1 & \cdots & \delta_n & 0 \end{bmatrix}$$

where the elasticities b_{ij} and σ_{ij} are given by the equations (2.12) and (2.14). The elasticities of demand for the input X_i with respect to output along the expansion path are noted with

$$(2.21) \qquad \delta_i = \left. \frac{@ \ln X_i}{@ \ln Y} \right|_{\bar{P}}$$

where the line above P means that input prices are kept constant. For a homogenous of degree one production function $\delta_i=1$ for all i.

When constant returns to scale is postulated, the last term in (2.19) vanishes and perfect duality emerges. This is the situation which is analysed by Hicks[1] as well as Sato and Koizumi[2].

In case of two inputs the relation between σ_{12} and b_{12} can be presented as follows. Since the cost function is homogenous of degree one in prices, we have

$$(2.22) \qquad \sigma_{11}*\theta_1 + \sigma_{12}*\theta_2 = 0$$

---

[1] Hicks (1970), pp.289-96
[2] R.Sato and T.Koizumi, pp.44-56

The weighted average of the scale elasticities δ_i equals one

$$(2.23) \qquad \delta_1 * \theta_1 + \delta_2 * \theta_2 = 1$$

Substituting into (2.19) for the case n=2 we can obtain

$$(2.24) \qquad b_{12} = b_{21} = b = \frac{\epsilon * \delta_1 * \delta_2}{\sigma} - \frac{@ \ln \Gamma}{@ \ln Y}$$

If the production function is homothetic, we have $\delta_1 = \delta_2 = 1/\epsilon$ and therefore

$$(2.25) \qquad b = \frac{1}{\sigma * \epsilon} - \frac{@ \ln \Gamma}{@ \ln Y}$$

If we further add the assumption of homogeneity then

$$(2.26) \qquad \frac{@ \ln \Gamma}{@ \ln Y} = \frac{1 - \epsilon}{\epsilon}$$

and

$$(2.27) \qquad (b - 1) * \epsilon = (\frac{1}{\sigma} - 1)$$

where b is the elasticity of complementarity in two factor case and σ is the elasticity of substitution in two factor case. Finally, when we have constant returns to scale, then $\epsilon=1$ and

$$(2.28) \qquad b = 1/\sigma$$

which is the original situation analysed by John Hicks and Joan Robinson in the early thirties.

2.5 On the classification of general CES functions in the two factor case

The general family of CES production functions contains a large number of different types, actually an infinite number of functions, and the CES production functions are in general not expressible in explicit forms. Thus those who are accustomed to thinking of the production function as an explicit relationship between inputs and outputs, may consider the general family of non-homothetic CES functions very strange. However, usually this implicitness aspect presents no serious problems neither in theory nor in estimation, for the concept of the production function is simply the relationship, explicit or implicit, between the inputs and the maximum level of output resulting from them.

In the two factor case the elasticity of substitution[1] between inputs K (capital) and L (labor) is defined according to equation (2.18) as follows

$$(2.29) \qquad \sigma = \frac{K*Y_K + L*Y_L}{K*L*(2*Y_{KL} - \frac{Y_K*Y_{LL}}{Y_L} - \frac{Y_L*Y_{KK}}{Y_K})}$$

$$= \frac{K + L*m}{K*L*(\frac{@\ m}{@\ K} - \frac{@\ m}{@\ L} * m^{-1})}$$

where $m = Y_L/Y_K$ is the marginal rate of substitution between the inputs K and L. Rewriting (2.29) as

$$(2.30) \qquad \sigma*K*L*(\frac{@\ m}{@\ K}) - \sigma*K*L*m^{-1}*(\frac{@\ m}{@\ L}) - K - L*m = 0$$

and letting $m = e^u$, equation (2.31) may be expressed as

$$(2.31) \qquad e^u*\frac{@\ u}{@\ K} - \frac{@\ u}{@\ L} = \frac{1}{\sigma*L} + \frac{e^u}{\sigma*K}$$

where $\sigma = \sigma(K,L,Y)$. If the elasticity of substitution σ is constant, equation (2.31) reduces to a rather simple partial differential equation whose solution is

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[1] In case of two inputs the constancy of the elasticities described in chapters 2.2, 2.3 and 2.4 imply equivalent production structures.

$$(2.32) \qquad m = (K/L)^{1/\sigma} * D(Y)$$

where D(Y) is an implicit function of production level Y and D(Y)>0. This is the general expression for the marginal rate of substitution corresponding to the general class of CES production functions[1].

To obtain the general CES family from (2.32) we need to solve (2.32) by setting $m = Y_L/Y_K$ and then

$$(2.33) \qquad K^{1/\sigma} * D(Y) * \frac{@\,Y}{@\,K} - L^{1/\sigma} * \frac{@\,Y}{@\,L} = 0$$

where σ is constant and $\infty > \sigma > 0$. The general solution to the partial differential equation in (2.33) is found to be

$$(2.34) \qquad \begin{array}{lll} D_1(Y) * K^{-p} + D_2(Y) * L^{-p} = 1 & (p = \frac{1}{\sigma} - 1) & (\sigma >< 1) \\ D_1(Y) * \ln K + D_2(Y) * \ln L = 1 & (p = 0) & (\sigma = 1) \end{array}$$

Thus the general class of nonhomothetic CES functions is, in general, the implicit relationship between K, L and Y defined by (2.34)[2]. The latter equation in (2.34) presents the nonhomothetic Cobb-Douglas production function.

Equation (2.34) shows the most general class of CES production functions, which can be presented only implicitly. Unless it can be determined that $D_1(Y)$ and $D_2(Y)$ are related in some particular manner, Y cannot be explicitly expressed as a function of K and L.

An obvious classification manner based on equations (2.34) is to divide general CES functions on one hand into homothetic CES functions and on the other hand into non-homothetic CES functions. When $D_1(Y) = h * D_2(Y)$ where h is a constant, we have the homothetic, the ordinary, family of CES functions. All other relationships between $D_1(Y)$ and $D_2(Y)$ define the non-homothetic family of CES functions, with the exception, of course, that $D_1(Y)$ and $D_2(Y)$ must be chosen in such a way that Y satisfies the usual properties of a production function.

In fact there are many possible classification methods depending upon the specific purposes in mind. It has been shown by R.Sato that separable types of non-homothetic CES functions can always be written as[3]

$$(2.35) \qquad Y = Y\left(\frac{\beta_1 * K^{-p} + \beta_3}{\beta_2 * L^{-p} + \beta_4}\right)$$

---

[1] Sato (1975) and Sato (1977 I), p.2
[2] Sato (1977 I), p.3
[3] Sato (1977 II), p.562

where β_i`s (i=1,2,3,4) are constants. R. Sato has analysed the non-homothetic type of CES functions further with the Lie theory of transformation groups[1]. He has named the following subgroups to the separable type of CES family[2]:

$$
\begin{aligned}
&\text{Homothetic type:} && Y = Y(\beta_1 * L^{-p} + \beta_2 * K^{-p}) \\
(2.36)\quad &\text{Capital-homothetic type:} && Y = Y\left(\frac{\beta_1 * L^{-p} + \beta_3}{\beta_2 * K^{-p}}\right) \\
&\text{Labor-homothetic type:} && Y = Y\left(\frac{\beta_2 * K^{-p} + \beta_4}{\beta_1 * L^{-p}}\right)
\end{aligned}
$$

where K refers to capital input and L to labor input.

The implicit way of defining the general CES production functions does not present any insurmountable difficulty from the point of view of both theoretical and empirical production analysis. For example the relationship in (2.34) may always be looked at as an explicit or implicit formulation of the capital (or labor) requirement function:

$$
(2.37)\quad
\begin{aligned}
K^{-p} &= \frac{1}{D_1(Y)} - \frac{D_2(Y)}{D_1(Y)} * L^{-p} = R_1(Y,L) \\
L^{-p} &= \frac{1}{D_2(Y)} - \frac{D_1(Y)}{D_2(Y)} * K^{-p} = R_2(Y,K)
\end{aligned}
$$

Equations (2.37) define explicitly the amount of capital (or labor) required to produce a given level of output in cooperation with a given amount of labor (or capital).

It has been proved that it is not possible to obtain a functional form for a production function which has an arbitrary set of constant elasticities of substitution if the number of factors is greater than two. This result is contained in the impossibility theorems of Uzawa[3] and McFadden[4]. No doubt CES production functions (more generally VES production functions as well) are still useful production functions in economic theory and practice.[5]

---

1 Lie M.S., Transformationsgruppen, Vol. I,II,III
2 Sato R. (1977 I), p.13
3 Uzawa (1962)
4 McFadden (1962,1963)
5 It is worth noting that the flexible functional forms are increasingly used in production analysis. See footnotes in chapter 1.

2.6 On the theory of income distribution

Although, in the case of non-homothetic production functions, the sum of factor income shares will not necessarily add up to the total output value, the basic postulates of marginal productivity theory are met if (1) the first and second order conditions for profit maximization are fulfilled and (2) the entrepreneur`s maximum profit equals zero. If these points are assumed, then each input is paid the value of its marginal product and the total output value is just exhausted. Since these conditions are satisfied by homothetic production functions, it was mistakenly earlier assumed that all production functions must be of this type.

In the case of general CES functions the behaviour of each factor s income share is not directly related to the factor ratio nor to substitution elasticity. However, if the movements of income distributions are expressed in the form of an income shares ratio, then the factor ratio and the substitution elasticity play important determining roles. For the non-homothetic CES case, the ratio of labor`s income to capital`s income from (2.32) is

$$\text{(2.38)} \qquad \frac{w_L}{w_K} = \frac{(@\ Y/@\ L)*L}{(@\ Y/@\ K)*K} = k^{1/\sigma-1}*D(Y)$$

where w_K and w_L are the income shares of capital and labor, respectively, under competition and k=K/L. Differentiating (2.38) partially with respect to k and Y, we obtain

$$\text{(2.39)} \qquad \begin{aligned} \frac{@\ (w_L/w_K)}{@\ k} &= (\frac{1}{\sigma} - 1)*k^{1/\sigma-2}*D(Y) \\ \frac{@\ (w_L/w_K)}{@\ Y} &= k^{1/\sigma-1}*D`(Y) \end{aligned}$$

From (2.39) we can see that as long as the substitution elasticity is greater (less) than unity, capital`s income relative to labor`s income rises (falls) when the capital-labor ratio increases and that as long as the non-homotheticity coefficient D`(Y) is positive (negative), labor`s income relative to capital`s income rises (falls) whenever output increases[1].

Although the elasticity of substitution is unitary in the nonhomothetic Cobb-Douglas situation, the relative income distribution varies depending upon the non-homotheticity coefficient.

---

[1] The former part of the conclusion can be found already in J.Hicks (1964, 1st ed. 1932) as well as Samuelson (1968), p.468. The latter part of the analysis is based on the investigation by R.Sato (1977,II), p.562.

When we calculate the limiting values of (2.38) we get

$$
(2.40)\qquad
\begin{aligned}
\lim_{k\to\infty} (w_K/w_L) &= \infty && \text{when } \sigma > 1\\
&= 0 && \text{when } \sigma < 1\\
\lim_{k\to 0} (w_K/w_L) &= 0 && \text{when } \sigma > 1\\
&= \infty && \text{when } \sigma < 1
\end{aligned}
$$

These limiting values suggest that the CES production technology cannot be fully representative with very high or very low values of the input ratio k. In many cases it is enough to find the tendency of changes in income share ratio with respect to the changes in input ratio k. However, when analysing relative shares in more general situations and when forming complete systems for relative shares we could use technologies where the corresponding limiting values are positive constants. In situations where such more practical limiting values are needed, the theory should be based on other postulates than the CES production technology.

It is worth noting that the nature of the technological progress has itself a direct impact on the relative factor shares. Only when the technical progress is neutral, either Hicks-neutral, Harrod-neutral or Solow-neutral, it has no effect on income distribution ceteris paribus[1]. Both capital-saving and labor-saving technical change effect the factor shares depending upon the value of the substitution elasticity in question[2].

Economical situations where it is in fact realistic to assume that the shares of various factors are limited to some extent are, for example

1. Shares of the various factors in the cost of production[3].
2. Share of income or total expenditure allocated to the various consumer or investment goods[4].
3. Share of full income devoted to leisure[5].
4. Market shares of various firms or products in an industry[6].

In the technical production analysis it is a common practice to assume that all inputs are needed in production. That means the production function is assumed to fulfill the property of strict essentiality. See chapter 2.1, the property 3b. If that assumption is made, then it is more realistic to assume finite limiting values of the income share ratio to exist than to assume it to be infinite or zero on the other side.

---

[1] Heathfield, pp.64-67
[2] Ferguson, pp.235-250 and pp.336-350
[3] See, for example, Berndt and Christensen (1973,1974) and Fuss (1977)
[4] See, for example, Christensen and Manser (1977) and Christensen, Jorgenson and Lau (1975) and Berndt, Darrough and Diewert (1977)
[5] For example, Wales and Woodland (1976,1977)
[6] See, for example, Rao (1972) and Weiss (1968)

To refer to an analogical situation in the consumer theory, it would be very realistic to assume that the preferences are such that only a limited part of the total expenditure will ever be devoted to leisure no matter what the non-zero prices are. On the other hand, the total expenditure share of non-leisure time will never achieve unity at least not on the macro level.

Of course, the market shares of products and firms in an industry depend on many endogenous and exogenous factors which include the marketing and firm strategies in the prevailing competitive situation as well as technical progress, financial situation in the relevant economic area, changes of preferences, etc. The distribution channels for certain homogenous groups of products vary considerably by country. These are determined by historical facts and change very slowly. In some countries a certain line of business is done by thousands of small entrepreneurs, in others the same business is transacted by large firms with a large market share and high level of vertical integration with other stages in the distribution.
I think it is very natural to assume that the limiting values of the ratios of market shares are in general finite and controlled by things which are partly exogenous to the firms in the line.

These are arguments for a technology which implies finite limiting values to the ratios of income shares.[1] This technology is presented in chapter 3.

---

[1] The idea of restrictive inputs shares has been used to develop classes of "restricted share production functions", see Ferguson and Pfouts (1962) pp.328-337, Newman and Read (1961) pp.127-133 and Tsang (1973) pp.456-463. Restricted marginal product production function has been examined by Sato, Koizumi and Wolkowitz (1975), pp.331-342. The restricted share production functions are based on a postulate of invariance of the factor share of an input with respect to the same input or to the other input. Tsang has assumed that labor`s share is a linear function of the capital-labor ratio. These formulations imply generalized Cobb-Douglas production function forms, which can be used to test whether relative factor scarcity affects the income distribution significantly.

3. THE PAR PRODUCTION TECHNOLOGY AND THE NEOCLASSICAL THEORY OF PRODUCTION

3.1 The par production function defined

Think of a two factor mean function having the form

$$(3.1) \qquad Y_{-1} = \frac{K - L}{\ln(K/L)} \qquad (K><L) \quad (K>0, L>0)$$

$$= K \qquad (K=L)$$

It is obvious that this is the logarithmic mean function which has been used, for example, in statistical index number theory (See e.g. Vartia 1976).
In spite of its many useful properties, the logarithmic mean has not been an active tool in economic analysis, as it is a simple mean function for two variables only. Here we are going to broaden the view by analysing (3.1) as a production function, analogically to the CES production structure, to get a more general form for economic analysis and neoclassical production theory.

Theoretically in (3.1) we have

$$(3.1B) \qquad \text{Mean} = \frac{\text{Absolute difference}}{\text{Relative difference}}$$

The logarithmic mean for two variables is defined by calculating relative difference as a difference of natural logarithms. This is a different kind of mean function from many other mean functions, because it cannot be derived from the quite general power mean function form[1]. There is not an acceptable generalisation of the logarithmic mean for more than two variables, neither.

The relative difference can, of course, be defined in many different ways. This has been thoroughly analysed by Yrjö Vartia (Vartia pp.9-25). However, to define a more general form for relative difference, we have to define a function, which is continuous and one valued over the chosen limits of a relevant parameter.

---

1 The power mean function includes, for example, geometric mean and harmonic mean. (Compare with Vartia p.11). In fact, the constant elasticity of substitution (CES) production function is based on the power mean function. (See Arrow-Chenery-Minhas-Solow pp.229-231).

Let the definition for a general relative difference be[1]

$$(3.2) \qquad H(K,L) = \int_{1}^{(K/L)} t^{c-1}dt \qquad \text{when } K>L$$

$$= - \int_{(K/L)}^{1} t^{c-1}dt \qquad \text{when } K<L$$

which leads to

$$(3.3) \qquad H(K,L) = \frac{1}{c}\,[(K/L)^{c}-1] \qquad (c><0)$$

$$= \ln(K/L) \qquad (c=0) \quad (K>0, L>0)$$

It is convenient to limit the values of the parameter so that $-1 \leq c \leq 1$, which is based on the following.

When this general relative difference is used in (3.1) instead of its special case we get additionally the following special cases

$$(3.4) \qquad Y_{-1} = K \qquad (c=-1)$$

$$= L \qquad (c=1).$$

Thus, depending upon the value of parameter c, the mean function can get any value between its maxima and minima. However, there is no certain constant value of c which in every case would lead to the same mean value as, for example, the harmonic mean does.[2]

To make the mean function more general we will make a simple monotonic transformation to each of the variables in (3.1). Thus Y,K and L shall be replaced by Y^{-a},K^{-a} and L^{-a}. This kind of monotonic transformation is used in constant elasticity of substitution production functions as well.

---

1 This is a generalization of the relative change used for example by Vartia 1976 (pp.11-13 and 124).

2 In any special case, when K and L are fixed ex ante, there is a given c which leads to a mean value corresponding to any of the relevant and known mean function values in the two factor case.

These transformations lead to the definition of the par production function, which in the two factor case is a generalization of the logarithmic mean. This function will include, as a special case, the Cobb-Douglas production function and as a limiting case the Leontief production function as well. The function is a stereo-type because it is non-continuous with zero values of parameters.

Let Y be the production output, K be capital input and L labor input in the production process. The efficiency parameter can be made equal to one by appropriate choice of output units.

The par production function is

$$(3.5) \qquad Y_a = \left\{ c*\frac{K^{-a}-L^{-a}}{(K/L)^{-a*c}-1} \right\}^{(-1/a)} \qquad (a><0) \quad (c><0) \quad (K><L)$$

$$= \left\{ \frac{K^{-a}-L^{-a}}{-a*\ln(K/L)} \right\}^{(-1/a)} \qquad (a><0) \quad (c=0) \quad (K><L)$$

$$= K^{(\frac{1-c}{2})} L^{(\frac{1+c}{2})} \qquad (a=0) \quad (-1\leq c\leq 1) \quad (K><L)$$

$$= K \qquad (K=L)$$

If a=-1, then

$$Y_{-1} = c*\frac{K-L}{(K/L)^{c}-1} \qquad (a=-1) \quad (c><0) \quad (K><L)$$

$$= \frac{K-L}{\ln(K/L)} \qquad (a=-1) \quad (c=0) \quad (K><L)$$

The Cobb-Douglas production function is a special case of the par function when a=0. This is proved in appendix A. In the above form the par function is homogenous of degree one and it is convex on the relevant area of variables. The convexity of the par production function is examined in appendix H. The limiting case, when $a \to \infty$, is the Leontief production technology. In the following we will drop the subindex from Y_a to simplify the notations.

Parameter c can be called the distribution limit parameter. Parameter a has a lot to do with the substitution and complementarity of the inputs. Anyhow, it is not the elasticity of substitution. To differentiate a from the elasticity of substitution it can here be called the substitution parameter.

3.2 Optimizing conditions

The first order condition for the profit maximum (under the common neoclassical assumptions of costless substitution and optimal choice of scale) when the production function is of the par type, is

$$(3.6) \qquad \frac{R}{W} = \frac{L}{K} * \frac{K^{-a} - (K/L)^{-a*c}*Y^{-a}}{(K/L)^{-a*c}*Y^{-a} - L^{-a}} \qquad (a><0)$$

$$= \frac{L}{K} * \frac{1-c}{1+c} \qquad (a=0) \quad (c>-1)$$

where R is the price for one capital input unit and W is the price for one labor input unit. In general we can conclude that the nonlinearities are here so difficult that in general an explicit and one-valued solution for input demand functions is not possible[1].

Forming the partial input elasticities we get

$$(3.7) \qquad \Phi \equiv \frac{@\ Y}{@\ K} * \frac{K}{Y} = \frac{K^{-a} - (K/L)^{-a*c}*Y^{-a}}{K^{-a} - L^{-a}} \qquad (a><0)$$

$$= (1-c)/2 \qquad (a=0)$$

$$(3.8) \qquad \phi \equiv \frac{@\ Y}{@\ L} * \frac{L}{Y} = \frac{(K/L)^{-a*c}*Y^{-a} - L^{-a}}{K^{-a} - L^{-a}} \qquad (a><0)$$

$$= (1+c)/2 \qquad (a=0)$$

---

[1] However, the input ratio can be solved approximately as a function of the input prices as follows

$$\ln(K/L) \approx \left(\frac{3}{3+a}\right)*\left[\ln\left(\frac{1-c}{1+c}\right)-\ln\left(\frac{R}{W}\right)\right] \qquad (a>-3)$$

More about input demand, see appendix I.

Solving $(K/L)^{-a*c}*Y^{-a}$ from (3.5) we can write the input elasticities as follows

$$(3.9) \qquad \Phi = \frac{K^{-a} - Y^{-a}}{K^{-a} - L^{-a}} - c \qquad (a><0)$$

$$= (1-c)/2 \qquad (a=0)$$

$$\phi = \frac{Y^{-a} - L^{-a}}{K^{-a} - L^{-a}} + c \qquad (a><0)$$

$$= (1+c)/2 \qquad (a=0)$$

Using (3.9) we can write the par production function in a simple form

$$(3.10) \qquad Y^{-a} = (1-\Phi-c)*K^{-a} + (1-\phi+c)*L^{-a} \qquad (a><0)$$

$$\ln(Y) = \left(\frac{1-c}{2}\right)*\ln(K) + \left(\frac{1+c}{2}\right)*\ln(L) \qquad (a=0)$$

where Φ and ϕ are the input elasticities of capital and labor. When a=0, (3.10) is the trivial Cobb-Douglas production function. In the optimum, when a><0, (3.10) gives an interesting form for further analysis

$$(3.11) \qquad Y^{-a} = (1-w_K-c)*K^{-a} + (1-w_L+c)*L^{-a} \qquad (a><0)$$

where w_K and w_L are the income shares of capital and labor. Equation (3.11) can be called the implicit form of the par production function under optimization.

3.3 The interpretation of the parameters

Using the definition (3.5) we can write the input elasticities of the par function as follows

$$(3.12) \qquad \Phi = \frac{K^{-a} - Y^{-a}}{K^{-a} - L^{-a}} - c$$

$$\phi = \frac{Y^{-a} - L^{-a}}{K^{-a} - L^{-a}} + c \qquad (K><L)$$

From (3.12) we can see, that the parameter c is a distribution limit parameter, because, as is proved in appendix B, the limiting values of the income elasticities are closely related to the value of parameter c as shown below. We will make a notation $k = K/L$ for the input ratio. Using that we get the following results for the distribution limits, when a><0:

$$
\begin{aligned}
\lim_{k\to\infty} (\Phi) &= 0 && (c>0)\ (a>0)\\
&= 1-c && (c>0)\ (a<0)\\
&= -c && (c<0)\ (a>0)\\
&= 1 && (c<0)\ (a<0)\\
\lim_{k\to 0} (\Phi) &= 1-c && (c>0)\ (a>0)\\
&= 0 && (c>0)\ (a<0)\\
&= 1 && (c<0)\ (a>0)\\
&= -c && (c<0)\ (a<0)
\end{aligned}
$$

(3.13)

$$
\begin{aligned}
\lim_{k\to\infty} (\phi) &= 1 && (c>0)\ (a>0)\\
&= c && (c>0)\ (a<0)\\
&= 1+c && (c<0)\ (a>0)\\
&= 0 && (c<0)\ (a<0)\\
\lim_{k\to 0} (\phi) &= c && (c>0)\ (a>0)\\
&= 1 && (c>0)\ (a<0)\\
&= 0 && (c<0)\ (a>0)\\
&= 1+c && (c<0)\ (a<0)
\end{aligned}
$$

Thus the input elasticity and the income share of K will vary between 1 and -c when k varies from 0 to ∞, in case c<0 and a>0, for example.

The parameter c can be called the distribution limit parameter.

In appendix C there are some curves showing the income share of inputs as a function of ln(K/L). There the distribution limits can easily be observed.

The limiting values of the ratio of the input shares are

$$
(3.14)\quad
\begin{aligned}
\lim_{k \to \infty} (w_K/w_L) &= 0 && \text{when } a > 0 \text{ and } c > 0 \\
&= \frac{1-c}{c} && \text{when } a < 0 \text{ and } c > 0 \\
&= \frac{-c}{1+c} && \text{when } a > 0 \text{ and } c < 0 \\
&= \infty && \text{when } a < 0 \text{ and } c < 0 \\
\lim_{k \to 0} (w_K/w_L) &= \frac{1-c}{c} && \text{when } a > 0 \text{ and } c > 0 \\
&= 0 && \text{when } a < 0 \text{ and } c > 0 \\
&= \infty && \text{when } a > 0 \text{ and } c < 0 \\
&= \frac{-c}{1+c} && \text{when } a < 0 \text{ and } c < 0
\end{aligned}
$$

The par function form in (3.5) is the base for interpreting the parameter a. Totally differentiating and letting dlnY be equal to 0 on an isoquant, we get

$$
(3.15)\quad
\begin{aligned}
\text{dlnY} &= [K^{-a}-(K/L)^{-a*c}*A]*\text{dlnK} + [(K/L)^{-a*c}*A-L^{-a}]*\text{dlnL} && (a><0) \\
&= \left(\frac{1-c}{2}\right)*\text{dlnK} + \left(\frac{1+c}{2}\right)*\text{dlnL} && (a=0)
\end{aligned}
$$

And thus the ratio of relative input changes on an isoquant will be

$$
(3.16)\quad
\begin{aligned}
\frac{\text{dlnK}}{\text{dlnL}} &= \frac{L^{-a} - A*(K/L)^{-a*c}}{K^{-a} - A*(K/L)^{-a*c}} && (a><0) \\
&= \frac{1+c}{c-1} && (a=0)
\end{aligned}
$$

where A is the constant level of Y^{-a} on an isoquant. Specifically, when c=0, we get

$$
(3.17)\quad
\begin{aligned}
\frac{\text{dlnK}}{\text{dlnL}} &= \frac{L^{-a} - A}{K^{-a} - A} && (a><0) \quad (c=0) \\
&= -1 && (a=0) \quad (c=0)
\end{aligned}
$$

Compared with the rather simple form of relative rate of marginal technical substitution in the CES technology we have to conclude that the form (3.17), when $a><0$, is a bit more complicated than the corresponding CES form. On the basis of forms (3.16) and (3.17) it can be seen that the parameter a is a substitution parameter in the par technology.

In appendix D there are illustrations of some isoquants for various values of the parameters a and c. When $a \rightarrow \infty$ the substitution will be minimized. Actually the Leontief production function with constant proportion of the inputs is a special case of the par function when $a \rightarrow \infty$. The substitution seems to grow highest when $a \approx -3$. See appendix H.

3.4 The cost function

The nonlinearities in the par production function are so difficult that it is not possible to solve the total cost function explicitly as a product of the output Y and the unit cost function that depends only on the factor prices and constant parameters[1]. However an interesting implicit form for the cost function can be derived from the implicit par function form under optimization

$$(3.18) \qquad Y^{-a} = (1-w_K-c)*K^{-a} + (1-w_L+c)*L^{-a}$$

where w_K and w_L are the income shares of the inputs.
We have defined the total cost function C to be a product of the output Y and the unit cost function f

$$(3.19) \qquad C = Y*f$$

where f is a function of R, the factor price of K and W, the factor price of L as follows

$$(3.20) \qquad f = f(R,W)$$

Substituting (3.19) into (3.20) and using Shephard`s lemma which says that[2]

$$(3.21) \qquad \frac{@\ C}{@\ R} = K$$

$$\frac{@\ C}{@\ W} = L$$

we shall have the implicit form for the unit cost function

$$(3.22) \qquad f^a = (1-w_K-c)*(w_K)^{-a}*R^a + (1-w_L+c)*(w_L)^{-a}*W^a$$

---

[1] When a=0, we can solve the cost function explicitly for the Cobb-Douglas technology as commonly known.
[2] Shephard (1970), p.170-171

It is notable that in (3.22) the weights are functions only of the income shares and the distribution limit and the substitution parameters.

By substituting both (3.18) and (3.22) into (3.19) the results can easily be verified.

3.5 The case of several variables

The difficulty in generalizing the logarithmic mean of the form

$$(3.23) \qquad Y = \frac{K - L}{\ln(K/L)}$$

to the case of n variables (n>2) has been known for a long time. Vartia reports of some trials which have been done[1]. Anyhow, there is no accepted generalized form for the logarithmic mean neither in statistics nor econometrics.

Here we will present a generalization for the par function. This function includes the logarithmic mean as a special case. The generalization is done in the implicit form. This implicit form is, however, very practical and can be used in empirical analysis as it is.

We start with the implicit par function form under optimization (3.11) and (3.18)

$$Y^{-a} = (1-w_K-c)*K^{-a} + (1-w_L+c)*L^{-a}$$

This form, as it is quasi separable[2] compared with the base par function form in (3.5), can be written for n variables (n>2) by invention. That requires some assumptions. We will set a premium that

a. The weights add up to unity[3] and

b. Each of the weights is nonnegative and smaller or equal to unity.

The rest of the explication is invention.

---

[1] Vartia p.12

[2] Separability as it is defined by Leontief (1947 I and II) or, for example, by Berndt and Christensen (1974) is not met here.

[3] In fact, the consistency in aggregation demands that the sum of the weights can differ slightly from unity. Vartia has noted this same property when he has defined his indexes. However this assumption helps us in inventing the generalized form for the par function.

The par function for n variables (n>2) can be written as

$$(3.24) \qquad Y^{-a} = \sum_{i=1}^{n} (1-w_i+c_i)*X_i^{-a} \qquad (a><0)$$

Because the income shares w_i (i=1,...,n) add naturally up to unity we have to demand that

$$(3.25) \qquad \sum_{i=1}^{n} c_i = 2-n \qquad (n=1,2,...)$$

This guarantees that the weights in (3.24) add up to unity

$$(3.26) \qquad \sum_{i=1}^{n} (1-w_i+c_i) = 1$$

The corresponding unit cost function form would be[1]

$$(3.27) \qquad f^a = \sum_{i=1}^{n} \frac{(1-w_i+c_i)}{w_i^a} * P_i^a \qquad (a><0)$$

where the parameters a and c_i (i=1,...,n) are the same as in (3.24) and P_i is the unit price of the input X_i (i=1,...,n). Note that the income shares can be written

$$(3.28) \qquad w_i = \frac{P_i*X_i}{C} \qquad (i=1,...,n)$$

as well. Substituting (3.27), (3.28) and (3.24) into (3.19) the consistency of the definitions can easily be verified.

When comparing the forms in (3.24) and (3.27) we see that the par production function (3.24) is a mean function with unitary sum of weights and (3.27) is a unit cost function with a sum of weights which is more or less than unity depending on whether a<0 (sum of weights less than unity) or a>0 (sum of weights more than unity).

---

[1] The corresponding form for Cobb-Douglas case is

$$\ln f = A + \sum_{i=1}^{n} w_i*\ln P_i \qquad (a=0)$$

where A is a constant depending only on the original distribution parameters in the production function.

The other point of our premium demands that

$$(3.29) \qquad w_i - 1 < c_i < w_i \qquad (i=1,\ldots,n).$$

The estimation of the distribution limit parameters has to be done with these premiums in mind. When we take a closer look at the methods of fitting the par function to data, the guidelines of choosing c_i`s will be chosen as well.

To provide a basis for the discussion of estimation methods we will note here that when the production function is of the very basic par function form (3.23), then the first order optimization conditions suggest that the input elasticities are[1]

$$(3.30) \qquad \Phi = \frac{K - Y}{K - L} \qquad (a=-1)\ (c=0)\ (K><L)$$

$$\phi = \frac{Y - L}{K - L}$$

This is actually based on the normal logarithmic mean situation. The condition (3.30) in fact includes the presumption that under optimization the price ratio is

$$(3.31) \qquad R/W = (L/K)*\frac{K - Y}{Y - L}$$

and it suggests that there is a direct connection between the price ratio and the parameter c, which is 0 in (3.31). Can we in this case generalize the logarithmic mean simply by writing (3.23) first in the form[2]

$$(3.32) \qquad Y = (1-\Phi)*K + (1-\phi)*L$$

and then generalizing to n variables as follows

$$(3.33) \qquad Y = \sum_{i=1}^{n} (1-w_i+\frac{2-n}{n})*X_i$$

where the parameters c_i are equal for all i? I think not. The reasoning is that (3.33) neglects the connection between the parameters c_i and the price relations as well as other possible circumstances, for example, consistency in aggregation.

---

[1] The prevailing income shares can be calculated without knowing the prices when we assume optimization to take place.

[2] The corresponding cost function is $f^{-1} = \Phi*\phi*(R^{-1}+W^{-1})$.

A simple test shows that when the number of variables is more than two, the first order conditions do not give exact consistent results for the price ratios nor the income shares unless the sum of the parameters c_i does not slightly differ from 2-n. In general, the par function and the corresponding cost function in case of n variables can be presented with (3.24) and (3.27).

3.6 On the generalization possibilities of the par function

Until this stage we have assumed the par production function to be linearly homogeneous. If we assume the scale elasticity to be constant, we can write the par production function in a general homogeneous form as follows

$$(3.34) \qquad Y = A^* \left\{ c^* \frac{K^{-a} - L^{-a}}{(K/L)^{-a*c_{-1}}} \right\}^{-(v/a)} \qquad \begin{matrix} (a><0) \\ (c><0) \\ (K><L) \end{matrix}$$

where v is the degree of homogeneity and v=1 in case of the linear homogeneity.

Referring to the CES classification (2.36) given by Sato R.[1] we could state the general homothetic case of the par technology as follows[2]

$$(3.35) \qquad \text{Homothetic type:} \quad Y = Y\left(\left\{ c^* \frac{K^{-a} - L^{-a}}{(K/L)^{-a*c_{-1}}} \right\}^{-(1/a)} \right)$$

$$(a><0) \quad (c><0) \quad (K><L)$$

where Y(·) is a continuous, twice-differentiable, finite, nonnegative and nondecreasing function of the argument.

---

[1] Sato R. (1977 I), p.13

[2] Whether the Sato-forms in (2.36) could directly be used to generalize (3.35) to the non-homothetic "labor- and capital-homothetic" cases can later be analysed separately. The similarity of the basic separable CES form and the quasi separable implicit par function form (3.10) could then be utilized.

Another possibility to generalize the par production function is to use the method presented by K.Sato[1]. K.Sato has used a two-level CES production function, where the lower level CES functions serve as inputs to the higher level CES production function. Using this kind of formulation would quarantee that the par substitution parameters a_{ij} could vary between the input combinations. However, we need to notice that the par function is not consistent in aggregation using this kind of calculation method[2].

If the par function form (3.5) was formulated to mean the unit cost function

$$(3.36) \qquad f = \left\{ c^{*} \frac{R^{-a} - W^{-a}}{(R/W)^{-a*c} - 1} \right\}^{(-1/a)} \qquad \begin{matrix} (a><0) \\ (c><0) \\ (R><W) \end{matrix}$$

we could have got other kind of interesting results. For example, using this formulation we can solve the log of the input ratio as a Taylor expansion of the input prices.

The par technology is not self-dual which means that the production function and the unit cost function have not the same form. That is why assuming (3.36) implies a production function which is not of the original par type.

In chapter 5.1 we have used an iterative calculation method to estimate the par function in case of several inputs and one observation. In case of several observations this iterative method can be generalized using some kind of definition for a criterion function[3] to be optimized according to the general rules of the nonlinear estimation procedures.

Criteria for the design of functional forms in production theory has been hamdled by Fuss, McFadden and Mundlak (1978)[4]. The five criteria presented by them are parsimony in parameters, ease of interpretation, computational ease, interpolative robustness and extrapolative robustness. The par production function is not very easy to compute. The other criteria seem to be fulfilled by the par function. Especially, estimation of the distribution limits, which are outside the range of the observed data, can be done with the par technology (meaning extrapolative robustness).[4]

---

1 Sato K. (1967) pp.201-218

2 That is why this method can not, in general, be used to calculate the logarithmic mean for n variables (n>2).

3 Criterion functions are handled, for example, by Walsh (1975).

4 See Fuss, McFadden and Mundlak (1978), pp.224-225. Criteria for the selection of functional forms in econometrics, in general, is presented by Lau (1986), pp.1515-1566.

4. ON THE THEORY OF INCOME DISTRIBUTION

4.1 The CES and par income distribution compared

In case of the CES production function we have (compare with equation (2.38))

$$(4.1) \qquad \ln\left(\frac{R}{W}\right) = \ln\left(\frac{\delta}{1-\delta}\right) - (1+p)*\ln(K/L)$$

which is an exact equation based on the first order profit maximum condition[1]. The first order condition for the profit maximum (3.6) in the par production function situation can approximately, in the neighbourhood of a=0, be written as

$$(4.2) \qquad \ln\left(\frac{R}{W}\right) = \ln\left(\frac{1-c}{1+c}\right) - [1+a/3]*\ln(K/L) - (1/18)*c*a^2*[\ln(K/L)]^2$$

in which equations R and W are the input prices of inputs K and L, respectively. p is the CES substitution parameter and a is the par substitution parameter, δ is the CES distribution parameter and c is the par distribution limit parameter. Equation (4.2) is proved in appendix E.

As the elasticity of substitution (2.14) can in the optimum be written

$$(4.3) \qquad \sigma = \frac{@\ \ln(W/R)}{@\ \ln(K/L)}$$

we can on the basis of (4.1) and (4.2) conclude that a very good approximation for the interrelation of the substitution parameter a and the elasticity of substitution is given by

$$(4.4) \qquad a/3 \approx p \qquad \sigma \approx \frac{3}{3+a} \qquad a \approx 3*\left(\frac{1-\sigma}{\sigma}\right) \qquad (a>-3) \quad (\sigma>0)$$

where σ is the (direct) elasticity of substitution. According to the information in appendix E and F the approximation (4.2) yields on a wide range of |a| and |lnk|.

---

[1] See, for example, Chiang p.419. The equation (4.1) has been derived from the linearly homogeneous CES form $Y^{-a} = \delta*K^{-a} + (1-\delta)*L^{-a}$.

A table for the approximation errors of equation (4.2) in percents is given in appendix F. The good fit is due to the fact that the second order term in the Taylor expansion is zero when c=0. See appendix E. If the form (4.2) is equipped with the third and fourth order terms we get

$$(4.5) \qquad \ln\left(\frac{R}{W}\right) = \ln\left(\frac{1-c}{1+c}\right) - [1+a/3]*x - (1/18)*c*a^2*x^2 + \left(\frac{2-3*c^2}{810}\right)*a^3*x^3 + \left(\frac{c^3+c}{3240}\right)*a^4*x^4$$

where x=ln(K/L). Using the first and the third order terms in the expansion a better fit is given, even in the area of $|a|\leq 4$ and $|\ln k|\leq 2$. A table for the approximation errors in percents in this case (when c=0) is given in appendix F as well.[1]

Writing (4.2) as follows

$$(4.6) \qquad \ln\left(\frac{w_K}{w_L}\right) = \ln\left(\frac{1-c}{1+c}\right) - (a/3)*\ln(K/L)$$

and by differentiating we get[2]

$$(4.7) \qquad \frac{@\ \ln(w_K/w_L)}{@\ \ln(K/L)} = -\ (a/3)$$

which tells us that as long as the substitution parameter a is greater (less) than zero, capital`s income relative to labor`s income falls (rises) when the capital-labor ratio increases. When a=0, we have constant income shares in the Cobb-Douglas situation. Equation (4.6) is an approximation, but with extreme values of the parameter a and the variable (K/L) the same dependence can generally be calculated from (3.6).

---

[1] The exact form of the third and the fourth order coefficients of the Taylor expansion of the form (3.6) was pointed out to me by Antti Kanto after I had practically given up the huge derivation.

[2] In the neighbourhood of c=0 the form (4.7) is exact enough. In case c<0 or c>0 the impact of the values of the parameter c and the variable x=ln(K/L) can be analysed using a relevant number of higher order terms in the derivation as well, resulting from the more exact approximation in (4.5). The corresponding CES differentiation leads to

$$\frac{@\ \ln\ (w_K/w_L)}{@\ \ln\ (K/L)} = -p\ .$$

In general the linearized forms of the par production function seem to be close to the corresponding CES forms. When only the first order term in the Taylor expansion of the par production equation is examined, we in fact have an equation implied by the CES production technology. However, the CES production technology is not a special case of the par production technology. The biggest difference lies in the fact that the limits of variation of the income share of L in par case are[1]

$$(4.8)\qquad \begin{array}{ll} c \le w_L \le 1 & \text{when } c>0 \\ 0 \le w_L \le 1+c & \text{when } c<0 \end{array}$$

while in CES case the corresponding values are always 0 and 1.

Many of the par equations which can be solved with the method of linearization, are obviously technically very near to the so called translog production function forms.
We can present the following example. Neglecting the second order term in (4.2) we can solve ln(K/L) as a function of ln(R/W) as follows

$$(4.9)\qquad \ln\left(\frac{K}{L}\right) = \left(\frac{3}{3+a}\right)*\ln\left(\frac{1-c}{1+c}\right) - \left(\frac{3}{3+a}\right)*\ln(R/W) \qquad (a>-3)$$

When (4.9) is substituted into (4.2) we get after some manipulation

$$(4.10)\qquad \ln\left(\frac{w_K}{w_L}\right) = A + B*\ln(R/W) + C*[\ln(R/W)]^2$$

where

$$A = \ln\left(\frac{1-c}{1+c}\right) - \left(\frac{a}{3+a}\right)*\ln\left(\frac{1-c}{1+c}\right) - (1/2)*\left[\frac{c*a^2}{(3+a)^2}\right]*\left[\ln\left(\frac{1-c}{1+c}\right)\right]^2$$

$$B = \left(\frac{a}{3+a}\right) + \left[\frac{c*a^2}{(3+a)^2}\right]*\ln\left(\frac{1-c}{1+c}\right)$$

$$C = -(1/2)*\left[\frac{c*a^2}{(3+a)^2}\right] \qquad (a>-3)$$

The same kind of second order form is implicated by, for example, a third order translog cost function[2].

---

[1] Look at the limiting values shown in (3.14).
[2] See Dalal, pp.355-360

An approximate unit cost function can also be solved in par case with the Taylor method by first substituting the input demand functions into C=R*K+W*L keeping in mind that f=C/Y and then deriving the expansion. Thus, in general, the par unit cost function can be presented in a linearized form

$$(4.11) \quad \ln f = \alpha + \ln W + \beta * \ln(R/W) + \mu * [\ln(R/W)]^2 + R_n$$

in the convex and convergent area of the par production function[1]. In (4.11) R_n is the remainder in the Taylor series. When the coefficients α, β and μ are presented in terms of the original parameters a and c, the result is somewhat complicated.

---

[1] See chapter 5.2 and appendix H of this study.

4.2 The dispersion interpretation of the elasticity of substitution

4.2.1 The Pearson equation and the Burr-Hatke approach

Most frequency functions of the well-known statistical distributions satisfy a differential equation

$$(4.12) \qquad \frac{df}{dx} = \frac{(x - a)*f}{b_0 + b_1*x + b_2*x^2}$$

where f(x) is the frequency function and a and b_i`s are constants. It can be easily verified that this is true, for example, in the case of the normal distribution, the $\chi^2$ distribution, Student`s distribution, the distribution of Fisher`s ratio $e^{2z}$, the Beta distribution and Pareto`s distribution. Equation (4.12) can be derived from the hypergeometric series as a limiting case and it forms the base of the system of frequency curves introduced by K.Pearson[1].

The fitting of distributions to observational data is usually done on basis of the frequency curves. However, we usually do need the theoretical frequencies for comparison with observations. The question naturally arises whether we could not directly fit the distribution function to data and obtain the frequency function, if we need it, by the relative simple process of differentiation.

Such an approach has been considered by Burr and Hatke[2]. Using a generalization of the Pearson equation (4.12) consider

$$(4.13) \qquad dF = F*(1-F)*g(x)*dx$$

where g(x) is some convenient function, which must be non-negative in $0 \le F \le 1$ and in the range of x. The solution of (4.13) can be derived as

$$(4.14) \qquad dF*\left(\frac{1}{F} + \frac{1}{1-F}\right) = g(x)*dx$$

and the distribution function F(x) is given by

$$(4.15) \qquad F(x) = [1 + e^{\{-G(x)\}}]^{-1}$$

The function G(x) will later be called the Burr-Hatke equation and it is defined by

$$(4.16) \qquad G(x) = \int_{-\infty}^{x} g(t)dt$$

---

[1] See H.Cramer pp.248-249
[2] Kendall-Stuart p.173

where G(x) is a continuously increasing function of x with limiting values $\lim_{x \to 0} G(x) = -\infty$ and $\lim_{x \to \infty} G(x) = \infty$.

For example, if we choose

$$(4.17) \qquad G(x) = \ln[(1+x^{\alpha})^{\beta} - 1]$$

then the distribution function will be

$$(4.18) \qquad F(x) = 1 - \frac{1}{(1+x^{\alpha})^{\beta}}$$

where $0 \leq x \leq \infty$ and $\alpha > 0$ and $\beta > 0$. Of course many other forms can be chosen for (4.17) to find a convenient form for the distribution function to fit the observational data in question.

The fitting of the distribution function to data can be done by several methods. Kendall and Stuart mention, for example, the method of cumulative moments, the method of range frequencies and the method of frequency-moments or probability-moments. Of course nonlinear estimation methods can in general be used[1].

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[1] Kendall-Stuart pp. 173-174

4.2.2 The elasticity of substitution as a dispersion parameter

If we consider the variable $x=\ln(K/L)$ as an output from a certain random process, we can examine its statistical distribution function theoretically. Although the random process which generates variable x usually includes specific optimization stages, it always has exogenous random elements as well. When input prices are considered random exogenous factors, then x becomes random in nature as well.[1]

In the case of the CES production function, the log of the ratio of inputs is a linear function of the log of the input prices in the optimum. That is why the distribution of variable x is totally defined by the distribution of the log of the ratio of input prices. When the dispersion of the log of the input price ratio is one, then the dispersion of x is $\sigma=1/(1+p)$. Assume the production process and the distribution of $\ln(W/R)$ to be such that the distribution function of $x=\ln(K/L)$ is defined by

$$(4.19) \qquad G(x) = \left(\frac{x-m}{\sigma}\right) - \ln\left(\frac{1-\delta}{\delta}\right) \qquad (\sigma>0) \quad (\delta>0)$$

where G(x) refers to the Burr-Hatke equation (4.16). Equation (4.19) is linear with respect to x. In that case, from (4.15), the distribution function of $x=\ln(K/L)$ will be

$$(4.20) \qquad F(x) = \frac{1}{1 + e^{\ln[(1-\delta)/\delta] - [(x-m)/\sigma]}}$$

The form (4.20) is the statistical distribution function of the logistic distribution which has been used, for example, to represent growth functions. It has been shown that the logistic distribution arises in a purely statistical manner as a limiting distribution (as $n \to \infty$) of the standardized mid-range (average of largest and smallest sample value) of random samples of size n. The logistic distribution is obtained as the limiting distribution of an approriate multiple of the extremal quotient which is the ratio of the largest and the smallest value in a sample as well[2].

According to Johnson and Kotz the logistic distribution has a similar shape as the normal distribution. That property makes it profitable, on suitable occasions, to replace the normal distribution by the logistic to simplify the analysis without too great discrepancies in the theory. Johnson and Kotz have compared

---

1 In this situation the entrepreneur is assumed to maximaze the expected profit of the firm. The stochastic specification of production models has been analysed by many writers, for example, Zellner, Kmenta and Dreze (1966), pp.784-795, Schim van der Loeff and Harkema (1981), pp.33-53.

2 See Johnson-Kotz, ch.22

the standardized normal and the logistic distribution functions and found out that the maximum difference between these two distribution functions, with proper parameter values, is under 0.01.

If we first assume that the variable ln(W/R), which is the log of the input price ratio, is logistically distributed having the distribution function

$$(4.21) \qquad H\{\ln(W/R)\} = \frac{1}{1 + e^{-\ln(W/R)}}$$

then substituting the first order optimizing condition $\ln(W/R)=(p+1)*\ln(K/L)-\ln[\delta/(1-\delta)]$ into (4.21) we get the distribution function of $x=\ln(K/L)$

$$(4.22) \qquad F(x) = \frac{1}{1 + e^{\ln[\delta/(1-\delta)] - x/\sigma}}$$

In equation (4.22) the location parameter m is zero. When the input variable K is replaced with $e^{-m}*K$ in the original CES production function, then the distribution function form (4.20) will be the result. In general (4.22) is the same form as (4.20).

In case we have $\delta=0.5$ the first two moments of the distribution function (4.22) are

$$(4.23) \qquad \begin{aligned} E(x) &= m \\ Var(x) &= \sigma^2*\pi^2/3 \end{aligned}$$

where σ is the elasticity of substitution and $\pi=3.14159..$

The above formulation shows that the elasticity of substitution has a clear dispersion interpretation in the distribution of $x=\ln(K/L)$. Another interesting feature of this interpretation and its connections to the theory of income distribution can be found by transforming the income share of L as follows

$$(4.24) \qquad w_L = \frac{W*L}{W*L + R*K} = \frac{1}{1 + e^{\ln(K/L)-\ln(W/R)}}$$

and further substituting the first order optimum condition $\ln(W/R)=(p+1)*\ln(K/L)-\ln[\delta/(1-\delta)]$ into (4.24) we get

$$(4.25) \qquad w_L = \frac{1}{1 + e^{\ln[\delta/(1-\delta)]-p*x}}$$

$$= \frac{1}{1 + e^{\ln[\delta/(1-\delta)]+[1-1/\sigma]*x}}$$

As the elasticity of substitution σ in terms of the CES substitution parameter p is $\sigma=1/(1+p)$, by comparing the equations (4.22) and (4.25), we immediately note that with the assumptions[1] we have made the distribution function of $x=\ln(K/L)$ can be derived from the income share equation of w_L by replacing the substitution parameter p with the elasticity of complementarity[2] $b=p+1$ in the income share equation. It can be shown that this leads to the following relationship[3]

$$(4.26) \qquad \ln\left(\frac{w_K}{w_L}\right) = x - G(x)$$

$$= x - \ln\left[\frac{F(x)}{1 - F(x)}\right]$$

where w_K and w_L are the income shares of K and L, G(x) is the Burr-Hatke equation in (4.16) and F(x) is the statistical distribution function of $x=\ln(K/L)$.

The result in (4.26) has not been presented in the theory of income distribution before. The Burr-Hatke equation G(x) seems to have clear implications for the distributive shares. This interdependence is estimated with simulated data in chapter 6.5 of this study.

As the Burr-Hatke equation is linear in the CES case, see equation (4.19), it is no wonder that the log of the CES input share ratio is unlimited as we have already noted in equations (2.24) and (2.26).

When distribution limits exist, the Burr-Hatke equation G(x) must limit the statistical distribution function of $x=\ln(K/L)$ either from below and/or from above.

Under proper circumstances the matching of the CES income distribution function and the statistical distribution function of ln(K/L) can be utilized. If the assumptions concerning the distribution of ln(W/R) holds, then either distribution function of ln(K/L) can be solved directly from the income distribution

---

[1] The assumptions of the logistic or nearly normal distribution of ln(W/R) and the CES production structure
[2] See equation (2.28)
[3] Proof in appendix J

function or on the other hand the methods of fitting the cumulative distribution function can be used to fit the transformed income distribution function. The latter possibility also supplies new estimation opportunities of the CES production function.

The par income share equations can be written exactly in the following way

$$
\begin{aligned}
w_K &= \frac{1}{1 - e^{+a*x}} + \frac{c}{1 - e^{-a*c*x}} - c && (c><0) \\
&= \frac{1}{1 - e^{+a*x}} + \frac{1}{a*x} && (c=0) \\
w_L &= \frac{1}{1 - e^{-a*x}} + \frac{c}{e^{-a*c*x} - 1} + c && (c><0) \\
&= \frac{1}{1 - e^{-a*x}} - \frac{1}{a*x} && (c=0)
\end{aligned}
\tag{4.27}
$$

However, the transformation used in the CES case cannot be succesfully used here, without further consideration, if we need an absolute fit in the Cobb-Douglas case.

The linear form of (4.26) in the CES case and a corresponding linearized form in the par case were presented and analysed in chapter 4.4. Based on the approximation (4.5), an approximation of the statistical distribution function of ln(K/L) in the par case could be achieved by substituting a/3 by 1+a/3 that is writing a+3 instead of the parameter a in the equation of w_L in (4.27).

5. METHODS OF FITTING THE PAR FUNCTION TO DATA

5.1 The log-mean generalization

5.1.1 The income shares are known

If the income shares of n inputs are known, then the output (which can be interpreted as the log-mean for n variables in case a=-1) can be calculated using the equation (3.24). At the same time we need to calculate the parameters c_i (i=1,...,n).

In the following we will present an iterative calculation method to fit the par production function to data in case of n variables (n>2) and one observation. The method is developed with the system of "forward from the end". In general the unknown variables here can be either[1]

(1) the total output $Y_{123...n}$ and the distribution limit parameters c_i (i=1,...,n) or
(2) the substitution parameter a and the distibution limit parameters c_i (i=1,...,n).

In case (1) above the substitution parameter a is given. In that case we in fact need only one iteration round (with all of its n-1 stages) to calculate the total output $Y_{123...n}$ and to simultaneously solve the distribution limit parameters.

In case (2) above where the total output is given, the exact match can be estimated by a direct search procedure. In that case the value of the substitution parameter a will be varied. One iteration round (with all of its n-1 stages) is needed for each value of the parameter a. The value of parameter a will be varied according to general iteration rules until the exactness of the chosen stopping rule is attained.

When we demand consistency in aggregation, then, in the last stage of calculation, the group of equations to be solved is of the form

$$
\begin{aligned}
Y_{123...n}^{-a} &= (1-w_1+c_1)*X_1^{-a} + (w_1-c_1)*Y_{234...n}^{-a} \\
&= (1-w_2+c_2)*X_2^{-a} + (w_2-c_2)*Y_{134...n}^{-a} \\
&\quad \vdots \\
&= (1-w_n+c_n)*X_n^{-a} + (w_n-c_n)*Y_{123...n-1}^{-a} \\
&= \Sigma\ (1-w_i+c_i)*X_i^{-a}
\end{aligned}
\tag{5.1}
$$

where there are n+1 unknown variables (parameters c_i and the total output $Y_{123...n}$) and n+1 equations.

---

[1] The notation $Y_{123...n}$ means the output in the par technology when all of the n inputs X_i (1=1,...,n) are in use. The concept of "subset output" is an assisting concept which is connected only to the stages in the calculation, meaning the "subset output" when there is only a subset of the inputs (noted with subscripts of Y) in calculation.

In the second to last stage of calculation we have n groups of equations to solve

$$
\begin{aligned}
Y_{234\ldots n}{}^{-a} &= (1-w_2+c_2)*X_2{}^{-a} + (w_2-c_2)*Y_{345\ldots n}{}^{-a} \\
&= (1-w_3+c_3)*X_3{}^{-a} + (w_3-c_3)*Y_{245\ldots n}{}^{-a} \\
&\;\;\vdots \\
&= (1-w_n+c_n)*X_n{}^{-a} + (w_n-c_n)*Y_{234\ldots n-1}{}^{-a} \\
&= \Sigma\ (1-w_i+c_i)*X_i{}^{-a}
\end{aligned}
\tag{5.2}
$$

$$
\begin{aligned}
Y_{134\ldots n}{}^{-a} &= \ldots \\
&\;\;\vdots \\
Y_{123\ldots n-1}{}^{-a} &= \ldots
\end{aligned}
$$

in which the income shares are calculated within the corresponding subset of inputs and thus the parameters c_i differ in each of the n groups of equations as well as in respect to the other stages of the calculation. In each of these n groups of equations we have n unknown variables (n-1 bits of parameters c_i and the corresponding subset output $Y_{12\ldots i-1,i+1\ldots n}$) and n equations.

In the k`th last stage we have $n!(n-1)!\ldots(n-k+2)!$ groups of equations each of which has (n-k+2) unknown variables and (n-k+2) equations.

Thus in the second stage, when k=n-2, we have $n!(n-1)!(n-2)!\ldots3!$ groups of equations with 4 equations and 4 unknown variables in each. The total amount of stages needed is n-1.

In the first stage we have only

$$\frac{n!}{(n-2)!2!}$$

pieces of subset outputs and corresponding pairs of inputs. The calculation is done with (3.5) for the each two inputs in question. When a=-1, these are the weighted log-means for two inputs. The parameters c_i must, in the first stage, be chosen so that for each two inputs we have

$$\frac{P_i*X_i}{P_j*X_j} = \frac{X_i{}^{-a} - Y_{ij}{}^{-a}*(X_i/X_j)^{-a*c_i}}{Y_{ij}{}^{-a}*(X_i/X_j)^{-a*c_i} - X_j{}^{-a}} \tag{5.3}$$

where Y_{ij} is the subset output of inputs X_i and X_j. When the income shares are known (which means that prices and quantities are known), there is only one value for c_i which satisfies (5.3) because the right hand side of (5.3) is a continously increasing or decreasing function of c_i depending on the value of the parameter a.

Of course, the calculation will start from the first stage and then continues through the needed n-1 stages until the total output is found. This method is consistent in aggregation.

The corresponding cost function can, of course, be used when necessary. However, one can always use the identity C=f*Y and get the same unit cost result on basis of the postulates described in chapters 3.3 and 3.4.

Empirical calculations using this method show that, typically, the sum of parameters c_i in each stage is slightly above 2-j, where j is the number of variables in the calculation stage in question. Thus the sum of the weights in (3.24) is correspondingly slightly under unity.

The value of the parameter a has no effect on the calculation procedure, unless it is not zero. So any relevant values of parameter a can be used with this procedure. When a=0 we have the Cobb-Douglas (geometric mean) situation with direct calculation possibilities. When a=-1 this procedure supplies the weighted logarithmic mean for the n inputs in question.

5.1.2 Income shares are unknown

As noticed already in chapter 3.4, there is a direct connection between the prices and the distribution limit parameters. When there are no income shares at all in the calculation situation, it is best that we choose a set of help variables in order to be able to use the generalized calculation equations (3.24) and (3.27) and the calculation procedure in chapter 5.1.1.

Let us denote the help variables $\pi_i = w_i - c_i$. Then the equation (3.24) can be written

$$(5.4) \qquad Y^{-a} = \Sigma\ (1-\pi_i)*X_i^{-a}$$

and in the case of two inputs it reduces to $Y_{ij}^{-a}=(1-\pi)*X_i^{-a}+\pi*X_j^{-a}$ $(i,j=1,...,n;i=j)$. These together with (5.4) can be directly substituted into the procedure introduced in chapter 5.1.1. The iteration procedure can be used both when the income shares are known and when they are unknown.
In the first stage the pairs of inputs and the corresponding subset output must be calculated directly by using the base logarithmic mean form (3.1) or the corresponding π_i`s must be solved with (3.30) and then the linear form (3.32) can be used. The same recursive calculation method can then be used. This way we can calculate the unweighted logarithmic mean (given the parameter a) for n inputs.

One finds at once that this is a mean which is pretty complicated to calculate. However, I think there is a certain analogy between the calculation procedure presented in chapter 5.1.1 and the methods which are used to choose the so called best linear form for linear regression. These can, in general, be divided into methods which consider all possible regressions and into methods which stepwise choose the best model which can be attained with changing only one regressand at a time. The former always lead to the best solution according to the chosen criteria, but extensive calculations are required. The latter are easier to calculate, but the result is not necessarily the best.

Our calculation procedure is laborious, but it always leads to the same result which is right according to our criteria consistency in aggregation, the invented form for the generalization, the role of the distribution limit parameters and the meaning of income shares under optimization. If the analogy is there, I expect easier methods for calculation can later be developed according to the same guidelines.

5.2 The linear approximation method

A linear approximation for the separable homothetic type of CES production functions was introduced by professor J.Kmenta in 1967[1]. This power series approximation can be estimated directly by single-equation least squares. Later J.Thursby and K.Lovell have shown that in the Kmenta approximation the estimates are somewhat biased and therefore the CES parameters are estimated consistently only under favorable circumstances[2].

The CES function in question can be written as

$$(5.5) \qquad \ln Y = \ln\mu - (v/p)*\ln(\delta*K^{-p}+(1-\delta)*L^{-p}) + u$$

where u is the stochastic error term assumed to be independently and normally distributed with zero mean and constant variance. δ is the distribution parameter, p is the substitution parameter as in (2.34) while v is the degree on homogeneity.

Equation (5.5) can be derived by using Taylor`s formula for expansion around p=0. After disregarding the terms of third and higher orders, the expansion leads to

$$(5.6) \qquad \ln Y = \ln\mu + v*\delta*\ln K + v*(1-\delta)*\ln L - (1/2)*p*v*\delta*(1-\delta)*(\ln K-\ln L)^2 + u$$

which can according to Kmenta be separated into two parts, one corresponding to the Cobb-Douglas form and one representing a correction due to the departure of p from zero. The latter part will disappear, when p=0.

It is proved in appendix G that the corresponding linear approximation to the par production function, if it is originally written

$$(5.7) \qquad Y = A*\left\{c*\frac{K^{-a} - L^{-a}}{(K/L)^{-a*c} - 1}\right\}^{(-v/a)} * e^{u} \qquad (a><0;c><0)$$

where v is the degree of homogeneity and u is the random term as above, can be written in linear in parameter combinations fashion as follows

$$(5.8) \qquad \ln Y = \ln A + v*\left(\frac{1-c}{2}\right)*\ln K + v*\left(\frac{1+c}{2}\right)*\ln L - a*v*\left(\frac{1-c^2}{24}\right)*(\ln K-\ln L)^2 + u$$

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[1] Kmenta, pp.180-189
[2] Thursby and Lovell, pp.363-377

which is very similar to (5.6). As shown already by M.McCarthy, the power series approximation to the CES production function is an approximation to functions other than CES as well[1]. What is relevant here is how well (5.8) approximates (5.7) within some range of practical importance.

The experimental tests have shown that in the CES case the bias of using the linear approximation seems to increase in the case of extreme values of the elasticity of substitution and extreme values of input ratios. In fact the study by Thursby and Lovell suggests that better estimation results can be attained by centering the observations. Another result of theirs is that satisfactory estimates for other parameters than the substitution parameter can be attained when the estimates for the substitution parameter are unsatisfactory. Their critiques suggest that the linear approximation should only be used with values of the substitution parameter which are near to zero. This is due to the area of convergence of the Taylor series, which in the CES case demands that[2]

$$(5.9) \qquad |\ln(K/L)| < |1/(p*\delta)|$$

Correspondingly the area of convergence in the case of the par production function is

$$(5.10) \qquad |\ln(K/L)| < |2/(a*(1-c))| \qquad (-1<c<1)$$

These conditions show that when the variance of the ratio of inputs is increased, the area of convergence of the approximation will be worse. In this case the estimates will be more biased because the goodness of fit of the approximation with extreme values of the input ratio will decrease.

Referring to the distribution function of $x=\ln(K/L)$ we can use the Bienayme-Tchebycheff inequality[3] and note that

$$(5.11) \qquad P(|x-m|\geq k*\sigma) \leq 1/k^2$$

If m=0, then, for example, in the CES case, the probability of $|\ln(K/L)|$ being bigger than $1/(p*\delta)$ is smaller than $p^2*\delta^2/(1+p)^2$.

In chapter 6.3 we test the linear approximations (5.6) and (5.8) in the case of CES production technology and par production technology to examine and compare the properties of the approximations in both cases.

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[1] M.McCarthy, pp.190-192
[2] J.Thursby and K.Lovell, pp.363-377
[3] See H.Cramer, p.182-183 and p.256

According to equation (4.5) we have

$$\text{(4.5B)} \quad \ln\left(\frac{R}{W}\right) = \ln\left(\frac{1-c}{1+c}\right) - [1+(1/3)*a]*\ln(K/L) - (1/18)*c*a^2*[\ln(K/L)]^2$$

which is an approximation made with the Taylor series expansion. (4.5), (4.5B) as well as (4.6) can be used to estimate the parameters in the par production function. Because equation (4.5B) gives an especially good fit, we can develop the relation further to get another proper form for empirical estimation situations.

The Cobb-Douglas unit cost function is of the form[1]

$$\text{(5.12)} \quad \ln f = \left(\frac{1-c}{2}\right)*\ln R + \left(\frac{1+c}{2}\right)*\ln W - \left(\frac{1+c}{2}\right)*\ln\left(\frac{1+c}{2}\right) - \left(\frac{1-c}{2}\right)*\ln\left(\frac{1-c}{2}\right)$$

where f is the unit cost for one unit of output Y, see equation (3.19). If we transform (5.12) so that

$$\text{(5.13)} \quad \ln f = \ln W + \left(\frac{1-c}{2}\right)*\ln\left(\frac{R}{W}\right) - \left(\frac{1+c}{2}\right)*\ln\left(\frac{1+c}{2}\right) - \left(\frac{1-c}{2}\right)*\ln\left(\frac{1-c}{2}\right)$$

and then solving for ln(R/W) and substituting into (4.6) and neglecting the higher order terms we get after some manipulation

$$\text{(5.14)} \quad \ln\left(\frac{W}{f}\right) = \ln\left(\frac{1+c}{2}\right) + [1+(1/3)*a]*\ln\left(\frac{Y}{L}\right)$$

Here we have the real wage rate[2] as a function of the labor productivity. Of course, (5.14) is an approximation with only a limited number of terms in the expansion as well as (4.5) and (4.6). In empirical applications (5.14) is very good as many studies confirm[3].

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[1] Henderson-Quandt, p.85, compare with ch. 3.4 of this study.
[2] Assuming a competetive situation, where the output price is equal to the average costs.
[3] A similar form for the CES production technology has been widely used, see, for example, Intriligator pp.275-276

5.4 Nonlinear estimation of the share equations

The highly interesting difference in the implications of the CES and par production technologies lies in the fact that in the par case the distributive shares are limited by the distribution limit parameters while in the CES case they are not limited in such a way. The problem in an empirical examination situation can actually be whether there exist real distribution limits in the case in question. One of the estimation possibilities in that case is to use nonlinear estimation techniques to estimate the share equations implied by the CES and par production technologies.

In general, the estimation of the share equations demands that there exists some systematic variation in the distributive shares. Theoretically, the estimation results with the nonlinear techniques should be better, when the variation in the distributive shares is large. The increased variance in the regressand decreases the variance of the estimates. On the other hand the nonlinear estimation can be used in case there is a big variance in the explaining variable $x=\ln(K/L)$. In fact the estimation results should be better when the variance of x is increased. This is due to the fact that the variance of the distributive shares is increased simultaneously. This is just the opposite of the case of linearized forms, which do not allow the increased variance of x without worse estimation results caused by the limited area of convergency of the linear approximations. Thus, when the variance of the variable $x=\ln(K/L)$ is large, it is worth trying to fit the share equations with nonlinear methods of estimation.

In the CES case the income share of L can be written, after substituting the first order optimizing condition $\ln(W/R)=-\ln[\delta/(1-\delta)]+(1+p)*\ln(K/L)$ into the equation $w_L=W*L/(R*K+W*L)$, as follows

$$(5.15) \qquad w_L = \frac{1-\delta}{(1-\delta) + \delta*(K/L)^{-p}}$$

In the par case, substituting (3.5) into (3.9) and setting the input elasticity of L equal to the income share of L, the corresponding equation is

$$(5.16) \qquad w_L = c + \frac{1}{1-(K/L)^{-a}} + \frac{c}{(K/L)^{-a*c} - 1}$$

which both are the income shares of L when p><0, a><0 and c><0. It is worth noting that the equation (5.16) has a discontinuing point in a=0 and/or K=L. In some cases this can cause difficulties in estimation. However, if the possible difficulties are caused only by some specific observations, these difficulties can, in general, be avoided.

Nonlinear techniques have widely been used in econometrics, in estimation of the CES production function parameters as well. Actually there are a lot of methods developed just for the purpose. To name some there are the Kumar&Kapinski method[1], the Corbo method[2] and the Thursby&Lovell method[3], which all have been used to estimate the CES production function parameters. An comparative study of the estimation characteristics of these has been made by Thursby[4]. A wide survey of the nonlinear methods is given by Walsh[5]. The method of Davidon-Fletcher-Powell[6], which is described by Walsh as well, is used in this study when making the sampling experiments.

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1 Kumar&Kapinski, pp.563-567
2 Corbo, pp.1466-1477
3 Thursby&Lovell, pp.363-377
4 Thursby, pp.295-299
5 Walsh (1978)
6 See Walsh, pp.110-120

5.5 Some notes on the stochastic specification of the share equations

The estimation of n shares should in general be restricted to a n-1 dimensional equation system because of the singularity of the equation system. This is the case with most of the studies made[1].

Another important feature in the share equations is that the specification should respect the fact that the shares cannot be negative, nor can they exceed unity. By assuming that the shares have a multivariate normal distribution there is a positive probability that shares will not respect the constraints. Since the mean will generally be different for each observation and since the shares are constrained, it is highly unlikely that the true density functions for all observations are symmetric with a common covariance matrix. This is why, in general, it can be argued that the normal distribution is invalid as a stochastic specification for share equations.

According to Woodland[2], who has tested the Dirichlet distribution (multivariate beta distribution) against the normal distribution, the normal model performs rather well even when the true model is not normal but is the Dirichlet model. If there is no heteroscedasticity, the normal specification seems to work good enough in share equations as well.[3]

However, the discussion of the normality of the random terms should be done. Although the assumption of the normality of the random terms seems to be harmless, wrong estimation results will be given if autocorrelation, heteroscedasticity etc. are strong[4].

As we estimated the income share equations with the Monte Carlo techniques in chapter 6.6, most of the empirical estimation problems are neglected here.

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1 See the references in chapter 2.6
2 Woodland, p.362
3 This problem can be examined with the logit-model as well, see Considine and Mount (1984), pp.434-443.
4 Woodland, pp.381-383.

6. THE SAMPLING EXPERIMENTS

6.1 The target of the experiments

As we have now introduced a quite general par production function form, there is of course an interest to examine the estimation properties of that function. Especially we are interested in checking the accuracy of the linear approximations. On the other hand the examination of the statistical properties should not be disturbed in the first stage with the common econometric problems of autocorrelation, heteroskedasticity or lack of data. That is why we try to make general conclusion on randomized Monte Carlo data, which is generated exactly for that purpose. When we generate the sampling data on one hand from the CES production technology and on the other hand from the par production technology, we have an ideal testing situation for comparing the mentioned technologies with the estimation results. As the "right" technology is known, the estimations reveal the possible inaccuracy in conclusions based either on the estimation techniques, the chosen approximation forms or on the wrong specification of the production technology.

The targets for the sampling experiments can thus be expressed as follows:

a. Do the used estimation techniques discriminate the CES and par production techniques from each other in this laboratory sampling situation so clearly that the model specification can be done reliably.
b. Can the estimation of the par production function parameters be based on the described equations and methods. Which method should be used for each parameter.
c. Is there any estimation method which could be suggested to be used to test in empirical estimation situations whether the production technology is of the CES or the par type.
d. Are there any situations where the par production technology should be preferred to CES technology in empirical estimation situations.

Of course these problems cannot be thoroughly analysed in this study. However, we expect answers to these questions by making some sampling experiments with Monte Carlo techniques.

All the analyses were done with the SURVO 84C statistical program[1]. The used hardwear was a Toshiba T5200/100 personal computer with PostScript and a HP Laserjet Series II printer.

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[1] See Mustonen (1987, 1988 I and II)

6.2 The design of the Monte Carlo data

In collecting our Monte Carlo data we artificially constructed 15 different data sets for each experiment, consisting of 50 observations. Each of the 15 data sets was generated with the same values of chosen production function parameters. The random term distribution parameters were varied between the different parameter combinations to create a reasonable comparing situation between the CES and par estimation results. This was done by trial and error. Within the 15 data sets all parameters including the random term dispersion and location parameter were kept constant. The three used parameter combinations are the following

A.	$p=-0.33333$	$a=-1$	$c=0.6$	$\delta=0.2$	$x \sim N(0,1)$
B.	$p=+0.66666$	$a=+2$	$c=-0.5$	$\delta=0.75$	$x \sim N(0,1)$
C.	$p=+2$	$a=+6$	$c=0.8$	$\delta=0.1$	$x \sim N(0,1)$

In generating the data we first randomized the variable $x=\ln(K/L)$ using the normal distribution as noted above. Then the variable K was randomized by the equation $K=2*e^{5*u}$, where u is rectangularly distributed between 0 and 1. After that L was calculated from $L=K*e^{-x}$. Values for the output variable were calculated in the CES case with

$$(6.1) \qquad YC = e^{g}*[\delta*K^{-p} + (1-\delta)*L^{-p}]^{(-1/p)}$$

where YC is the CES output and e^{g} is a log-normal random term. In the par case the output variable values were calculated with

$$(6.2) \qquad YP = e^{g}*[c*\frac{K^{-a} - L^{-a}}{(K/L)^{-a*c} - 1}]^{(-1/a)}$$

where YP is the par output and g is a random term as above.

The input price for L is set to be $W=10*e^{u}$, where u is rectangularly distributed between 0 and 1. For CES technology the input price for K is then calculated assuming the optimizing situation

$$(6.3) \qquad RC = W*[\delta/(1-\delta)]*e^{-(1+p)*x}$$

where RC is the input price for K in the CES case. In the par case the input price values for K were calculated with

$$(6.4) \qquad RP = W*(L/K)*[\frac{K^{-a} - YP^{-a}*(K/L)^{-a*c}}{YP^{-a}*(K/L)^{-a*c} - L^{-a}}]$$

In equation (6.4) the par output variable YP was calculated without the random term.

After these the income shares for L were calculated as

$$WLC = e^{g}*[W*L/(W*L+RC*K)]$$
$$WLP = e^{g}*[W*L/(W*L+RP*K)] \quad (6.5)$$

where WLC is the income share of L in the CES case and WLP in the par case respectively. The unit prices for the outputs were calculated by

$$fC = e^{g}*(W*L+RC*K)/YC$$
$$fP = e^{g}*(W*L+RP*K)/YP \quad (6.6)$$

where the letters C and P refer to the CES and par technology. Lastly we calculated the real wage variable values by

$$WfC = W/fC$$
$$WfP = W/fP \quad (6.7)$$

and the output per worker or labor productivity variable values by

$$VC = YC/L$$
$$VP = YP/L \quad (6.8)$$

The estimated equations were all estimated with each of the parameter combinations. The standardized normal distribution was used for variable x=ln(K/L) in the case of the linear approximation forms, the side relation estimation forms and in case of the ratio of the income shares forms while the higher value of the dispersion [$x \sim N(0,4)$] was additionally used in the case of nonlinear estimation of the share equations.

The random number generator used was given by the Survo 84C statistical system. Survo uses the C-language functions rand and srand in the microsoft C run-time library. This gives a series of pseudorandom numbers from various seeds. The seed number in each separate simulation situation was taken depending on current time and on the preceding random numbers. This guaranteed the fact that the produced random numbers were different in each trial. The normally distributed random numbers were produced with the help of the inverse distribution function of the normal distribution using the rectangularly distributed random numbers as parameters. This transformation function is supplied by the Survo 84C system.[1]

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[1] See Mustonen (1987), pp.44-45 and pp.103-104

6.3 Tests with the linear approximation forms

In tables 1-6 there are the estimation results from the sampling experiments which we made to estimate the equations (5.6) and (5.8).

The fit measured with the R^2 is equally good in both the CES and par case. The parameter values in combinations A and B were estimated consistently in both the CES and par case. When the value of the substitution parameter was increased the estimation results became biased: In combination C the substitution parameters p and a were both estimated to be smaller than the real parameter value was, the bias being bigger in the par case than in the CES case. The distribution parameter δ and the distribution limit parameter c was estimated quite consistently in all cases. However, when the substitution parameter had a high value (p=2), the distribution parameter δ was clearly biased in the CES case. See table 5.

The estimation results in tables 1-6 confirm the results of Thursby and Lovell about the biasness of the CES estimates given by the linear approximation method[1]. We can note the same fact concerning the par production function as well.
In tables 1 and 2 (parameter combination A) most of the second order coefficients are not significant. In tables 5 and 6 (parameter combination C), where the substitution parameters have the highest values, most of the second order coefficients in the CES case are significant (table 5), while in the par case they are not (table 6).

The real value of the degree of homogeneity was v=1, which was estimated quite consistently in all cases.

We calculated the coefficient of variation[2], the ratio of the estimated dispersion and the estimated mean, for each of the original parameters δ, p, c and a from the samples presented in tables 1-6. The results are below:

Parameter combination	Estimated values δ	V(δ)	p	V(p)	c	V(c)	a	V(a)
A	0.198	0.136	-0.368	-0.457	0.577	0.083	-0.881	-0.806
	(0.027)		(0.168)		(0.048)		(0.710)	
B	0.739	0.047	0.573	0.400	-0.511	-0.127	1.912	0.454
	(0.035)		(0.229)		(0.065)		(0.869)	
C	0.192	0.255	1.473	0.303	0.785	0.068	1.961	0.370
	(0.049)		(0.447)		(0.053)		(0.726)	

The dispersions of the original parameters are put in brackets above. The dispersions were calculated from the samples in tables 1-6. The coefficients of variation of the substitution parameters, V(p) and V(a), are smaller in the CES case than in the par case. In case of the parameters δ and c the coefficients of variation seem to be smaller and tolerably equal in size.

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[1] See Thursby and Lovell, pp.363-377
[2] See, for example, H.Cramer, pp.357-358

Table 1: CES linear approximation, parameter combination A:
Real values $p=-0.333$ $a=-1$ $c=0.6$ $\delta=0.2$
$x \sim N(0,1)$ $g \sim N(0,0.03)$

Sample	Estimated values $v*\delta$	disp.	v	δ	$(1/2)*v*p*\delta*(1-\delta)$	disp.	p	R^2
1	0.167	0.0240	0.952	0.175	-0.0269	0.0134	-0.392	0.990
2	0.229	0.0241	1.041	0.220	-0.0374	0.0147	-0.419	0.993
3	0.194	0.0361	1.021	0.190	-0.0446	0.0196	-0.479	0.991
4	0.165	0.0286	0.972	0.170	-0.0197	0.0197	-0.287	0.992
5	0.224	0.0269	1.010	0.222	-0.0254	0.0146	-0.291	0.989
6	0.160	0.0320	0.972	0.165	-0.0298	0.0168	-0.445	0.990
7	0.233	0.0342	1.021	0.228	-0.0463	0.0259	-0.515	0.985
8	0.223	0.0401	1.013	0.220	-0.0458	0.0244	-0.527	0.988
9	0.177	0.0406	1.007	0.176	-0.0386	0.0246	-0.529	0.990
10	0.173	0.0346	0.972	0.178	-0.0011	0.0226	-0.015	0.982
11	0.209	0.0233	1.025	0.204	-0.0472	0.0136	-0.567	0.995
12	0.219	0.0289	1.015	0.216	-0.0226	0.0232	-0.263	0.992
13	0.244	0.0332	1.007	0.242	-0.0234	0.0214	-0.253	0.993
14	0.155	0.0261	0.995	0.156	-0.0307	0.0178	-0.469	0.995
15	0.207	0.0244	1.015	0.204	-0.0057	0.0142	-0.069	0.991
Average	0.199	0.0305	1.003	0.198	-0.0297	0.0191	-0.368	0.990

Table 2: Par linear approximation, parameter combination A:
Real values $p=-0.333$ $a=-1$ $c=0.6$ $\delta=0.2$
$x \sim N(0,1)$ $g \sim N(0,0.03)$

Sample	Estimated values $v*(1-c)/2$	disp.	v	c	$a*v*(1-c^2)/24$	disp.	a	R^2
1	0.2480	0.0255	1.033	0.520	-0.0301	0.0144	-0.990	0.992
2	0.1896	0.0295	1.001	0.621	-0.0196	0.0186	-0.748	0.984
3	0.2130	0.0292	0.997	0.573	-0.0250	0.0225	-0.893	0.990
4	0.2180	0.0265	0.993	0.561	-0.0202	0.0130	-0.706	0.989
5	0.2141	0.0371	0.997	0.571	-0.0367	0.0195	-1.306	0.987
6	0.2401	0.0336	0.994	0.517	-0.0357	0.0156	-1.169	0.989
7	0.2135	0.0398	0.977	0.563	-0.0008	0.0312	-0.028	0.989
8	0.2229	0.0338	0.993	0.551	-0.0171	0.0200	-0.588	0.984
9	0.1912	0.0252	0.984	0.612	+0.0120	0.0157	+0.460	0.994
10	0.1718	0.0360	0.994	0.654	-0.0390	0.0266	-1.635	0.989
11	0.1691	0.0249	0.970	0.651	-0.0618	0.0193	-2.576	0.993
12	0.2480	0.0244	1.001	0.504	-0.0261	0.0137	-0.841	0.990
13	0.2308	0.0274	1.007	0.542	-0.0059	0.0189	-0.202	0.990
14	0.2071	0.0359	1.004	0.587	-0.0357	0.0190	-1.309	0.988
15	0.1821	0.0349	0.975	0.626	-0.0173	0.0165	-0.683	0.987
Average	0.2106	0.0309	0.995	0.577	-0.0239	0.0190	-0.881	0.989

Table 3: CES linear approximation, parameter combination B:
Real values p=0.666 a=2 c=-0.5 δ=0.75
$x \sim N(0,1)$ $g \sim N(0,0.03)$

Sample	Estimated values $v*\delta$	disp.	v	δ	$(1/2)*v*p*\delta*(1-\delta)$	disp.	p	R^2
1	0.781	0.027	1.001	0.780	0.0672	0.0152	0.783	0.989
2	0.715	0.037	0.998	0.716	0.0762	0.0225	0.751	0.989
3	0.731	0.028	1.003	0.729	0.0455	0.0172	0.459	0.990
4	0.799	0.027	1.028	0.777	0.0688	0.0161	0.772	0.992
5	0.675	0.034	0.992	0.681	0.0461	0.0184	0.428	0.989
6	0.758	0.033	1.011	0.750	0.0444	0.0251	0.468	0.990
7	0.739	0.036	1.003	0.737	0.0654	0.0227	0.673	0.988
8	0.701	0.024	0.984	0.695	0.0637	0.0173	0.610	0.990
9	0.735	0.033	0.999	0.736	0.0663	0.0200	0.683	0.987
10	0.767	0.031	0.992	0.774	0.0693	0.0203	0.799	0.988
11	0.771	0.040	0.997	0.773	0.1142	0.0286	1.306	0.985
12	0.770	0.035	1.001	0.769	0.0485	0.0191	0.545	0.989
13	0.694	0.021	0.966	0.718	0.0511	0.0103	0.523	0.989
14	0.676	0.037	0.993	0.681	0.0380	0.0195	0.352	0.984
15	0.753	0.028	0.985	0.765	0.0541	0.0133	0.611	0.988
Average	0.738	0.031	0.997	0.739	0.0633	0.0190	0.573	0.988

Table 4: Par linear approximation, parameter combination B:
Real values p=0.666 a=2 c=-0.5 δ=0.75
$x \sim N(0,1)$ $g \sim N(0,0.03)$

Sample	Estimated values $v*(1-c)/2$	disp.	v	c	$a*v*(1-c^2)/24$	disp.	a	R^2
1	0.8047	0.0365	1.027	-0.566	0.0727	0.0178	2.500	0.982
2	0.7689	0.0420	1.000	-0.537	0.0762	0.0306	2.573	0.983
3	0.7707	0.0267	1.015	-0.519	0.0572	0.0145	1.851	0.985
4	0.8217	0.0286	1.007	-0.632	0.0712	0.0177	2.846	0.990
5	0.7155	0.0296	0.990	-0.431	0.0493	0.0180	1.476	0.987
6	0.7511	0.0288	1.005	-0.495	0.0736	0.0226	2.338	0.986
7	0.7398	0.0292	1.002	-0.480	0.0457	0.0163	1.419	0.988
8	0.7096	0.0248	0.984	-0.443	0.0670	0.0180	2.000	0.988
9	0.7680	0.0336	0.995	-0.544	0.0228	0.0202	0.779	0.979
10	0.7366	0.0346	1.008	-0.461	0.0309	0.0184	0.943	0.987
11	0.8299	0.0366	1.026	-0.618	0.1047	0.0241	4.069	0.984
12	0.7624	0.0450	1.030	-0.481	0.0630	0.0237	1.968	0.983
13	0.6901	0.0276	0.986	-0.400	0.0308	0.0120	0.881	0.986
14	0.7564	0.0375	0.993	-0.524	0.0415	0.0140	1.372	0.983
15	0.7668	0.0284	0.997	-0.539	0.0495	0.0165	1.673	0.990
Average	0.7595	0.0326	1.004	-0.511	0.0571	0.0190	1.912	0.985

Table 5: CES linear approximation, parameter combination C:
Real values p=2 a=6 c=0.8 δ=0.1
$x \sim N(0,1)$ $g \sim N(0,0.03)$

Sample	Estimated values $v*\delta$	disp.	v	δ	$(1/2)*v*p*\delta*(1-\delta)$	disp.	p	R^2
1	0.342	0.033	1.008	0.339	0.2063	0.025	1.828	0.983
2	0.180	0.034	1.002	0.179	0.1436	0.018	1.950	0.986
3	0.187	0.036	1.002	0.186	0.0749	0.027	0.987	0.987
4	0.160	0.031	0.986	0.163	0.1445	0.023	2.149	0.990
5	0.254	0.037	1.047	0.242	0.1163	0.026	1.212	0.986
6	0.185	0.026	0.972	0.191	0.1155	0.015	1.537	0.989
7	0.152	0.034	0.979	0.155	0.0613	0.022	0.957	0.989
8	0.193	0.031	0.998	0.193	0.0960	0.021	1.235	0.987
9	0.227	0.033	1.020	0.222	0.1107	0.024	1.257	0.988
10	0.161	0.037	0.988	0.162	0.1479	0.023	2.201	0.990
11	0.187	0.040	0.979	0.191	0.0820	0.021	1.082	0.984
12	0.183	0.031	0.995	0.184	0.1707	0.019	2.285	0.989
13	0.129	0.031	1.004	0.129	0.0892	0.029	1.582	0.988
14	0.179	0.030	0.997	0.180	0.1286	0.022	1.749	0.985
15	0.164	0.027	0.981	0.167	0.1000	0.017	1.465	0.990
Average	0.192	0.033	0.997	0.192	0.1192	0.022	1.473	0.987

Table 6: Par linear approximation, parameter combination C:
Real values p=2 a=6 c=0.8 δ=0.1
$x \sim N(0,1)$ $g \sim N(0,0.03)$

Sample	Estimated values $v*(1-c)/2$	disp.	v	c	$a*v*(1-c^2)/24$	disp.	a	R^2
1	0.127	0.0346	0.988	0.743	0.0620	0.0247	3.326	0.990
2	0.073	0.0340	0.986	0.853	0.0187	0.0183	1.632	0.992
3	0.065	0.0345	1.000	0.870	0.0279	0.0196	2.746	0.988
4	0.136	0.0236	1.031	0.737	0.0263	0.0108	1.378	0.993
5	0.120	0.0470	1.016	0.765	0.0436	0.0347	2.521	0.989
6	0.129	0.0292	0.997	0.741	0.0294	0.0259	1.561	0.991
7	0.116	0.0291	1.002	0.768	0.0393	0.0221	2.298	0.986
8	0.124	0.0257	1.013	0.754	0.0290	0.0166	1.615	0.989
9	0.085	0.0304	0.972	0.825	0.0201	0.0148	1.547	0.988
10	0.124	0.0245	0.985	0.749	0.0257	0.0171	1.406	0.982
11	0.080	0.0246	0.991	0.838	0.0277	0.0131	2.234	0.990
12	0.129	0.0237	1.026	0.748	0.0285	0.0176	1.595	0.992
13	0.105	0.0291	0.973	0.784	0.0111	0.0077	0.692	0.993
14	0.060	0.0398	0.999	0.880	0.0296	0.0125	3.141	0.987
15	0.140	0.0283	1.004	0.722	0.0342	0.0132	1.715	0.993
Average	0.108	0.0305	0.999	0.785	0.0302	0.0179	1.961	0.990

6.4 The side relation estimation

In the side relation estimation situation we estimated the CES side relation equation

$$(6.9) \qquad \ln\left(\frac{W}{f}\right) = \ln(1-\delta) + (1+p)*\ln\left(\frac{Y}{L}\right)$$

and the corresponding par approximation form (5.14). The dispersion of the random term g was varied according to the parameter combination in order to set the R^2 for CES around 0.98-0.99. Then the same random term dispersion was used in the par case as well. The results are on the next three pages.

The fit in the CES case is better in all parameter combinations measured with R^2. On the basis of the results in tables 7-12 one can say that the side relation equation is not just the right one to be used, when one expects the production technology to be of the par type. The formulation (5.14) gives consistent results for low values of the substitution parameter but the fit is worse than in the CES case. That is why one can expect that in empirical estimation situations the estimation difficulties will increase.

In the CES case the estimation results seem to be unbiased even in the case of p=+2. In general, the dispersions of the par estimates are higher than the dispersions of the CES estimates. In the CES case these estimation results seem to be unbiased and consistent.

We calculated the coefficient of variation[1], the ratio of the estimated dispersion and the estimated mean, for each of the original parameters δ, p, c and a from the samples in tables 7-12. The results are below:

Parameter combination	Estimated values δ	V(δ)	p	V(p)	c	V(c)	a	V(a)
A	0.200	0.015	-0.331	-0.076	0.609	0.011	-0.938	-0.076
	(0.003)		(0.025)		(0.007)		(0.071)	
B	0.749	0.004	0.667	0.021	-0.535	-0.026	1.827	0.087
	(0.003)		(0.014)		(0.014)		(0.159)	
C	0.103	0.087	1.991	0.011	0.849	0.012	1.958	0.288
	(0.009)		(0.022)		(0.012)		(0.288)	

The dispersions of the original parameters are put in brackets above. The dispersions were calculated from the samples in tables 7-12. The higher the value of the par substitution parameter a, the higher the value of V(a), the coefficient of variation. The higher the value of the CES substitution parameter p, the smaller the value of V(p), the coefficient of variation. In case of the parameters δ and c the coefficients of variation seem to be tolerably equal in size.

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[1] See, for example, H.Cramer, pp.357-358

Table 7: CES side relation estimation, parameter combination A:
Real values p=-0.333 a=-1 c=0.6 δ=0.2
$x \sim N(0,1)$ $g \sim N(0,0.0003)$

Sample	Estimated values ln(1-δ)	disp.	δ	(1+p)	disp.	p	R^2
1	-0.2162	0.0024	0.194	0.655	0.0098	-0.345	0.989
2	-0.2245	0.0026	0.201	0.673	0.0155	-0.327	0.975
3	-0.2267	0.0022	0.203	0.650	0.0098	-0.350	0.989
4	-0.2214	0.0023	0.199	0.658	0.0102	-0.342	0.989
5	-0.2242	0.0025	0.201	0.675	0.0121	-0.325	0.984
6	-0.2235	0.0025	0.200	0.682	0.0125	-0.318	0.984
7	-0.2209	0.0026	0.198	0.683	0.0123	-0.317	0.985
8	-0.2210	0.0025	0.198	0.658	0.0101	-0.342	0.989
9	-0.2274	0.0025	0.203	0.669	0.0126	-0.331	0.983
10	-0.2233	0.0030	0.200	0.669	0.0164	-0.331	0.972
11	-0.2215	0.0024	0.199	0.686	0.0140	-0.314	0.980
12	-0.2219	0.0021	0.199	0.677	0.0088	-0.323	0.992
13	-0.2260	0.0026	0.202	0.669	0.0159	-0.331	0.974
14	-0.2263	0.0025	0.203	0.662	0.0103	-0.338	0.989
15	-0.2196	0.0026	0.197	0.669	0.0111	-0.331	0.987
Average	-0.2303	0.0025	0.200	0.669	0.0121	-0.331	0.984

Table 8: Par side relation estimation, parameter combination A:
Real values p=-0.333 a=-1 c=0.6 δ=0.2
$x \sim N(0,1)$ $g \sim N(0,0.0003)$

Sample	Estimated values ln[(1+c)/2]	disp.	c	(1+a/3)	disp.	a	R^2
1	-0.2139	0.0034	0.615	0.646	0.0173	-1.062	0.967
2	-0.2250	0.0025	0.597	0.709	0.0124	-0.873	0.986
3	-0.2173	0.0035	0.609	0.707	0.0176	-0.879	0.971
4	-0.2190	0.0028	0.607	0.721	0.0132	-0.837	0.984
5	-0.2174	0.0033	0.609	0.663	0.0158	-1.011	0.973
6	-0.2103	0.0031	0.621	0.670	0.0145	-0.990	0.978
7	-0.2240	0.0036	0.599	0.682	0.0160	-0.954	0.974
8	-0.2185	0.0037	0.607	0.695	0.0323	-0.915	0.970
9	-0.2103	0.0041	0.621	0.721	0.0153	-0.837	0.979
10	-0.2163	0.0031	0.611	0.644	0.0176	-1.068	0.965
11	-0.2173	0.0035	0.609	0.694	0.0162	-0.918	0.974
12	-0.2163	0.0030	0.611	0.688	0.0140	-0.936	0.981
13	-0.2156	0.0033	0.612	0.681	0.0152	-0.957	0.977
14	-0.2205	0.0036	0.604	0.694	0.0179	-0.918	0.969
15	-0.2165	0.0032	0.611	0.694	0.0166	-0.918	0.973
Average	-0.2172	0.0033	0.609	0.687	0.0168	-0.938	0.975

Table 9: CES side relation estimation, parameter combination B:
Real values p=0.666 a=2 c=-0.5 δ=0.75
$x \sim N(0,1)$ $g \sim N(0,0.005)$

Sample	Estimated values ln(1-δ)	disp.	δ	(1+p)	disp.	p	R^2
1	-1.3950	0.0122	0.752	1.685	0.0179	0.685	0.995
2	-1.3816	0.0119	0.749	1.677	0.0156	0.677	0.996
3	-1.3985	0.0131	0.753	1.668	0.0183	0.668	0.994
4	-1.3694	0.0112	0.746	1.685	0.0164	0.685	0.995
5	-1.4006	0.0117	0.754	1.657	0.0159	0.657	0.996
6	-1.3857	0.0133	0.750	1.640	0.0167	0.640	0.995
7	-1.3994	0.0130	0.753	1.673	0.0166	0.673	0.995
8	-1.3850	0.0101	0.750	1.680	0.0124	0.680	0.997
9	-1.3790	0.0123	0.748	1.658	0.0194	0.658	0.993
10	-1.3773	0.0124	0.748	1.667	0.0174	0.668	0.995
11	-1.3603	0.0128	0.743	1.646	0.0182	0.646	0.994
12	-1.3714	0.0136	0.746	1.679	0.0190	0.679	0.994
13	-1.3890	0.0129	0.751	1.671	0.0154	0.671	0.996
14	-1.3745	0.0135	0.747	1.668	0.0224	0.668	0.991
15	-1.3768	0.0129	0.748	1.654	0.0154	0.654	0.996
Average	-1.3829	0.0125	0.749	1.667	0.0171	0.667	0.995

Tabel 10: Par side relation estimation, parameter combination B:
Real values p=0.666 a=2 c=-0.5 δ=0.75
$x \sim N(0,1)$ $g \sim N(0,0.005)$

Sample	Estimated values ln[(1+c)/2]	disp.	c	(1+a/3)	disp.	a	R^2
1	-1.4288	0.0212	-0.521	1.689	0.0331	2.067	0.982
2	-1.4737	0.0261	-0.542	1.638	0.0349	1.914	0.979
3	-1.4666	0.0222	-0.539	1.599	0.0297	1.797	0.984
4	-1.4189	0.0180	-0.516	1.716	0.0265	2.148	0.989
5	-1.4401	0.0286	-0.526	1.705	0.0308	2.115	0.985
6	-1.4367	0.0169	-0.525	1.657	0.0264	1.971	0.988
7	-1.4472	0.0264	-0.530	1.635	0.0418	1.905	0.970
8	-1.4995	0.0227	-0.554	1.616	0.0250	1.848	0.989
9	-1.4287	0.0202	-0.521	1.525	0.0316	1.575	0.980
10	-1.4611	0.0195	-0.536	1.660	0.0242	1.980	0.990
11	-1.4689	0.0292	-0.540	1.712	0.0353	2.136	0.980
12	-1.4565	0.0252	-0.534	1.617	0.0330	1.851	0.980
13	-1.4924	0.0230	-0.550	1.697	0.0305	2.091	0.985
14	-1.5233	0.0225	-0.564	1.670	0.0259	2.010	0.989
15	-1.4353	0.0196	-0.524	1.595	0.0283	1.785	0.985
Average	-1.4585	0.0228	-0.535	1.609	0.0305	1.827	0.984

Table 11: CES side relation estimation, parameter combination C:
Real values p=2 a=6 c=0.8 δ=0.1
$x \sim N(0,1)$ $g \sim N(0,0.0005)$

Sample	Estimated values $\ln(1-\delta)$	disp.	δ	(1+p)	disp.	p	R^2
1	-0.1114	0.0079	0.105	3.030	0.0276	2.030	0.996
2	-0.1055	0.0077	0.111	2.981	0.0225	1.981	0.997
3	-0.1110	0.0085	0.105	2.981	0.0435	1.981	0.990
4	-0.1189	0.0083	0.112	2.991	0.0233	1.991	0.997
5	-0.1092	0.0071	0.103	2.983	0.0212	1.983	0.998
6	-0.0982	0.0086	0.094	2.966	0.0267	1.966	0.996
7	-0.1170	0.0084	0.110	2.971	0.0509	1.971	0.986
8	-0.1048	0.0070	0.100	2.981	0.0357	1.981	0.993
9	-0.1156	0.0068	0.109	2.971	0.0243	1.971	0.997
10	-0.1048	0.0074	0.099	3.027	0.0184	2.027	0.998
11	-0.0833	0.0063	0.080	3.030	0.0171	2.030	0.998
12	-0.1173	0.0065	0.111	2.979	0.0208	1.979	0.998
13	-0.1091	0.0075	0.103	3.010	0.0304	2.010	0.995
14	-0.0975	0.0090	0.092	2.972	0.0377	1.972	0.992
15	-0.1131	0.0074	0.107	2.989	0.0355	1.989	0.993
Average	-0.1078	0.0076	0.103	2.991	0.0290	1.991	0.995

Table 12: Par side relation estimation, parameter combination C:
Real values p=2 a=6 c=0.8 δ=0.1
$x \sim N(0,1)$ $g \sim N(0,0.0005)$

Sample	Estimated values $\ln[(1+c)/2]$	disp.	c	(1+a/3)	disp.	a	R^2
1	-0.0768	0.0089	0.852	1.709	0.0887	2.126	0.885
2	-0.0868	0.0089	0.834	1.706	0.0919	2.117	0.878
3	-0.0770	0.0099	0.852	1.736	0.0967	2.208	0.870
4	-0.0776	0.0103	0.851	1.624	0.0822	1.871	0.891
5	-0.0834	0.0112	0.840	1.558	0.0804	1.673	0.887
6	-0.0838	0.0111	0.839	1.612	0.1007	1.837	0.842
7	-0.0745	0.0112	0.856	1.719	0.0955	2.158	0.871
8	-0.0773	0.0108	0.851	1.651	0.0918	1.952	0.871
9	-0.0795	0.0104	0.847	1.538	0.0715	1.613	0.906
10	-0.0799	0.0112	0.846	1.703	0.1070	2.108	0.841
11	-0.0745	0.0089	0.856	1.827	0.0901	2.480	0.895
12	-0.0602	0.0104	0.883	1.771	0.0947	2.314	0.879
13	-0.0756	0.0106	0.847	1.489	0.0674	1.467	0.911
14	-0.0856	0.0108	0.836	1.548	0.0954	1.643	0.846
15	-0.0819	0.0098	0.843	1.603	0.0751	1.808	0.905
Average	-0.776	0.0103	0.849	1.653	0.0886	1.958	0.879

Equation (4.6) was estimated in the par case and in the CES case the corresponding equation

$$(6.10) \qquad \ln\left(\frac{w_K}{w_L}\right) = \ln[\delta/(1-\delta)] - p*\ln(K/L)$$

was estimated[1]. The estimation results can be found on the next three pages. See tables 13-18. In general, the parameter estimates of the par production function are better than in the previous two cases. However, here the estimates are also somewhat biased when the values of the real substitution parameters (p and a) are increased.

The R^2 measure in tables 13-18, where the estimation results of the first order equation are presented, seems to suggest that the fit in the par case is worse than in the CES case. Here the dispersion values of the parameter estimates are higher in the par case than in the CES case.

Tables 19, 21 and 23 show the estimation results of the equation (4.6) when the second order term is included in the equation. The reliability of the estimates is clearly increased, when the second order term is added to the regression equation.

Tables 20, 22 and 24 present the corresponding estimation results for the CES production regressand when the second order term is added to the regression equation. This estimation is done with the same samples than we used in tables 19, 21 and 23. Theoretically the coefficient for the second order term should be zero in the case of the CES production function. This is what we found out in the samples as well. However, in 12 out of the 45 CES estimations presented in tables 20, 22 and 24, the second order coefficient in the CES equation was found significantly (1 % level) different from zero.
In the par case the corresponding significance level was practically 0% in all cases and thus in every case we found the second order coefficient different from zero.

When degrees in the regression were still increased (third and fourth degrees in the regression included) the bias in the estimates of the parameter a was vanished totally. These estimations are not reported in this study.

In the estimation there were no restrictions put on the coefficients of the second order terms.

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[1] This CES estimation method is introduced, for example, by Wallis (1973), pp.56-62

We estimated totally 100 samples for each of the reported parameter combinations using the second order equation in order to study the specification error with this method. The averages calculated from these estimations are below:

Par income share ratio estimation, second order equation (n=100)

Averages

Parameter comb.	$\ln(\frac{1-c}{1+c})$	disp.	c	(a/3)	disp.	a	$-(\frac{c}{18})*a^2$	disp.	R^2	Number of not signif. second order coeff.[1]
A	-1.3860	0.0069	+0.600	-0.329	0.0058	-0.987	-0.031	0.0044	0.987	None
B	+1.1040	0.0137	-0.502	+0.644	0.0118	+1.912	-0.095	0.0090	0.986	None
C	-2.5230	0.1041	+0.852	+1.709	0.0927	+5.127	-0.463	0.0726	0.883	None

CES income share ratio estimation, second order equation (n=100)

Averages

Parameter comb.	$\ln(\frac{\delta}{1-\delta})$	disp.	δ	p	disp.	second order coeff.	disp.	R^2	Number of significant second order coefficents[1]
A	-1.3863	0.0067	0.200	-0.329	0.0056	+0.0012	0.0042	0.990	6 out of 100
B	+1.0990	0.0139	0.750	+0.666	0.0121	+0.0000	0.0091	0.986	8 out of 100
C	-2.2168	0.0464	0.098	+1.995	0.0391	+0.0320	0.0305	0.985	34 out of 100

The theory suggests that the second order coefficient should be negative in case of the par production technology. Out of the 6, 8 and 34 (referring to parameter combinations A, B and C) significant second order coefficients in CES case only 1, 3 and 6 cases were significant and negative. Thus tested against the alternative par technology the CES estimation results are quite discriminative.

On the basis of the results in tables 19-24 we argue that one reliable method for discriminating the CES and par production technologies is to estimate both the first and second order equation for the log of the ratio of the distributive shares. When the first order equation fits better, the technique is of the CES type. When the second order equation fits better (judged by the statistical methods of regression fitting), the technique is of the par type.

In general, it can be noted that our results with this method seem to be more precise and reliable than can be achieved with the general translog approximation method in the CES approximation situation[2].

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[1] Tested on the 1% risk level

[2] Compare with the estimation results reported by Guilkey and Lovell, pp.137-147.

Table 13: CES income share ratio estimation, parameter combination A: Real values $p=-0.333$ $a=-1$ $c=0.6$ $\delta=0.2$ $x \sim N(0,1)$ $g \sim N(0,0.00005)$

Sample	Estimated values $\ln[\delta/(1-\delta)]$	disp.	δ	p	disp.	R^2
1	-1.3729	0.0054	0.202	-0.326	0.0052	0.987
2	-1.3840	0.0046	0.200	-0.330	0.0051	0.989
3	-1.3941	0.0054	0.199	-0.338	0.0059	0.985
4	-1.3953	0.0053	0.199	-0.338	0.0062	0.984
5	-1.3903	0.0069	0.199	-0.335	0.0073	0.978
6	-1.3900	0.0061	0.199	-0.337	0.0055	0.987
7	-1.3765	0.0048	0.202	-0.326	0.0052	0.988
8	-1.3820	0.0054	0.201	-0.333	0.0060	0.985
9	-1.3882	0.0045	0.200	-0.337	0.0047	0.991
10	-1.3961	0.0052	0.198	-0.340	0.0057	0.986
11	-1.3805	0.0052	0.201	-0.325	0.0050	0.989
12	-1.3860	0.0060	0.200	-0.330	0.0051	0.989
13	-1.3800	0.0061	0.201	-0.334	0.0054	0.987
14	-1.3938	0.0056	0.199	-0.334	0.0050	0.989
15	-1.3846	0.0050	0.200	-0.334	0.0057	0.986
Average	-1.3863	0.0054	0.200	-0.333	0.0055	0.987

Table 14: Par income share ratio estimation, parameter combination A: Real values $p=-0.333$ $a=-1$ $c=0.6$ $\delta=0.2$ $x \sim N(0,1)$ $g \sim N(0,0.00005)$

Sample	Estimated values $\ln[(1-c)/(1+c)]$	disp.	c	(a/3)	disp.	a	R^2
1	-1.4283	0.0102	0.613	-0.323	0.0091	-0.969	0.963
2	-1.4210	0.0070	0.611	-0.333	0.0072	-0.999	0.978
3	-1.4042	0.0065	0.606	-0.318	0.0066	-0.954	0.980
4	-1.4203	0.0082	0.611	-0.345	0.0088	-1.035	0.970
5	-1.4312	0.0076	0.614	-0.330	0.0069	-0.990	0.980
6	-1.3993	0.0060	0.604	-0.329	0.0080	-0.987	0.972
7	-1.4185	0.0070	0.610	-0.300	0.0067	-0.900	0.976
8	-1.4243	0.0085	0.612	-0.328	0.0083	-0.984	0.970
9	-1.4267	0.0102	0.613	-0.329	0.0093	-0.987	0.963
10	-1.4233	0.0090	0.612	-0.342	0.0091	-1.026	0.967
11	-1.4174	0.0072	0.610	-0.333	0.0072	-0.999	0.978
12	-1.4268	0.0107	0.613	-0.316	0.0100	-0.948	0.955
13	-1.4272	0.0074	0.613	-0.335	0.0071	-1.005	0.979
14	-1.4254	0.0091	0.612	-0.383	0.0080	-1.149	0.979
15	-1.4197	0.0092	0.611	-0.347	0.0086	-1.041	0.971
Average	-1.4209	0.0083	0.611	-0.333	0.0081	-0.998	0.972

Table 15: CES income share ratio estimation, parameter combination B: Real values $p=0.666$ $a=2$ $c=-0.5$ $\delta=0.75$ $x \sim N(0,1)$ $g \sim N(0,0.003)$

Sample	Estimated values $\ln[\delta/(1-\delta)]$	disp.	δ	p	disp.	R^2
1	1.0973	0.0139	0.750	0.663	0.0137	0.980
2	1.0925	0.0126	0.749	0.674	0.0130	0.983
3	1.0989	0.0107	0.750	0.663	0.0122	0.984
4	1.1013	0.0116	0.750	0.666	0.0115	0.986
5	1.0807	0.0104	0.747	0.680	0.0104	0.989
6	1.0961	0.0109	0.750	0.660	0.0129	0.982
7	1.1050	0.0092	0.751	0.657	0.0080	0.993
8	1.0900	0.0142	0.748	0.689	0.0141	0.980
9	1.1114	0.0109	0.752	0.657	0.0109	0.986
10	1.1050	0.0116	0.751	0.644	0.0128	0.981
11	1.0939	0.0104	0.749	0.664	0.0098	0.990
12	1.1044	0.0101	0.751	0.662	0.0104	0.988
13	1.0884	0.0112	0.748	0.674	0.0136	0.981
14	1.1331	0.0105	0.757	0.675	0.0095	0.991
15	1.1009	0.0099	0.750	0.675	0.0111	0.987
Average	1.0999	0.0112	0.750	0.667	0.0116	0.985

Table 16: Par income share ratio estimation, parameter combination B: Real values $p=0.666$ $a=2$ $c=-0.5$ $\delta=0.75$ $x \sim N(0,1)$ $g \sim N(0,0.003)$

Sample	Estimated values $\ln[(1-c)/(1+c)]$	disp.	c	(a/3)	disp.	a	R^2
1	1.2194	0.0185	-0.544	0.634	0.0185	1.902	0.961
2	1.2161	0.0205	-0.543	0.658	0.0194	1.974	0.960
3	1.2053	0.0224	-0.539	0.688	0.0206	2.064	0.959
4	1.2238	0.0323	-0.545	0.598	0.0272	1.794	0.910
5	1.2027	0.0178	-0.538	0.616	0.0185	1.848	0.956
6	1.2121	0.0203	-0.541	0.641	0.0194	1.923	0.958
7	1.1850	0.0215	-0.532	0.618	0.0236	1.854	0.935
8	1.1786	0.0222	-0.529	0.662	0.0218	1.986	0.951
9	1.1824	0.0175	-0.531	0.603	0.0189	1.809	0.955
10	1.2164	0.0217	-0.548	0.602	0.0225	1.806	0.937
11	1.1942	0.0210	-0.535	0.631	0.0220	1.893	0.945
12	1.2274	0.0268	-0.547	0.572	0.0246	1.716	0.919
13	1.2070	0.0277	-0.540	0.664	0.0260	1.992	0.931
14	1.1789	0.0175	-0.529	0.650	0.0178	1.950	0.965
15	1.2201	0.0207	-0.544	0.622	0.0202	1.866	0.952
Average	1.2046	0.219	-0.539	0.631	0.0214	1.892	0.946

Table 17: CES income share ratio estimation, parameter combination C: Real values p=2 a=6 c=0.8 δ=0.1 $x \sim N(0,1)$ $g \sim N(0,0.0001)$

Sample	Estimated values $\ln[\delta/(1-\delta)]$	disp.	δ	p	disp.	R^2
1	-2.2976	0.0319	0.091	2.113	0.0368	0.987
2	-2.1314	0.0559	0.106	1.891	0.0543	0.966
3	-2.1695	0.0536	0.103	1.921	0.0526	0.967
4	-2.1130	0.0485	0.108	1.811	0.0518	0.962
5	-2.0733	0.0860	0.112	1.761	0.0831	0.913
6	-2.2740	0.0527	0.093	2.062	0.0672	0.956
7	-2.1677	0.1005	0.103	1.855	0.1002	0.886
8	-2.2289	0.0780	0.097	2.043	0.0762	0.944
9	-2.0812	0.0661	0.112	1.769	0.0629	0.945
10	-2.1729	0.0855	0.102	1.932	0.0821	0.925
11	-2.1194	0.1115	0.107	1.840	0.1071	0.863
12	-2.4850	0.1222	0.077	2.285	0.1303	0.867
13	-2.0400	0.0736	0.115	1.637	0.0723	0.916
14	-2.1051	0.0437	0.109	1.913	0.0436	0.976
15	-2.1188	0.0675	0.107	1.850	0.0733	0.933
Average	-2.1719	0.0718	0.103	1.910	0.0729	0.934

Table 18: Par income share ratio estimation, parameter combination C: Real values p=2 a=6 c=0.8 δ=0.1 $x \sim N(0,1)$ $g \sim N(0,0.0001)$

Sample	Estimated values $\ln[(1-c)/(1+c)]$	disp.	c	(a/3)	disp.	a	R^2
1	-2.8376	0.1115	0.889	1.414	0.1180	4.242	0.764
2	-2.6128	0.1059	0.863	0.981	0.0913	2.943	0.748
3	-2.9859	0.1413	0.904	1.256	0.1289	3.768	0.688
4	-2.7223	0.1066	0.875	1.041	0.1070	3.123	0.668
5	-2.7411	0.0915	0.879	1.230	0.0989	3.690	0.775
6	-2.8262	0.1186	0.888	1.140	0.1168	3.420	0.670
7	-2.7259	0.0931	0.877	1.402	0.1077	4.206	0.790
8	-2.8345	0.1110	0.889	1.270	0.1098	3.810	0.766
9	-2.8453	0.1313	0.890	1.225	0.1429	3.675	0.642
10	-2.6658	0.1153	0.870	1.122	0.1156	3.366	0.696
11	-2.7398	0.1143	0.879	1.161	0.1163	3.483	0.684
12	-2.9302	0.1389	0.899	1.605	0.1568	4.815	0.709
13	-2.8655	0.1059	0.892	1.311	0.1135	3.933	0.748
14	-2.7376	0.1071	0.878	1.370	0.1252	4.110	0.718
15	-2.9633	0.1493	0.901	1.373	0.1416	4.125	0.667
Average	-2.8023	0.1161	0.885	1.260	0.1194	3.781	0.716

Table 19: Par income share ratio estimation, second order equation, parameter combination A: Real values p=-0.333 a=-1 c=0.6 δ=0.2 x~N(0,1) g~N(0,0.00005)

Sample	Estimated values $\ln(\frac{1-c}{1+c})$	disp.	c	(a/3)	disp.	a	$-(\frac{c}{18})*a^2$	disp.	R^2
1	-1.3921	0.0071	0.602	-0.324	0.0056	-0.972	-0.021	0.0038	0.987
2	-1.3883	0.0071	0.601	-0.331	0.0047	-0.993	-0.036	0.0044	0.991
3	-1.3886	0.0066	0.601	-0.324	0.0052	-0.972	-0.030	0.0037	0.988
4	-1.3914	0.0078	0.602	-0.334	0.0055	-1.002	-0.032	0.0033	0.988
5	-1.3953	0.0067	0.603	-0.319	0.0045	-0.957	-0.024	0.0046	0.991
6	-1.3885	0.0073	0.601	-0.324	0.0078	-0.972	-0.032	0.0060	0.980
7	-1.3747	0.0056	0.596	-0.329	0.0050	-0.987	-0.037	0.0042	0.989
8	-1.3764	0.0072	0.597	-0.321	0.0065	-0.963	-0.035	0.0047	0.987
9	-1.3924	0.0065	0.602	-0.330	0.0052	-0.990	-0.033	0.0042	0.989
10	-1.3811	0.0071	0.598	-0.315	0.0059	-0.945	-0.021	0.0043	0.984
11	-1.3657	0.0087	0.593	-0.333	0.0086	-0.999	-0.045	0.0055	0.986
12	-1.3827	0.0074	0.598	-0.332	0.0052	-0.996	-0.032	0.0037	0.990
13	-1.3764	0.0055	0.597	-0.323	0.0041	-0.969	-0.029	0.0023	0.992
14	-1.3883	0.0058	0.601	-0.323	0.0051	-0.969	-0.030	0.0038	0.990
15	-1.3838	0.0068	0.599	-0.325	0.0049	-0.975	-0.029	0.0028	0.990
Average	-1.3844	0.0069	0.599	-0.326	0.0059	-0.977	-0.031	0.0041	0.988

Table 20: CES income share ratio estimation, second order equation, parameter combination A: Real values p=-0.333 a=-1 c=0.6 δ=0.2 x~N(0,1) g~N(0,0.00005)

Sample	Estimated values $\ln(\frac{\delta}{1-\delta})$	disp.	δ	p	disp.	second order coeff.	disp.	R^2
1	-1.3903	0.0069	0.199	-0.328	0.0054	0.0097	0.0037	0.987
2	-1.3882	0.0068	0.200	-0.333	0.0046	*-0.0033	0.0043	0.992
3	-1.3884	0.0065	0.200	-0.327	0.0051	*-0.0022	0.0037	0.989
4	-1.3910	0.0074	0.199	-0.337	0.0052	* 0.0000	0.0031	0.989
5	-1.3942	0.0065	0.199	-0.322	0.0044	* 0.0077	0.0045	0.992
6	-1.3887	0.0072	0.200	-0.325	0.0077	* 0.0019	0.0058	0.984
7	-1.3748	0.0055	0.202	-0.331	0.0050	*-0.0038	0.0042	0.990
8	-1.3760	0.0069	0.202	-0.324	0.0063	* 0.0038	0.0045	0.986
9	-1.3925	0.0064	0.199	-0.333	0.0051	* 0.0000	0.0041	0.989
10	-1.3798	0.0067	0.201	-0.320	0.0055	0.0095	0.0040	0.986
11	-1.3654	0.0083	0.203	-0.334	0.0083	0.0127	0.0053	0.984
12	-1.3828	0.0072	0.201	-0.334	0.0051	*-0.0005	0.0036	0.990
13	-1.3759	0.0052	0.202	-0.328	0.0039	*-0.0024	0.0022	0.994
14	-1.3885	0.0057	0.200	-0.326	0.0050	* 0.0034	0.0037	0.991
15	-1.3835	0.0063	0.201	-0.330	0.0046	* 0.0019	0.0026	0.991
Average	-1.3840	0.0066	0.201	-0.329	0.0054	0.0023	0.0040	0.923

* = The coefficient is not significant on the 1% risk level

Table 21: Par income share ratio estimation, second order equation, parameter combination B: Real values p=0.666
a=2 c=-0.5 δ=0.75 x~N(0,1) g~N(0,0.003)

Sample	Estimated values $\ln(\frac{1-c}{1+c})$	disp.	c	(a/3)	disp.	a	$-(\frac{c}{18})*a^2$	disp.	R^2
1	1.0909	0.0122	-0.497	0.641	0.0110	1.923	0.017	0.0075	0.989
2	1.0981	0.0150	-0.500	0.669	0.0149	2.007	0.090	0.0115	0.981
3	1.1247	0.0132	-0.510	0.635	0.0106	1.905	0.098	0.0049	0.987
4	1.1028	0.0112	-0.502	0.663	0.0132	1.989	0.082	0.0092	0.989
5	1.1123	0.0151	-0.505	0.633	0.0117	1.899	0.086	0.0095	0.985
6	1.0980	0.0096	-0.500	0.621	0.0093	1.863	0.102	0.0048	0.990
7	1.1094	0.0150	-0.504	0.643	0.0124	1.929	0.101	0.0085	0.984
8	1.1008	0.0107	-0.501	0.646	0.0090	1.938	0.090	0.0066	0.993
9	1.1205	0.0130	-0.508	0.646	0.0103	1.938	0.097	0.0080	0.989
10	1.0960	0.0123	-0.499	0.632	0.0118	1.896	0.090	0.0067	0.989
11	1.1094	0.0139	-0.504	0.624	0.0113	1.872	0.104	0.0079	0.990
12	1.0983	0.0124	-0.500	0.632	0.0106	1.896	0.102	0.0078	0.989
13	1.1166	0.0161	-0.507	0.654	0.0134	1.962	0.091	0.0116	0.986
14	1.1063	0.0146	-0.503	0.619	0.0110	1.857	0.111	0.0111	0.985
15	1.1127	0.0132	-0.505	0.643	0.0119	1.929	0.099	0.0072	0.980
Average	1.1806	0.0132	-0.503	0.640	0.0115	1.920	0.091	0.0082	0.987

Table 22: CES income share ratio estimation, second order equation, parameter combination B: Real values p=0.666
a=2 c=-0.5 δ=0.75 x~N(0,1) g~N(0,0.003)

Sample	Estimated values $\ln(\frac{\delta}{1-\delta})$	disp.	δ	p	disp.	second order coeff.	disp.	R^2
1	1.0838	0.0115	0.747	0.665	0.0104	* 0.012	0.0071	0.989
2	1.0931	0.0154	0.749	0.696	0.0153	*-0.007	0.0118	0.979
3	1.1200	0.0157	0.754	0.660	0.0126	* 0.002	0.0058	0.987
4	1.0910	0.0119	0.749	0.679	0.0140	*-0.003	0.0098	0.986
5	1.1127	0.0160	0.753	0.656	0.0124	*-0.020	0.0101	0.985
6	1.0914	0.0090	0.749	0.649	0.0087	0.012	0.0045	0.992
7	1.1017	0.0150	0.751	0.666	0.0124	* 0.004	0.0085	0.984
8	1.0943	0.0116	0.749	0.674	0.0097	*-0.004	0.0071	0.991
9	1.1139	0.0135	0.753	0.672	0.0107	*-0.002	0.0083	0.988
10	1.0848	0.0122	0.747	0.664	0.0117	* 0.003	0.0067	0.988
11	1.1024	0.0143	0.751	0.650	0.0117	* 0.010	0.0081	0.988
12	1.0892	0.0123	0.748	0.655	0.0105	* 0.009	0.0077	0.989
13	1.1113	0.0170	0.752	0.677	0.0142	*-0.007	0.0123	0.983
14	1.1029	0.0147	0.751	0.634	0.0111	* 0.010	0.0111	0.986
15	1.1185	0.0143	0.754	0.669	0.0164	-0.027	0.0104	0.980
Average	1.1007	0.0136	0.750	0.664	0.0121	-0.001	0.0086	0.986

* = The coefficient is not significant on the 1% risk level

Table 23: Par income share ratio estimation, second order equation, parameter combination C: Real values p=2
a=6 c=0.8 δ=0.1 x~N(0,1) g~N(0,0.00001)

Sample	Estimated values $\ln(\frac{1-c}{1+c})$	disp.	c	(a/3)	disp.	a	$-(\frac{c}{18})*a^2$	disp.	R^2
1	-2.4833	0.0953	0.846	1.432	0.0990	4.296	-0.231	0.0652	0.847
2	-2.8533	0.1933	0.891	1.887	0.1503	5.661	-0.442	0.1460	0.793
3	-2.8006	0.1605	0.885	1.454	0.1397	4.362	-0.189	0.0740	0.742
4	-2.3761	0.0520	0.830	1.891	0.0588	5.673	-0.813	0.0591	0.960
5	-2.5100	0.0659	0.850	1.698	0.0621	5.094	-0.473	0.0425	0.945
6	-2.5787	0.1067	0.859	1.552	0.1064	4.656	-0.320	0.0898	0.842
7	-2.4295	0.1258	0.838	2.059	0.1016	6.177	-0.823	0.0823	0.899
8	-2.4625	0.0580	0.843	1.649	0.0521	4.947	-0.559	0.0472	0.957
9	-2.6333	0.1267	0.866	1.715	0.1066	5.145	-0.438	0.0712	0.859
10	-2.5129	0.0891	0.850	1.708	0.0706	5.124	-0.561	0.0599	0.930
11	-2.4988	0.1109	0.848	1.815	0.0839	5.445	-0.603	0.0728	0.921
12	-2.5620	0.1083	0.857	1.558	0.0924	4.674	-0.407	0.0633	0.874
13	-2.7999	0.1486	0.885	1.565	0.1270	4.695	-0.171	0.0827	0.808
14	-2.4534	0.0886	0.842	1.932	0.0949	5.796	-0.694	0.0659	0.902
15	-2.6091	0.1334	0.863	1.551	0.1082	4.653	-0.389	0.0779	0.827
Average	-2.5709	0.1109	0.857	1.698	0.0969	5.093	-0.474	0.0733	0.874

Table 24: CES income share ratio estimation, second order equation, parameter combination C: Real values p=2
a=6 c=0.8 δ=0.1 x~N(0,1) g~N(0,0.00001)

Sample	Estimated values $\ln(\frac{\delta}{1-\delta})$	disp.	δ	p	disp.	second order coeff.	disp.	R^2
1	-2.2393	0.0387	0.096	1.863	0.0393	1.140	0.0264	0.980
2	-2.2906	0.0516	0.092	2.036	0.0367	* 0.049	0.0371	0.985
3	-2.2885	0.0479	0.092	1.934	0.0385	0.082	0.0213	0.982
4	-2.1551	0.0404	0.104	2.151	0.0353	-0.154	0.0345	0.988
5	-2.2004	0.0133	0.100	1.938	0.0125	0.037	0.0086	0.998
6	-2.2490	0.0379	0.095	1.963	0.0356	0.081	0.0294	0.986
7	-2.2108	0.0275	0.099	1.957	0.0222	* 0.033	0.0180	0.995
8	-2.1948	0.0306	0.100	2.065	0.0215	*-0.039	0.0151	0.995
9	-2.1780	0.0720	0.102	2.078	0.0590	*-0.068	0.0544	0.964
10	-2.1792	0.0469	0.102	2.025	0.0350	*-0.029	0.0290	0.986
11	-2.2204	0.0269	0.098	1.952	0.0201	0.044	0.0176	0.995
12	-2.2315	0.0806	0.097	2.026	0.0659	* 0.006	0.0462	0.954
13	-2.3266	0.0526	0.089	1.975	0.0434	0.144	0.0319	0.979
14	-2.2066	0.0188	0.099	1.973	0.0180	* 0.020	0.0128	0.997
15	-2.2192	0.0463	0.098	1.966	0.0315	* 0.032	0.0225	0.988
Average	-2.2260	0.0421	0.098	1.993	0.0343	0.092	0.0270	0.985

* = The coefficient is not significant on the 1% risk level

6.6 Nonlinear estimation of the share equations

In the estimation of the share equations we have here used the Davidon-Fletcher-Powell method, which is a quite usual method for nonlinear estimation with good convergency properties[1].

The estimated equations were written in regression in the following forms. The CES type share equation was

$$(6.11) \qquad w_L = \frac{1 - \delta}{(1-\delta) + \delta*(K/L)^{-p}}$$

and the par type share equation was

$$(6.12) \qquad w_L = c + \frac{1}{1 - (K/L)^{-a}} + \frac{c}{(K/L)^{-a*c} - 1}$$

which are both income shares of L when a><0 and c><0. No difficulties were found with the discontinuing points of the denominators of (6.12).

The estimation was done in two stages: In the first stage we estimated both share equation types for both of the real regressands and chose the best share function form for both CES w_L and par w_L using R^2 as a criteria. In the second stage we solved the parameters for both share function types using the best regression form.

The results on the next six pages seem to suggest that, generally, the share function structure in CES and par technology is so different, especially on basis of the nature of the distribution limits in these two production techniques, that the income shares need to be handled with the right functional forms to be reliably estimated. In all but one of the 90 samples reported in tables 25-30, the R^2 qriteria gave the right functional form for the regressands. The conclusion here is that the analysis of the income shares (or shares in other contexts) can serve as a basis for discriminating the CES and par techniques from each other.[2]

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[1] See Walsh, pp.110-120

[2] The discrimination can be done statistically, for example, by means of the Pesaran test and the comprehensive classical F test, see Harvey (1977), pp.464-471. Several nonparametric tests have been used by Ramsey and Zarembka (1971), pp.471-477.

We estimated totally 100 samples for each of the reported parameter combinations in order to study the specification error in this kind of testing. The averages calculated from the estimations are below:

Parameter combination		R^2 values when the real regressand CES-type: Share function CES-type	Share function par-type	par-type: Share function CES-type	Share function par-type	Number of misspecifications
A	$x \sim N(0,1)$	0.977	0.960	0.960	0.973	2 out of 100
	$x \sim N(0,4)$	0.949	0.888	0.882	0.924	0 out of 100
B	$x \sim N(0,1)$	0.995	0.953	0.954	0.995	0 out of 100
	$x \sim N(0,4)$	0.970	0.869	0.860	0.965	0 out of 100
C	$x \sim N(0,1)$	0.986	0.691	0.653	0.882	0 out of 100
	$x \sim N(0,4)$	0.995	0.770	0.619	0.907	0 out of 100

		Estimates from the best regressions (averages): Best CES-type		Best par-type	
		p	δ	a	c
A	$x \sim N(0,1)$	-0.333	0.200	-1.000	0.600
	$x \sim N(0,4)$	-0.333	0.201	-1.000	0.600
B	$x \sim N(0,1)$	0.665	0.750	1.999	-0.500
	$x \sim N(0,4)$	0.667	0.750	2.002	-0.500
C	$x \sim N(0,1)$	1.998	0.100	6.007	0.801
	$x \sim N(0,4)$	1.995	0.101	6.010	0.800

The discriminating properties were better when the real values of the substitution parameters p and a were increased. The specification error practically vanished with the higher values of the substitution parameters (meaning low substitution). This is due to the fact that the income shares are constants in the case of the Cobb-Douglas production function and nearly constants in the neighbourhood of $a \approx 0$ and $p \approx 0$. With higher values of parameters p and a the dispersion of the random term could be increased. The increasing of the variance of $x=\ln(K/L)$ gave us the possibility to increase the random term dispersion as well. The used random term dispersions correspond well to the fact that the regressands are income shares limited from 0 to 1.

In all nonlinear regressions presented the estimated coefficients were significant on the 0.1 % risk level.

Table 25: Nonlinear estimation of the share equations, parameter combination A: Real values p=-0.333 a=-1 c=0.6 δ=0.2 x~N(0,1) g~N(0,0.0001)

	R^2 values when the real regressand				Estimates from the best regressions (marked with *)			
	CES-type		par-type		Best CES-type		Best par-type	
Sample	Share function CES-type	Share function par-type	Share function CES-type	Share function par-type	p	δ	a	c
1	*0.984	0.947	0.964	*0.977	-0.331	0.201	-0.977	0.599
2	*0.977	0.933	0.949	*0.969	-0.322	0.203	-0.939	0.599
3	*0.984	0.951	0.961	*0.980	-0.343	0.196	-0.977	0.602
4	*0.967	0.951	0.959	*0.964	-0.318	0.201	-0.944	0.594
5	*0.967	0.955	0.954	*0.965	-0.314	0.202	-0.935	0.597
6	*0.959	0.939	*0.954	0.953	-0.329	0.200	-0.914	0.604
7	*0.954	0.938	0.937	*0.948	-0.344	0.198	-1.035	0.602
8	*0.984	0.960	0.973	*0.980	-0.340	0.199	-1.014	0.602
9	*0.982	0.959	0.971	*0.971	-0.340	0.198	-1.015	0.604
10	*0.982	0.969	0.973	*0.981	-0.328	0.198	-0.980	0.605
11	*0.979	0.948	0.956	*0.975	-0.326	0.200	-0.967	0.600
12	*0.971	0.956	0.937	*0.968	-0.339	0.197	-1.027	0.605
13	*0.983	0.964	0.958	*0.979	-0.340	0.199	-1.030	0.602
14	*0.990	0.979	0.973	*0.988	-0.326	0.200	-0.975	0.602
15	*0.971	0.964	0.958	*0.968	-0.326	0.200	-0.979	0.600
Average	0.976	0.954	0.958	0.971	-0.331	0.199	-0.984	0.601

Table 26: Nonlinear estimation of the share equations, parameter combination A: Real values p=-0.333 a=-1 c=0.6 δ=0.2 x~N(0,4) g~N(0,0.001)

	R^2 values when the real regressand				Estimates from the best regressions (marked with *)			
	CES-type		par-type		Best CES-type		Best par-type	
Sample	Share function CES-type	Share function par-type	Share function CES-type	Share function par-type	p	δ	a	c
1	*0.974	0.916	0.922	*0.956	-0.347	0.195	-1.071	0.605
2	*0.920	0.884	-	*0.899	-0.316	0.205	-0.937	0.593
3	*0.953	0.908	0.923	*0.933	-0.316	0.205	-0.911	0.594
4	*0.975	0.873	0.866	*0.946	-0.329	0.199	-0.984	0.604
5	*0.962	0.911	0.891	*0.941	-0.352	0.194	-1.109	0.609
6	*0.962	0.912	0.869	*0.940	-0.328	0.199	-1.004	0.604
7	*0.957	0.922	0.899	*0.938	-0.329	0.204	-0.999	0.593
8	*0.949	-	0.861	*0.924	-0.337	0.201	-1.043	0.599
9	*0.892	0.856	0.829	*0.853	-0.342	0.199	-1.041	0.601
10	*0.966	0.923	0.902	*0.945	-0.319	0.205	-0.929	0.593
11	*0.948	0.877	0.898	*0.917	-0.353	0.238	-1.061	0.610
12	*0.972	0.871	-	*0.951	-0.329	0.199	-0.943	0.601
13	*0.897	0.870	0.881	*0.884	-0.348	0.202	-1.025	0.594
14	*0.957	0.900	-	*0.926	-0.323	0.204	-0.946	0.594
15	*0.961	0.921	0.878	*0.944	-0.325	0.258	-1.000	0.593
Average	0.950	0.896	0.885	0.926	-0.333	0.207	-1.000	0.599

- = The estimation was unsuccessful

Table 27: Nonlinear estimation of the share equations, parameter combination B: Real values p=0.666 a=2 c=-0.5 δ=0.75 $x \sim N(0,1)$ $g \sim N(0,0.001)$

	R^2 values when the real regressand				Estimates from the best regressions (marked with *)			
	CES-type		par-type		Best CES-type		Best par-type	
Sample	Share function CES-type	Share function par-type	Share function CES-type	Share function par-type	p	δ	a	c
1	*0.994	0.971	0.965	*0.993	0.658	0.749	1.989	-0.500
2	*0.995	0.957	0.955	*0.994	0.658	0.750	1.982	-0.501
3	*0.995	0.957	0.982	*0.994	0.690	0.752	2.042	-0.501
4	*0.997	0.977	0.973	*0.997	0.658	0.751	1.975	-0.502
5	*0.995	0.981	0.974	*0.995	0.660	0.751	1.990	-0.501
6	*0.993	0.944	0.948	*0.991	0.655	0.749	1.963	-0.500
7	*0.995	0.900	0.917	*0.993	0.669	0.750	2.007	-0.500
8	*0.993	0.962	0.950	*0.992	0.666	0.749	2.025	-0.498
9	*0.995	0.952	0.972	*0.994	0.681	0.751	2.020	-0.499
10	*0.995	0.940	0.948	*0.994	0.677	0.750	2.022	-0.499
11	*0.992	0.969	0.970	*0.992	0.669	0.750	2.006	-0.500
12	*0.993	0.970	0.965	*0.993	0.670	0.747	2.022	-0.495
13	*0.993	0.940	0.955	*0.993	0.668	0.748	1.995	-0.496
14	*0.994	0.983	0.973	*0.994	0.645	0.751	1.943	-0.504
15	*0.995	0.969	0.960	*0.993	0.652	0.749	1.966	-0.499
Average	0.994	0.958	0.960	0.993	0.665	0.750	1.995	-0.500

Table 28: Nonlinear estimation of the share equations, parameter combination B: Real values p=0.666 a=2 c=-0.5 δ=0.75 $x \sim N(0,4)$ $g \sim N(0,0.01)$

	R^2 values when the real regressand				Estimates from the best regressions (marked with *)			
	CES-type		par-type		Best CES-type		Best par-type	
Sample	Share function CES-type	Share function par-type	Share function CES-type	Share function par-type	p	δ	a	c
1	*0.983	0.899	0.886	*0.975	0.691	0.743	2.099	-0.487
2	*0.966	0.850	0.834	*0.957	0.662	0.746	1.907	-0.493
3	*0.982	0.912	0.892	*0.973	0.676	0.745	2.139	-0.495
4	*0.970	0.893	0.905	*0.970	0.703	0.746	1.993	-0.484
5	*0.977	0.911	0.894	*0.977	0.623	0.745	1.899	-0.503
6	*0.981	0.856	0.835	*0.970	0.694	0.752	2.061	-0.499
7	*0.961	0.920	0.853	*0.955	0.605	0.751	2.024	-0.521
8	*0.984	0.852	0.862	*0.976	0.688	0.754	2.027	-0.500
9	*0.977	0.890	0.882	*0.968	0.689	0.750	2.056	-0.496
10	*0.967	0.834	0.864	*0.950	0.687	0.751	1.971	-0.494
11	*0.973	0.872	0.820	*0.963	0.673	0.751	2.106	-0.504
12	*0.964	0.838	0.807	*0.946	0.659	0.749	1.987	-0.500
13	*0.979	0.853	0.792	*0.963	0.669	0.747	2.023	-0.495
14	*0.983	0.908	0.865	*0.978	0.620	0.755	1.902	-0.519
15	*0.975	0.885	0.865	*0.968	0.696	0.745	2.123	-0.491
Average	0.975	0.878	0.857	0.966	0.669	0.749	2.021	-0.499

Table 29: Nonlinear estimation of the share equations, parameter combination C: Real values p=2 a=6
c=0.8 δ=0.1 $x \sim N(0,1)$ $g \sim N(0,0.001)$

	R^2 values when the real regressand				Estimates from the best regressions (marked with *)			
	CES-type		par-type		Best CES-type		Best par-type	
Sample	Share function CES-type	Share function par-type	Share function CES-type	Share function par-type	p	δ	a	c
1	*0.980	0.696	0.653	*0.854	1.999	0.099	6.132	0.805
2	*0.983	0.675	0.594	*0.881	1.978	0.104	6.970	0.797
3	*0.983	0.638	0.622	*0.867	2.002	0.099	6.275	0.805
4	*0.989	0.619	0.618	*0.885	2.083	0.094	6.514	0.800
5	*0.992	0.676	0.473	*0.896	1.894	0.108	4.557	0.800
6	*0.990	0.733	0.780	*0.913	1.936	0.108	5.104	0.787
7	*0.984	0.724	0.670	*0.881	1.997	0.100	5.235	0.799
8	*0.985	0.765	0.797	*0.877	1.969	0.104	4.313	0.785
9	*0.982	0.688	0.559	*0.890	1.967	0.105	6.379	0.792
10	*0.982	0.738	0.747	*0.891	2.044	0.097	6.295	0.800
11	*0.988	0.717	0.706	*0.871	1.987	0.101	5.603	0.800
12	*0.995	0.626	0.595	*0.919	1.989	0.099	6.061	0.804
13	*0.990	0.634	0.624	*0.860	2.018	0.097	6.668	0.808
14	*0.983	0.651	0.642	*0.902	2.051	0.099	8.352	0.800
15	*0.984	0.737	0.724	*0.917	2.012	0.101	6.439	0.795
Average	0.986	0.688	0.654	0.887	1.995	0.101	6.193	0.798

Table 30: Nonlinear estimation of the share equations, parameter combination C: Real values p=2 a=6
c=0.8 δ=0.1 $x \sim N(0,4)$ $g \sim N(0,0.001)$

	R^2 values when the real regressand				Estimates from the best regressions (marked with *)			
	CES-type		par-type		Best CES-type		Best par-type	
Sample	Share function CES-type	Share function par-type	Share function CES-type	Share function par-type	p	δ	a	c
1	*0.993	0.732	0.519	*0.896	1.960	0.102	5.605	0.804
2	*0.996	0.782	0.568	*0.920	1.979	0.104	6.863	0.804
3	*0.994	0.796	0.678	*0.895	1.944	0.107	4.958	0.794
4	*0.996	0.777	0.664	*0.926	2.024	0.097	7.132	0.807
5	*0.992	0.677	0.563	*0.854	1.892	0.111	4.497	0.786
6	*0.997	0.784	0.661	*0.908	1.911	0.107	4.060	0.798
7	*0.995	0.791	0.571	*0.901	2.042	0.094	5.332	0.798
8	*0.996	0.833	0.576	*0.923	2.049	0.097	7.314	0.795
9	*0.996	0.701	0.641	*0.920	1.969	0.105	5.638	0.794
10	*0.994	0.796	0.737	*0.932	2.018	0.098	9.086	0.805
11	*0.997	0.757	0.652	*0.929	1.977	0.104	6.424	0.799
12	*0.996	0.801	0.692	*0.913	1.982	0.100	3.915	0.797
13	*0.997	0.799	0.639	*0.952	1.991	0.101	7.346	0.804
14	*0.998	0.707	0.531	*0.919	1.973	0.105	6.630	0.798
15	*0.996	0.805	0.577	*0.910	1.895	0.113	5.470	0.796
Average	0.996	0.769	0.618	0.913	1.974	0.103	6.018	0.798

7. CONCLUSIONS

The most important property of the par production techniques compared with the CES production techniques is in the variation limits of the distributive shares. That is why we have specially examined these differences in the theory of income distribution. The analysis of the distributive shares deserves more attention in the production theory than it has got. Maybe the lack of analysis has been the result of the lack of proper tools and methods. In this study we have presented the par production technology, which is a practical addition to the methods of analysing the distributive shares. Broadly taken, the par technigues can be used to analyse the shares in many contexts to reveal the distribution limits on the markets. Based on the relatively simple mathematical forms and equations, however, the CES production technology seems to be more practical and useful for handling many of the other problems in production theory. In fact, the conclusion is that the par and CES production techniques are complementary to each other. Whether the par production technology should be used instead of the CES or vice versa depends on the target of the examination.
On the other hand the choice of the functional form for production technology can be based practically on the best fit in the empirical estimation. More generally, the distributive criteria based, for example, on the entropy measure could be adopted[1].

The substitution parameters in both the par and CES cases seem to have a similar role in defining the substitution between the inputs of production. Approximately, the interrelationship can even be expressed as $a \approx 3*p$. Thus the dependency, which the relative income shares have with the substitution parameters in question, applies in both the par and CES situation. The Cobb-Douglas special case is included in both of the compared production techniques.

As the impossibility theorem, proved by Uzawa[2] in 1968, implies that the CES production function cannot be generalized to n (n>2) variables with arbitrary values of the partial elasticities of substitution, the basic reason for this conclusion may lie in the fact that for all of the input variables in question the relative income shares cannot vary simultaneously "unlimitedly" between 0 and 1. To have constancy in the arbitrary values of the partial substitution parameters in question, it could be inferred, the relative income shares should have limited areas which are more narrow than 0 and 1 in case the number of input variables is bigger than 2. Whether this is the case, can be examined in the future.

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[1] About efficient functional forms in regression see, for example, Maasoumi (1986), pp.301-309. See also Harvey (1977) and Ramsey and Zarembka (1971), who have used several tests in order to discriminate functional forms in regression.

[2] See Uzawa (1968)

The par production function can be used in theoretical and empirical analysis both in the basic nonlinear and the linearized form. The utility of using par techniques is, of course, totally dependent on the situation. The sampling experiments we have made in this study show clearly that there is a clear difference between using CES and par techniques in different situations.

In the following we try to answer the questions put in chapter 6.1 concerning the sampling experiments.

a. The used estimation techniques discriminate the par and CES production techniques clearly when the nonlinear estimation of the share equations is used. The fit of the "right" equation is better in all but 1 of the 90 experiments we made. The nonlinear method clearly estimates the values of the parameters with the greatest unbiasedness of the estimation methods we used.

   Another method which discriminates the par and CES situations clearly is the input share ratio estimation made with the linear and linearized forms of the analysed relationships in question.

b. When the estimation is made with the linearized forms of the production functions, there seems to be bias in the values of the estimates concerning both production functions. The same kind of bias seems to exist in the values of the distribution limit parameter estimates when the side relation method is used to estimate the par production function. However the bias seems to exist only with extreme values of the parameters.

   The tests we made with the income share ratio equation seem to be practical in two senses: The test discriminates the par and CES production technologies from each other and when the higher degrees in the linearized form are included in the estimated equation of the par case, the bias in the estimates seems to vanish. Thus the method of estimating the income share ratio can be used with high reliability.

c. Based on items a and b above, the method of estimating the income share ratio equation (the higher degrees included) and the method of estimating the share equations with nonlinear methods can be suggested.

d. When the problem to be examined clearly involves distribution limits and this fact affects the handling of the problem and the making of conclusions for further activities, the par production technology should be preferred. Where such things do not exist, the CES production technology should be preferred, because of the easier handling and simpler mathematical form.

As the par production structure is convex only for a given area of parameter values, albeit the relevant area, difficulties in the compact theory of the par production structure arise unless simplifications are done. In this study these have been accomplished by the method of linearizing. The result seems to be practical and useful. Whether the linearized forms or the basic nonlinear forms are used is totally up to the examiner, as the par production technology seems to work not only in the linear least squares situation, but in nonlinear estimation situations as well.

The methods of econometric analysis of the technical change are rather advanced in the case of the CES production structure. As we have not analysed technical change in this study at all, it is worth noting that when the linear approximation forms for the par production structure functionally correspond to the CES structure in their simplest forms, the same general methods of measuring the technical change in estimation can be used in the par case as well.

This study is a study in the method of neoclassical theory of production with implications to the theory of income distribution. The production structure is the basic tool in many of the problems in economics. This study must leave the wide field of possible applications for later efforts.

APPENDIX A: Proof that the Cobb-Douglas production function form is a limiting form of the par production function when $a \rightarrow 0$

First we have to assume that in the par function form

$$(A.1) \qquad Y = \left\{ c * \frac{K^{-a} - L^{-a}}{(K/L)^{-a*c} - 1} \right\}^{(-1/a)}$$

we have K>L and c><0 and a>0. When K<L and/or a<0 the same result will be attained, but in derivation the signs have to be separately noticed. When c=0 the derivation is somewhat easier.

Dividing both sides of (A.1) with L we get

$$(A.2) \qquad \frac{Y}{L} = \left\{ c * \frac{k^{-a} - 1}{k^{-a*c} - 1} \right\}^{(-1/a)}$$

where k=K/L and taking the natural logarithm further

$$(A.3) \qquad \ln(Y/L) = \frac{\ln(k^{-a}-1) - \ln\{(1/c)*(k^{-a*c}-1)\}}{a}$$

We will make an obvious notation

$$(A.4) \qquad \ln(Y/L) = \frac{m(a)}{n(a)}$$

and derive the limiting value of (A.4) with L`Hopital`s rule as follows

$$(A.5) \qquad \begin{aligned} n`(a) &= 1 \\ m`(a) &= \frac{-k^{-a}*\ln k}{k^{-a} - 1} + \frac{c*k^{-a*c}*\ln k}{k^{-a*c} - 1} \end{aligned}$$

Further we find that

$$(A.6) \qquad m`(a) = \frac{-k^{-a-a*c}*\ln k + k^{-a}*\ln k + c*k^{-a-a*c}*\ln k - c*k^{-a*c}*\ln k}{(k^{-a} - 1)*(k^{-a*c} - 1)}$$

and make a notation again

$$(A.7) \qquad m`(a) = \frac{g(a)}{h(a)}$$

Using L`Hopital`s rule we get

$$\text{(A.8)} \qquad \lim_{a \to 0} m`(a) = \lim \frac{g''(a)}{h''(a)}$$

$$= \frac{c*(\ln k)^3 - c^2*(\ln k)^3}{2*c*(\ln k)^2}$$

and further by substituting (A.8) and $n`(a) = 1$ into (A.4) we get

$$\text{(A.9)} \qquad \lim_{a \to 0} \{\ln(Y/L)\} = \left(\frac{1-c}{2}\right)*\ln k$$

which gives us the needed limiting form

$$\text{(A.10)} \qquad \ln Y = \left(\frac{1-c}{2}\right)*\ln K + \left(\frac{1+c}{2}\right)*\ln L$$

representing the Cobb-Douglas production function.

APPENDIX B: The limiting values of the par input elasticities

According to (3.12) the partial input elasticity of capital is

$$\text{(B.1)} \qquad \Phi = \frac{K^{-a} - Y^{-a}}{K^{-a} - L^{-a}} - c \qquad (K><L) \quad (a><0)$$

Substituting (3.5) into (B.1) we get

$$\text{(B.2)} \qquad \Phi = \frac{k^{-a}}{k^{-a} - 1} - \frac{c}{k^{-a*c} - 1} - c \qquad (K><L) \quad (a><0) \quad (c><0)$$

where k=K/L. From (B.2) the limiting values lim Φ in (3.13) can directly be calculated for various values of a and c when $k \rightarrow \infty$ and $k \rightarrow 0$.

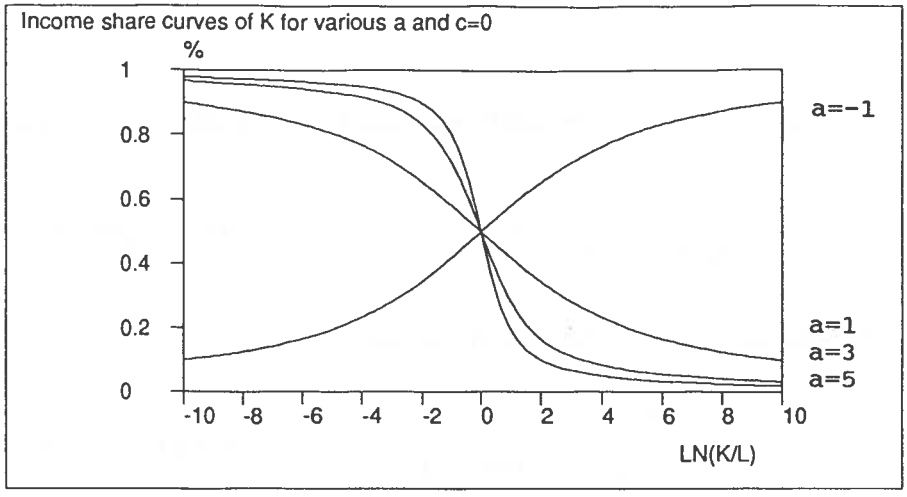
When c=0 we get correspondingly

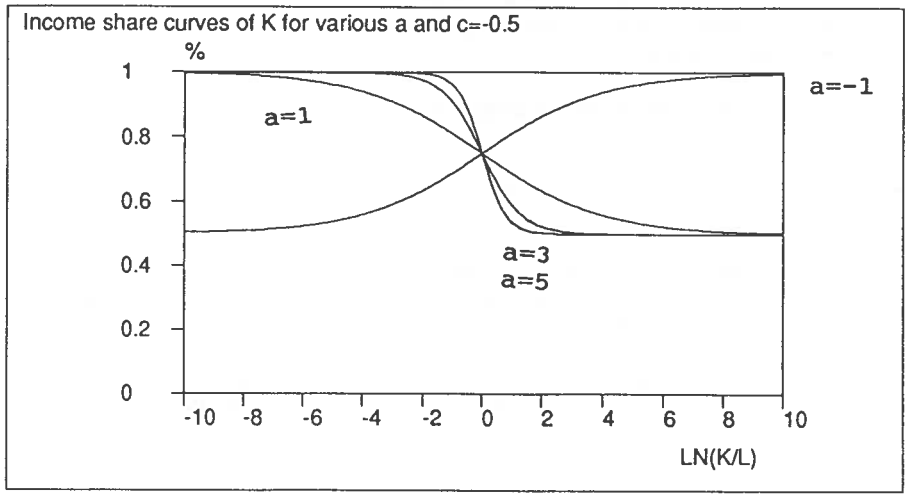
$$\text{(B.3)} \qquad \Phi = \frac{k^{-a}}{k^{-a} - 1} + \frac{1}{a*\ln k} \qquad (K><L) \quad (a><0) \quad (c=0)$$

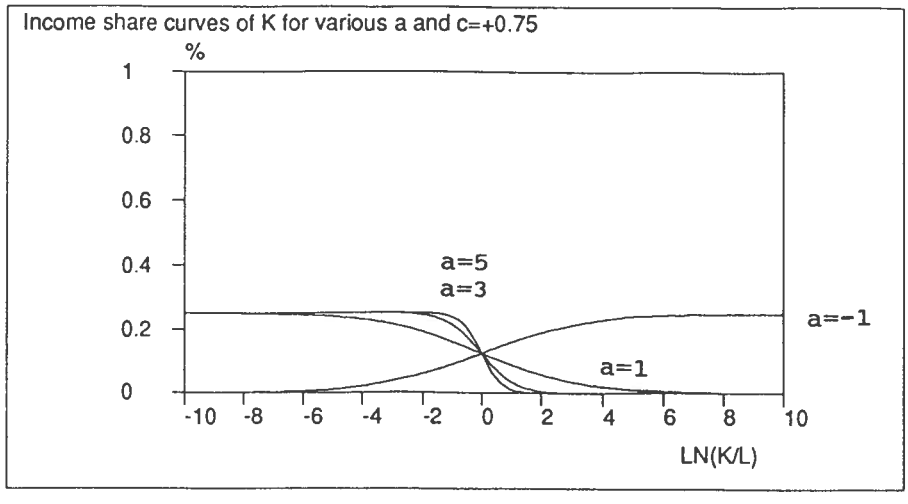
from which the limiting values can be noted in case c=0. When a=0 we have the Cobb-Douglas production function with constant partial elasticities, $\Phi=(1-c)/2$.

For the partial input elasticity of labor ϕ the limiting values can be calculated with $\phi=1-\Phi$ according to (3.12).

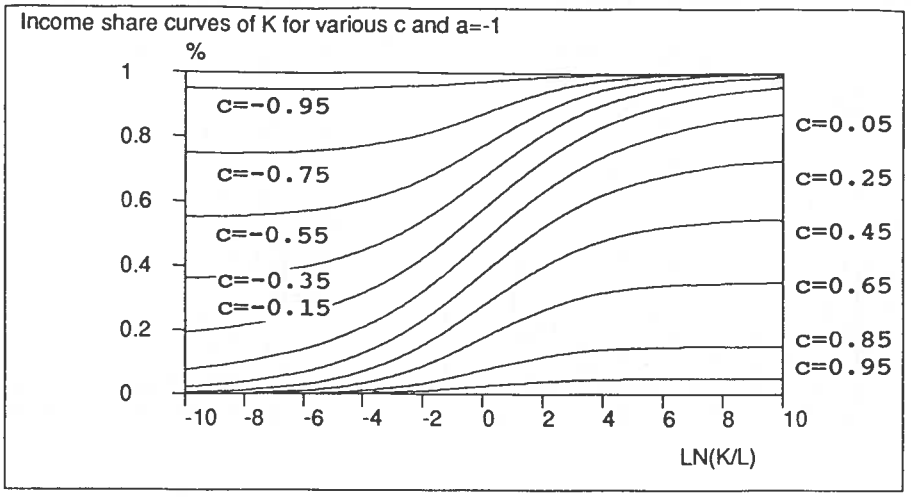
APPENDIX C: Par income share curves as a function of x=ln(K/L)

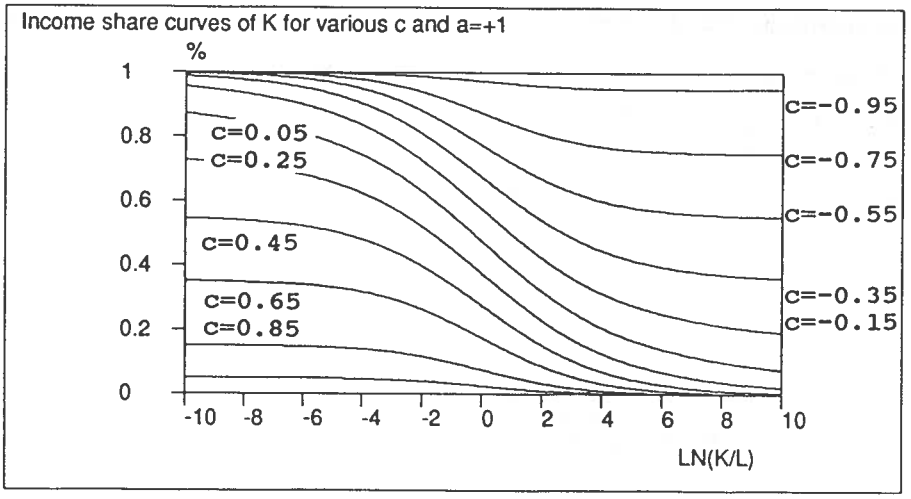


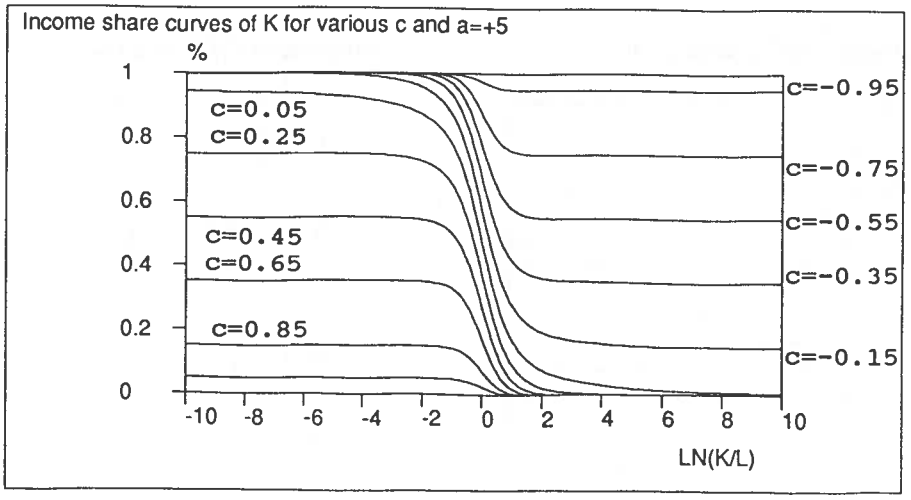




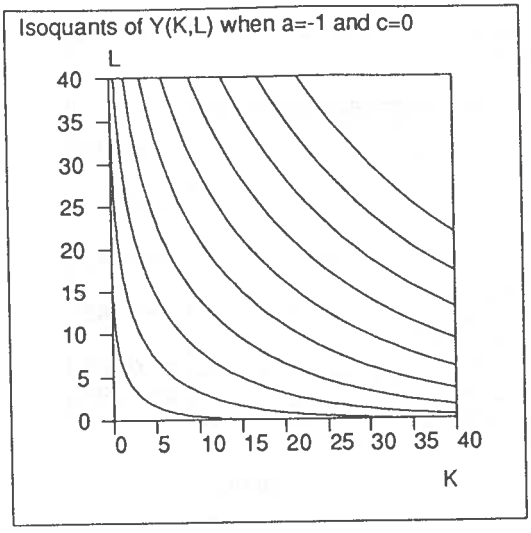
APPENDIX C: Par income share curves as a function of x=ln(K/L)

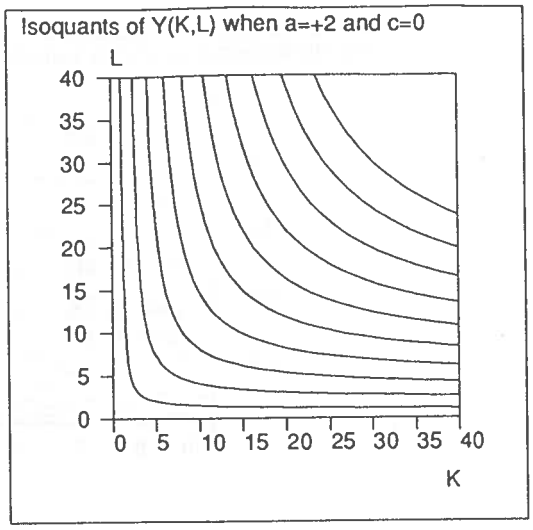


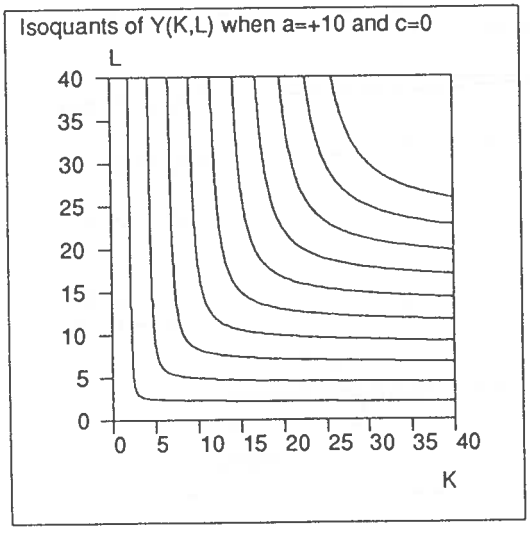


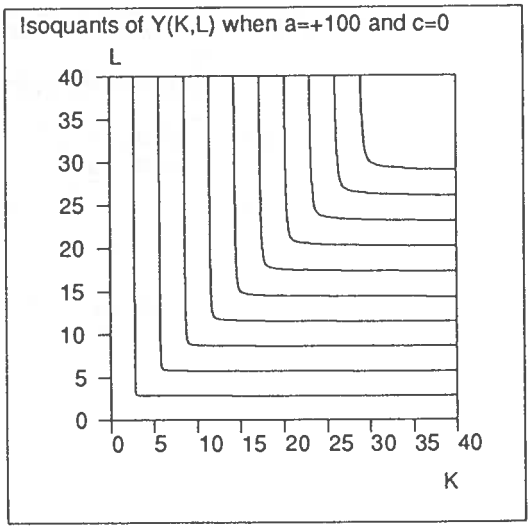


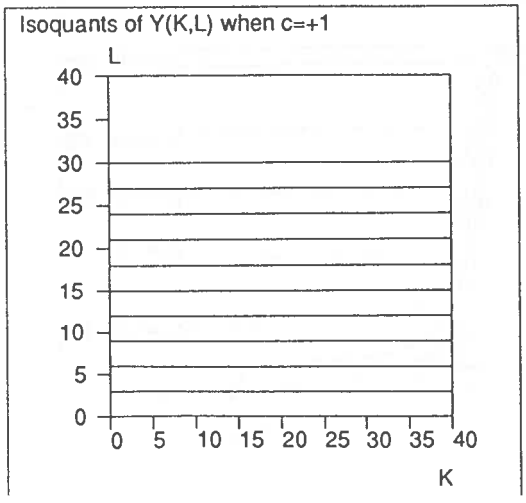
APPENDIX D: Par production function isoquant curves for various values of parameters a and c

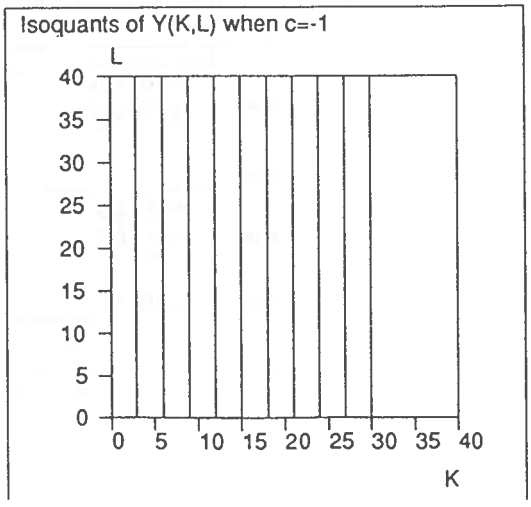




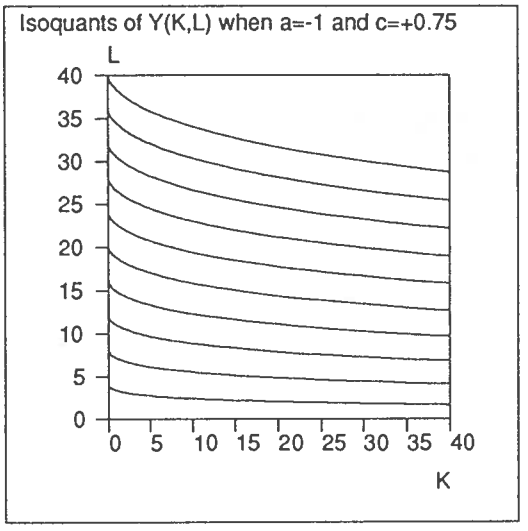


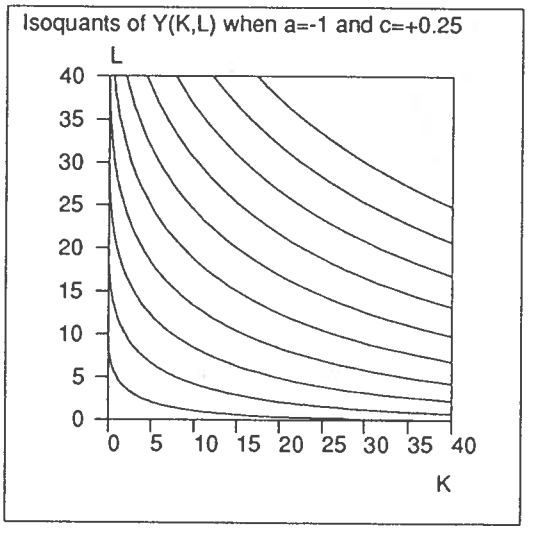


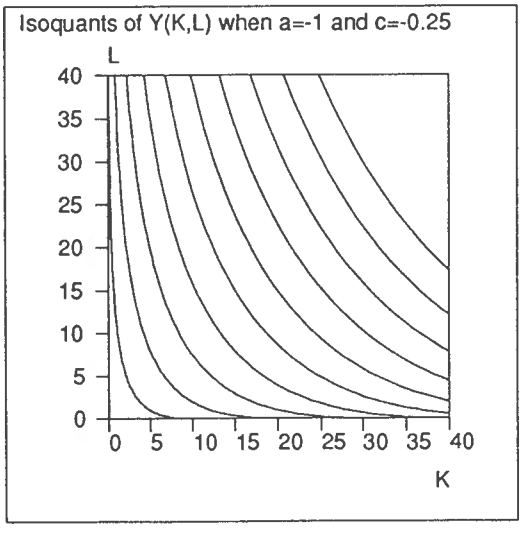


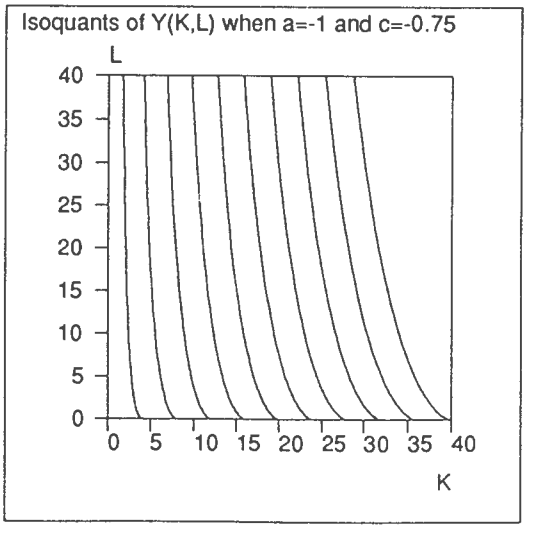


APPENDIX D: Par production function isoquant curves for various values of parameters a and c

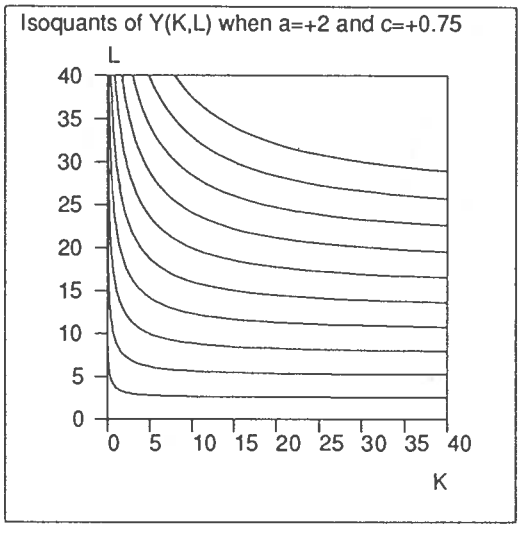


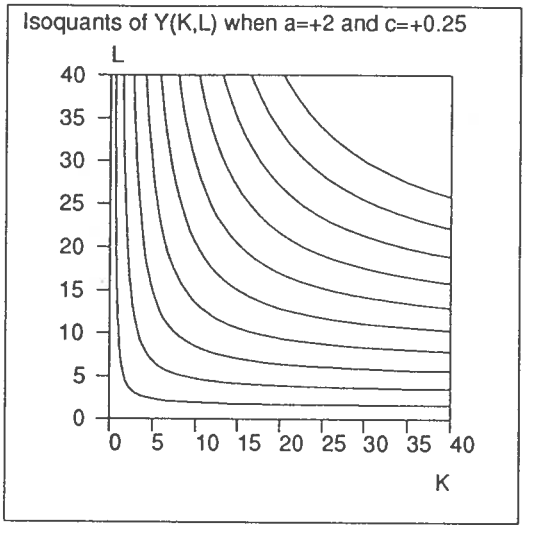


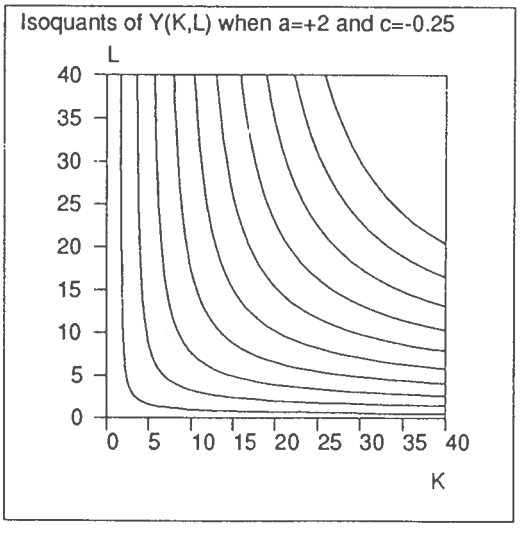


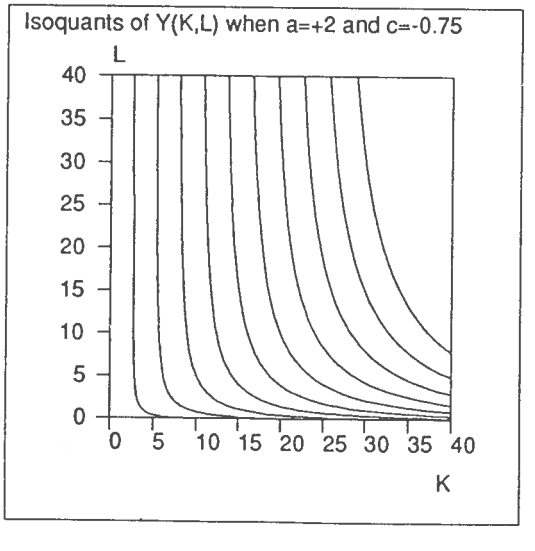


APPENDIX D: Par production function isoquant curves for various values of parameters a and c

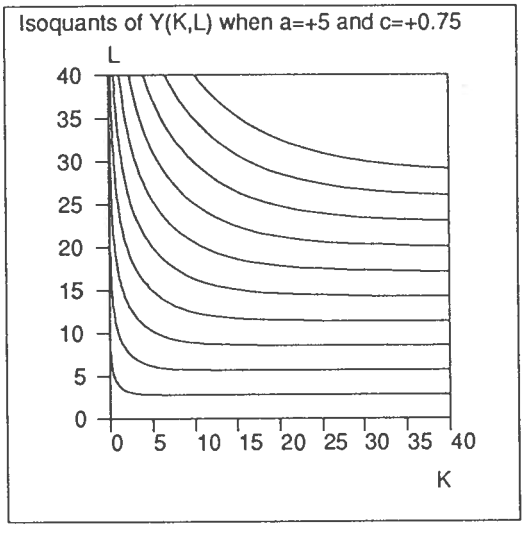


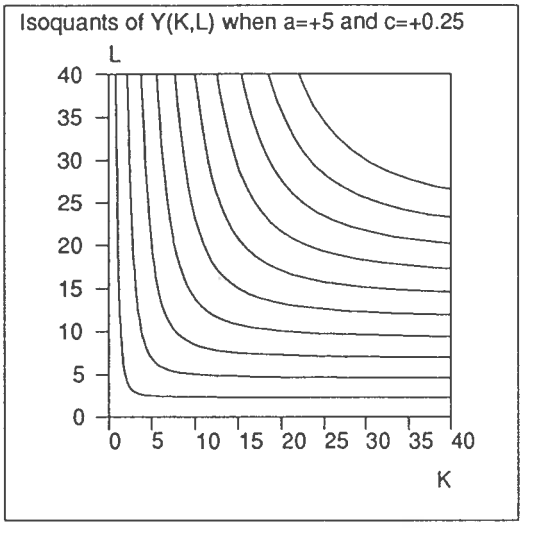


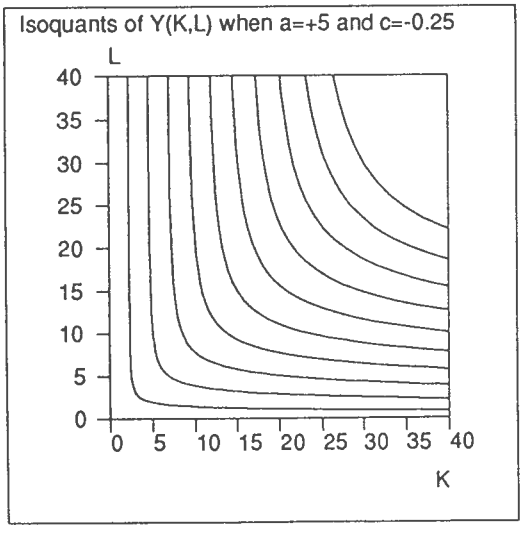


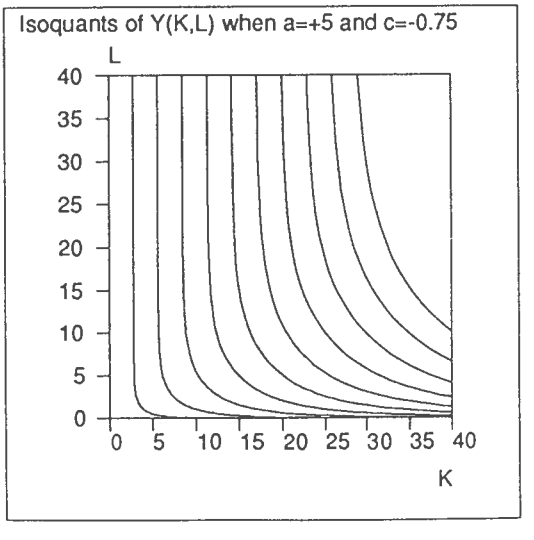


APPENDIX D: Par production function isoquant curves for various values of parameters a and c

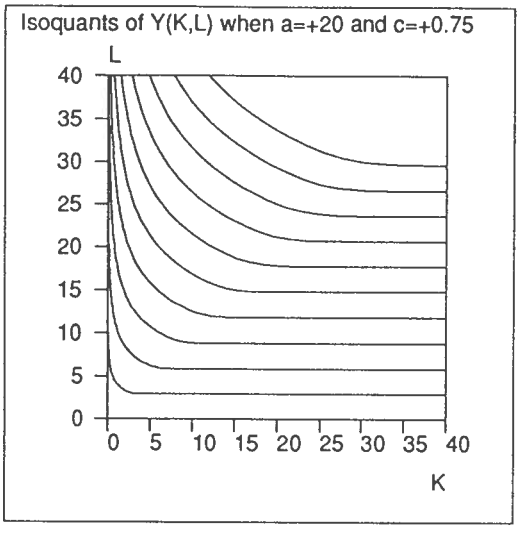


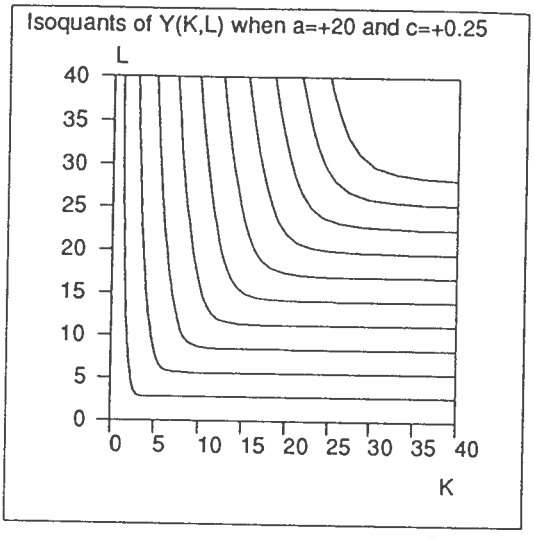


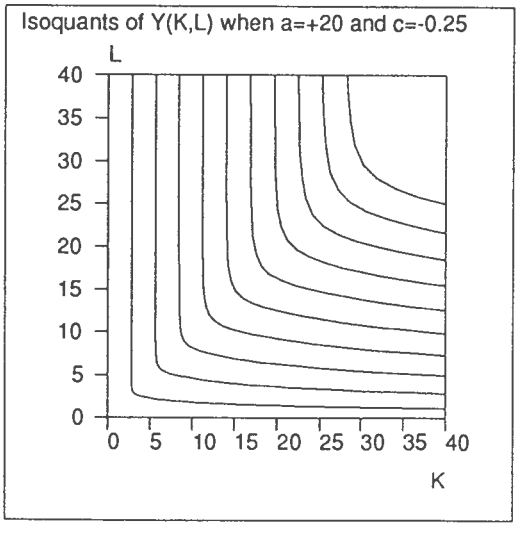


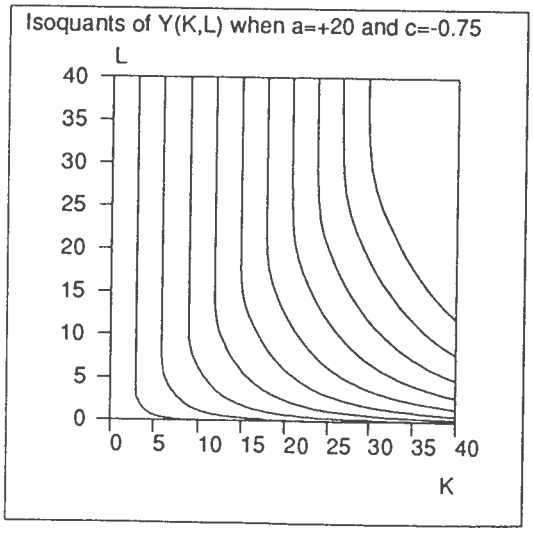


APPENDIX D: Par production function isoquant curves for various values of parameters a and c

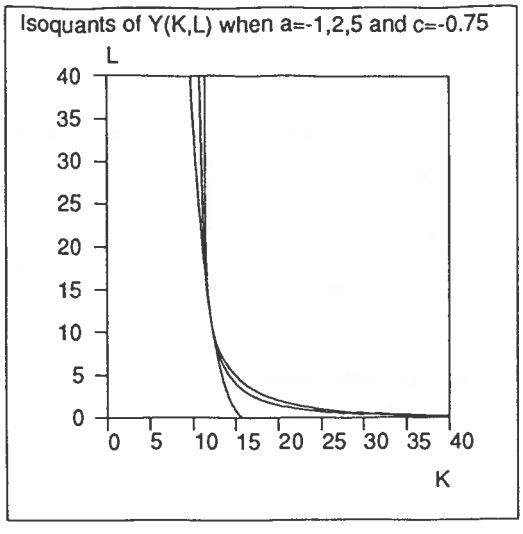


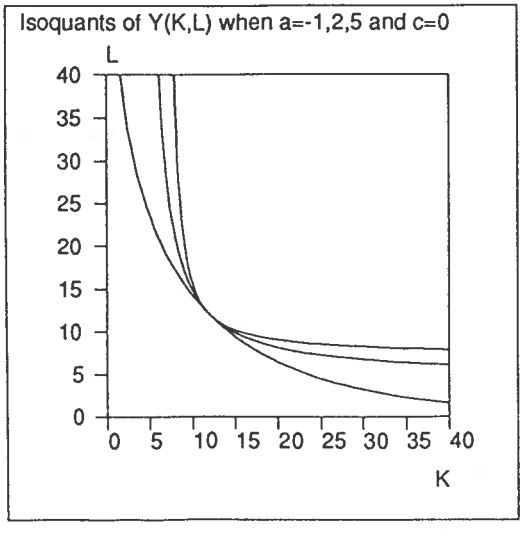


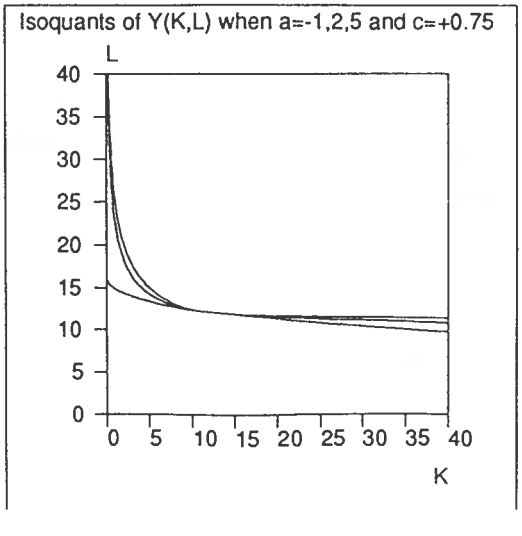




APPENDIX D: Par production function isoquant curves for various values of parameters a and c







APPENDIX E: Derivation of the linear approximation forms in chapter 4.1

In equation (3.6) we have the first order profit maximum condition for the par production function form

$$\text{(E.1)} \qquad \frac{R}{W} = \frac{L}{K} * \frac{K^{-a} - (K/L)^{-a*c*Y^{-a}}}{(K/L)^{-a*c*Y^{-a}} - L^{-a}} \qquad (K><L) \quad (a><0)$$

Writing (E.1) first as follows[1]

$$\text{(E.2)} \qquad \frac{w_K}{w_L} = (K/L)^{-a*}\frac{1 - (Y/K)^{-a} - c + c*(K/L)^{a}}{(Y/L)^{-a} + c*(K/L)^{-a} - c - 1}$$

and substituting further (3.5) into (E.2) we get

$$\text{(E.3)} \qquad \frac{w_K}{w_L} = (K/L)^{-a*}\frac{k^{-a*c} - 1 - c*k^{-a*c} + c*k^{a-a*c}}{1 + c*k^{-a-a*c} - c*k^{-a*c} - k^{-a*c}}$$

If we make an obvious notation

$$\text{(E.4)} \qquad \frac{w_K}{w_L} = k^{-a*}\Omega$$

where k=K/L, we can calculate the limiting value for Ω by noting the nominator and denominator of Ω as follows

$$\text{(E.5)} \qquad \Omega(a) = \frac{h(a)}{g(a)}$$

---

1 The variables w_K and w_L are the income shares

$$w_K = \frac{R*K}{R*K + W*L}$$

$$w_L = \frac{W*L}{R*K + W*L}$$

and then using the quantient derivation rule and L`Hopital`s rule we get

$$\lim_{a \to 0} \Omega(a) = \frac{1-c}{1+c}$$

(E.6) $$\lim_{a \to 0} \Omega`(a) = \frac{1-c}{1+c} * (2/3)*\ln k$$

$$\lim_{a \to 0} \Omega"(a) = \frac{1-c}{1+c} * (c/18)*(\ln k)^2$$

from which we get the form

(E.7) $$\frac{w_K}{w_L} = (\frac{1-c}{1+c})*\{1+(2/3)*a*x+(c/18)*a^2*x^2\}*k^{-a}$$

When k^{-a} is developed as

(E.8) $$k^{-a} = e^{-a*x}$$

$$\approx 1 - a*x + (1/2)*a^2*x^2 - (1/6)*a^3*x^3$$

we further get

(E.9) $$\ln(w_K/w_L) = \ln(\frac{1-c}{1+c}) + \ln\{1-(1/3)*a*x+(\frac{2+c}{36})*a^2*x^2\}$$

from which by Taylor expansion we get

(E.10)
$$\tau(a) = \ln\{1-(1/3)*a*x+(\frac{2+c}{36})*a^2*x^2\}$$
$$\tau(0) = 0$$
$$\tau`(0) = -(1/3)*x$$
$$\tau"(0) = -(1/18)*c*x^2$$

so that

(E.11) $$\ln(R/W) = \ln(\frac{1-c}{1+c}) - [1+(1/3)*a]*x - (1/18)*c*a^2*x^2$$

which we have in equations (4.2) and (4.5). Note that $\tau"(0)$ is zero when c=0 and thus (E.11) gives an especially good approximation.

APPENDIX F: Tables for the approximation errors of equations (4.2) and (4.5) in percents

In equation

$$\text{(F.1)} \qquad \ln(R/W) = \ln\left(\frac{1-c}{1+c}\right) - [1+(1/3)*a]*\ln k$$

the values of the right hand side differ from the left hand side as follows (c=0):

(%)	|lnk| =0	=1	=2	=3	=4	=5	=6
|a|=0.5	0	0.0	0.1	0.2	0.4	0.6	0.9
=1.0	0	0.2	0.7	1.5	2.5	3.7	4.9
=1.5	0	0.5	2.0	4.2	6.6	9.1	11.4
=2.0	0	1.1	4.1	8.1	12.1	15.8	19.1
=2.5	0	2.0	6.9	12.8	18.3	23.2	
=3.0	0	3.1	10.3	18.1	25.0		
=3.5	0	4.5	14.1	23.8			
=4.0	0	6.0	18.2	29.7			

In equation (4.5) we have

$$\text{(F.2)} \qquad \ln(R/W) = \ln\left(\frac{1-c}{1+c}\right) - [1+(1/3)*a]*\ln k - (1/18)*c*a^2*(\ln k)^2$$

$$+\left(\frac{2-3*c^2}{810}\right)*a^3*(\ln k)^3 + \left(\frac{c^3+c}{3240}\right)*a^4*(\ln k)^4$$

When c=0, the corresponding errors in percents are:

(%)	|lnk| =0	=1	=2	=3	=4	=5	=6
|a|=0.5	0	0.0	0.0	0.0	0.0	0.0	0.1
=1.0	0	0.0	0.0	0.2	0.5	1.1	2.1
=1.5	0	0.0	0.2	1.0	2.9	6.1	10.9
=2.0	0	0.1	0.8	3.5	9.2	18.5	
=2.5	0	0.1	2.1	8.6	21.5		
=3.0	0	0.3	4.4	17.3			
=3.5	0	0.7	8.2	30.7			
=4.0	0	1.2	13.8				
=4.5	0	1.9					
=5.0	0	2.9					

from which it can be concluded that the approximation does represent the right function value widely. However, when higher order terms are included in the approximation, the area of good representation is somewhat more narrow.

The theoretical values of the left hand side of the equations (F.1) and (F.2) are calculated as a rate of marginal technical substitution which is the ratio of the marginal productivities of the par production function (3.5).

APPENDIX G: A linear approximation form for the par production function in the neighbourhood of a=0

The par production function

$$(G.1) \qquad Y^{-a} = c*\frac{K^{-a} - L^{-a}}{(K/L)^{-a*c} - 1} \qquad (a><0) \quad (c><0) \quad (K><L)$$

can be written as follows

$$(G.2) \qquad (1/c)*V^{-a} = \frac{k^{-a} - 1}{k^{-a*c} - 1} \qquad (a><0) \quad (c><0) \quad (K><L)$$

where V=Y/L and k=K/L. We will note

$$(G.3) \qquad M(a) = \frac{k^{-a} - 1}{k^{-a*c} - 1} = \frac{h(a)}{g(a)} \qquad (a><0) \quad (c><0) \quad (K><L)$$

First we assume that c>0. When c<0, the same results can be attained by multiplying (G.2) with -1. Further we will note

$$(G.4) \qquad \epsilon(a) = \ln(1/c) - a*\ln V = \ln\left(\frac{k^{-a} - 1}{k^{-a*c} - 1}\right) = \ln M(a)$$

Using L`Hopital`s rule we get

$$\epsilon(0) = -\ln c$$

$$(G.5) \qquad \epsilon`(0) = \left(\frac{c-1}{2}\right)*\ln k$$

$$\epsilon"(0) = \left(\frac{1-c^2}{12}\right)*(\ln k)^2$$

So that approximately

$$(G.6) \qquad \epsilon(a) \approx -\ln c + \left(\frac{c-1}{2}\right)*a*\ln k + \left(\frac{1-c^2}{2*12}\right)*a^2*(\ln k)^2$$

From (G.6) we get further by using the definitions (G.3) and (G.4)

$$(G.7) \qquad \ln Y = \left(\frac{1-c}{2}\right)*\ln K + \left(\frac{1+c}{2}\right)*\ln L - \left(\frac{1-c^2}{24}\right)*a*[\ln(K/L)]^2$$

which is the needed second order linear approximation form for the par production function. The special case c=0 leads to the same formula. The Cobb-Douglas situation (a=0) is also of the form (G.7).

APPENDIX H: On the approximative convexity conditions of the par production function

In the equation (4.2) we have in fact

$$\text{(H.1)} \qquad \ln\left(\frac{dL}{dK}\right) = \ln\left(\frac{1-c}{1+c}\right) - [1+(1/3)*a]*x - (1/18)*c*a^2*x^2$$

where $x=\ln(K/L)=\ln k$. For the approximation to be convex we have to have

$$\text{(H.2)} \qquad \frac{@\ \ln(-dL/dK)}{@\ x} \leq 0$$

so that the convexity condition will be (approximately, as the equation (4.2) is only an approximation)

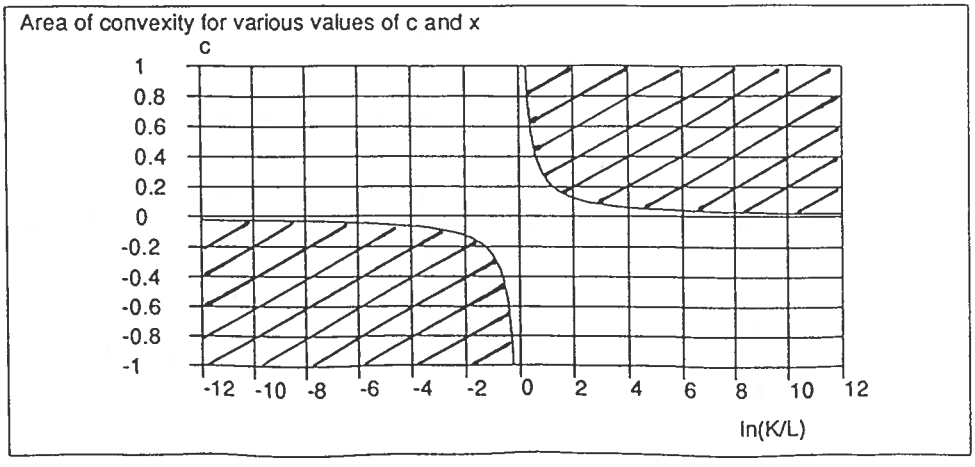
$$\text{(H.3)} \qquad -(1+a/3) - (1/9)*c*a^2*x \leq 0$$

<u>When c=0</u> we get a condition a>-3 for the approximation to be strictly convex.

<u>When c><0</u> we can solve the second order equation (H.3) to get

$$\text{(H.4)} \qquad a = \frac{-3 \pm 3*\sqrt{(1-4*c*x)}}{2*c*x}$$

from which it can be concluded that the approximation is convex for all values of the parameter a when c*x>1/4. This area is shaded in the following picture



In the following table there are the convexity limits for x according to various values of the parameters a and c, which are based on equation (H.3):

	c=0.9	c=0.5	c=0.1	c=-0.1	c=-0.5	c=-0.9
a=-2	x>-0.8	x>-1.5	x>-7.5	x<7.5	x<1.5	x<0.8
a=-1	x>-6.7	x>-12	x>-60	x<60	x<12	x<6.7
a=+1	x>-13.3	x>-24	x>-120	x<120	x<24	x<13.3
a=+3	x>-2.2	x>-4	x>-20	x<20	x<4	x<2.2
a=+5	x>-1.1	x>-1.9	x>-9.6	x<9.6	x<1.9	x<1.1

Because the equation (4.2) is an approximation, the limits calculated for higher values of the parameter a are not representative. The conclusion based on the above table is, however, that in the neighbourhood of a=0 the approximation equation is convex for the relevant area of values of the parameters a and c.

APPENDIX I: Approximative solution of the par input demand functions

According to the equation (4.5) we have

$$\text{(I.0)} \qquad \ln\left(\frac{R}{W}\right) = \ln\left(\frac{1-c}{1+c}\right) - [1+(1/3)*a]*\ln(K/L)$$

when higher order terms with respect to a are neglected.

Solving (I.0) for ln(K/L) we get

$$\text{(I.1)} \qquad \ln(K/L) \approx \left(\frac{3}{3+a}\right)* \left[\ln\left(\frac{1-c}{1+c}\right) - \ln\left(\frac{R}{W}\right)\right]$$

which is an approximative solution when $a>-3$. An one-valued solution is possible only under strict convexity. See appendix H. Solving (I.1) for lnK we get

$$\text{(I.2)} \qquad \ln K = \ln L + \left(\frac{3}{3+a}\right)*\ln\left(\frac{1-c}{1+c}\right) - \left(\frac{3}{3+a}\right)*\ln\left(\frac{R}{W}\right) \qquad (a>-3)$$

To get an approximation for the input demand functions we substitute the Cobb-Douglas production function condition

$$\text{(I.3)} \qquad \ln K = \left(\frac{2}{1-c}\right)*\ln Y - \left(\frac{1+c}{1-c}\right)*\ln L$$

into equation (I.2). Then we get the approximative demand functions for the inputs as follows

$$\text{(I.4)} \qquad \begin{aligned} \ln L &= \ln Y - \left(\frac{1-c}{2}\right)*A + \left(\frac{1-c}{2}\right)*\left(\frac{3}{3+a}\right)*\ln\left(\frac{R}{W}\right) \\ \ln K &= \ln Y + \left(\frac{1+c}{2}\right)*A - \left(\frac{1+c}{2}\right)*\left(\frac{3}{3+a}\right)*\ln\left(\frac{R}{W}\right) \end{aligned} \qquad (a>-3)$$

where $A=\left(\frac{3}{3+a}\right)*\ln\left(\frac{1-c}{1+c}\right)$.

A better approximation for the input demand functions is given, if the approximation form (G.7) is used instead of (I.3). In that case we get

$$\ln L = \ln Y - \left(\frac{1-c}{2}\right)*A + B + \left[\left(\frac{1-c}{2}\right)*\left(\frac{3}{3+a}\right) + C\right]*\ln(R/W) + D*[\ln(R/W)]^2$$

(I.5)

$$\ln K = \ln Y + \left(\frac{1+c}{2}\right)*A + B - \left[\left(\frac{1+c}{2}\right)*\left(\frac{3}{3+a}\right) + C\right]*\ln(R/W) + D*[\ln(R/W)]]^2$$

where

$$A = \left(\frac{3}{3+a}\right)*\ln\left(\frac{1-c}{1+c}\right)$$

$$B = \left(\frac{1-c^2}{8}\right)*\left[\frac{3*a}{(3+a)^2}\right]*\left[\ln\left(\frac{1-c}{1+c}\right)\right]^2$$

(I.6)

$$C = \left(\frac{1-c^2}{4}\right)*\left[\frac{3*a}{(3+a)^2}\right]*\ln\left(\frac{1-c}{1+c}\right)$$

$$D = \left(\frac{1-c^2}{8}\right)*\left[\frac{3*a}{(3+a)^2}\right]$$

APPENDIX J: Proof for equation (4.26)

In equations (4.22) and (4.25) we have

$$F(x) = \frac{1}{1 + e^{A-(x/\sigma)}}$$

(J.1)

$$w_L = \frac{1}{1 + e^{A+x-(x/\sigma)}}$$

where $A=\ln[\delta/(1-\delta)]$. From (J.1) we get

$$\ln[\frac{1}{F(x)} - 1] = A - (x/\sigma)$$

(J.2)

$$\ln[\frac{1}{w_L} - 1] = A - (x/\sigma) + x$$

Substitution of the former into the latter leads to

(J.3) $$e^x = \frac{[\frac{1}{w_L} - 1]}{[\frac{1}{F(x)} - 1]} = \frac{F(x)}{w_L} * \frac{1 - w_L}{1 - F(x)}$$

and noting that $G(x) = \ln[\frac{F(x)}{1 - F(x)}]$ and $w_K = 1 - w_L$ we get

(J.4) $$\ln(\frac{w_K}{w_L}) = x - G(x)$$

$$= x - \ln[\frac{F(x)}{1 - F(x)}]$$

which we have in equation (4.26).

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